

SUBORDINATIONS BY ALPHA-CONVEX FUNCTIONS

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Abstract

Let $H(U)$ be the space of analytic functions in the unit disk U and let $\mathcal{D} = \{\varphi \in H(U) : \varphi(0) = 1, \varphi(z) \neq 0, z \in U\}$. For the functions $\phi, \varphi \in \mathcal{D}$ we will determine simple sufficient conditions such that

$$\left[\frac{\varphi(z)}{\phi(z) + (1/\gamma)z\phi'(z)} \right]^{1/\beta} f(z) < k(z) \Rightarrow \mathbf{I}_{\phi, \varphi; \beta, \gamma}[f](z) < k(z),$$

for all $k \in \mathcal{M}'_{1/\beta}$, where

$$\mathbf{I}_{\phi, \varphi; \beta, \gamma}[f](z) = \left[\frac{\gamma}{z^\gamma \phi(z)} \int_0^z f^\beta(t) t^{\gamma-1} \varphi(t) dt \right]^{1/\beta}$$

and $\mathcal{M}'_{1/\beta}$ represents the class of $1/\beta$ -convex functions (not necessarily normalized).

In particular, we will give sufficient conditions on ϕ and φ so that the operators $\mathbf{I}_{\phi, \varphi; \beta, \gamma}$ are averaging operators on certain subsets of $H(U)$. In addition, some particular cases of the main result, obtained for appropriate choices of the ϕ and φ functions, will also be given.

1. Introduction

Let $H(U)$ be the space of analytic functions in the unit disk $U = \{z \in \mathbf{C} : |z| < 1\}$ and let $\mathcal{D} = \{\varphi \in H(U) : \varphi(0) = 1, \varphi(z) \neq 0, z \in U\}$. We denote by \mathcal{A} the class of analytic functions in U and usually normalized, i.e.

$$\mathcal{A} = \{f \in H(U) : f(0) = 0, f'(0) = 1\}.$$

If $\beta, \gamma \in \mathbf{C}$ with $\beta \neq 0$, for the functions $\phi, \varphi \in \mathcal{D}$ we will define the integral operator $\mathbf{I}_{\phi, \varphi; \beta, \gamma} : \mathcal{K}_{\phi; \beta, \gamma} \rightarrow H(U)$ of the form

$$(1.1) \quad \mathbf{I}_{\phi, \varphi; \beta, \gamma}[f](z) = \left[\frac{\gamma}{z^\gamma \phi(z)} \int_0^z f^\beta(t) t^{\gamma-1} \varphi(t) dt \right]^{1/\beta},$$

where $\mathcal{K}_{\phi; \beta, \gamma} \subset H(U)$ will be determined in Lemma 3.1, such that the integral operator (1.1) is well defined.

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The first major result concerning this operator was given in [8], and some particular cases were previously studied in a large number of papers.

For a set $E \subset \mathbf{C}$ let denote by $\text{co } E$ the *convex hull* of E . In [10] and [4] the authors introduced the concept of *averaging (or mean-value) operator* on an arbitrary set $K \subset H(\mathbf{U})$, like an operator $\mathbf{I} : K \rightarrow H(\mathbf{U})$ that satisfies

$$\mathbf{I}[f](0) = f(0) \quad \text{and} \quad \mathbf{I}[f](\mathbf{U}) \subset \text{co } f(\mathbf{U}), \quad \text{for all } f \in K.$$

For $f, g \in H(\mathbf{U})$ we say that the function f is *subordinate* to g , written $f(z) < g(z)$, if g is univalent in \mathbf{U} , $f(0) = g(0)$ and $f(\mathbf{U}) \subseteq g(\mathbf{U})$.

By using several results involving differential subordinations and subordination chains techniques, we will determine simple sufficient conditions on $\phi, \varphi \in \mathcal{D}$ such that

$$\left[\frac{\varphi(z)}{\phi(z) + (1/\gamma)z\phi'(z)} \right]^{1/\beta} f(z) < k(z) \Rightarrow \mathbf{I}_{\phi, \varphi; \beta, \gamma}[f](z) < k(z),$$

for all $k \in \mathcal{M}'_{1/\beta}$, where $\mathcal{M}'_{1/\beta}$ represents the class of $1/\beta$ -convex functions (not necessarily normalized) and is given by (2.2). In particular, we will give conditions on ϕ and φ so that the operators $\mathbf{I}_{\phi, \varphi; \beta, \gamma}$ are averaging operators on certain subsets of $H(\mathbf{U})$, and in addition, some special cases of the main result obtained for appropriate choices of the ϕ and φ functions will also be presented.

2. Preliminaries

In order to prove our main results, we will need the following definitions and lemmas presented in this section.

Let denote by \mathcal{Q} the set of functions q that are analytic and injective on $\overline{\mathbf{U}} \setminus E(q)$, where

$$E(q) = \left\{ \zeta \in \partial\mathbf{U} : \lim_{z \rightarrow \zeta} q(z) = \infty \right\},$$

and such that $q'(\zeta) \neq 0$ for $\zeta \in \partial\mathbf{U} \setminus E(q)$.

LEMMA 2.1 [1]. *Let $q \in \mathcal{Q}$, with $q(0) = a$, and let $p(z) = a + a_n z^n + \dots$ be analytic in \mathbf{U} with $p(z) \not\equiv a$ and $n \geq 1$. If p is not subordinate to q , then there exist points $z_0 \in \mathbf{U}$ and $\zeta_0 \in \partial\mathbf{U} \setminus E(q)$, and an $m \geq n \geq 1$ for which $p(|z| < |z_0|) \subset q(\mathbf{U})$, and*

- (i) $p(z_0) = q(\zeta_0)$,
- (ii) $z_0 p'(z_0) = m \zeta_0 q'(\zeta_0)$,
- (iii) $\text{Re} \frac{z_0 p''(z_0)}{p'(z_0)} + 1 \geq m \text{Re} \left[\frac{z_0 q''(z_0)}{q'(z_0)} + 1 \right]$.

For $\alpha \in \mathbf{R}$, a function $f \in H(\mathbf{U})$ with $f(0) = 0$ and $f'(0) \neq 0$ is called to be an α -convex function (not necessarily normalized), if

$$\operatorname{Re} \left[(1 - \alpha) \frac{zf'(z)}{f(z)} + \alpha \left(\frac{zf''(z)}{f'(z)} + 1 \right) \right] > 0, \quad z \in U,$$

and we denote this class by \mathcal{M}_α . The class of α -convex functions was introduced by P. T. Mocanu in [9]. Note that all α -convex functions are univalent and starlike, and moreover [7],

$$(2.1) \quad \mathcal{M}_\alpha \subset \mathcal{M}_\beta \subset \mathcal{M}_0, \quad \text{for } 0 \leq \frac{\beta}{\alpha} \leq 1.$$

For $\alpha \in \mathbf{R}$ we denote by

$$(2.2) \quad \mathcal{M}'_\alpha = \left\{ f \in H(U) : f'(0) \neq 0, \operatorname{Re} \left[(1 - \alpha) \frac{zf'(z)}{f(z)} + \alpha \left(\frac{zf''(z)}{f'(z)} + 1 \right) \right] > 0, z \in U \right\},$$

and then

$$\mathcal{K}' \equiv \mathcal{M}'_1 = \left\{ f \in H(U) : f'(0) \neq 0, \operatorname{Re} \frac{zf''(z)}{f'(z)} + 1 > 0, z \in U \right\}$$

represents the class of *convex functions* (not necessarily normalized) in U .

The next lemma gives us a necessary and sufficient condition for an operator to be an averaging operator.

LEMMA 2.2 [10], [4, Lemma 2]. *Let $K \subset H(U)$ and let an operator $I : K \rightarrow H(U)$ that satisfies $I[f](0) = f(0)$ for all $f \in K$. A necessary and sufficient condition for I to be an averaging operator on K is that*

$$f \in K, \text{ } k \text{ convex and } f(z) \prec k(z) \Rightarrow I[f](z) \prec k(z).$$

Let $c \in \mathbf{C}$ with $\operatorname{Re} c > 0$, and let $N = N(c) = (|c|\sqrt{1 + 2 \operatorname{Re} c} + \operatorname{Im} c) / \operatorname{Re} c$. If χ is the univalent function $\chi(z) = 2Nz/(1 - z^2)$, then we define the *open door function* R_c by

$$(2.3) \quad R_c(z) = \chi \left(\frac{z + b}{1 + \bar{b}z} \right), \quad z \in U,$$

where $b = \chi^{-1}(c)$.

Remark that R_c is univalent in U , $R_c(0) = c$ and $R_c(U) = \chi(U)$ is the complex plane slit along the half-lines $\operatorname{Re} w = 0$, $\operatorname{Im} w \geq N$ and $\operatorname{Re} w = 0$, $\operatorname{Im} w \leq -N$.

LEMMA 2.3 [3, Theorem 1]. *Let $\phi, \varphi \in \mathcal{D}$ and let α, β, γ and δ be complex numbers with $\beta \neq 0$, $\alpha + \delta = \beta + \gamma$ and $\operatorname{Re}(\alpha + \delta) > 0$. If $f \in \mathcal{A}$ satisfies*

$$\alpha \frac{zf'(z)}{f(z)} + \frac{z\varphi'(z)}{\varphi(z)} + \delta \prec R_{\alpha+\delta}(z),$$

and the function F is defined by

$$(2.4) \quad F(z) = \left[\frac{\beta + \gamma}{z^\gamma \phi(z)} \int_0^z f^\alpha(t) t^{\delta-1} \varphi(t) dt \right]^{1/\beta} = z + \dots,$$

then $F \in \mathcal{A}$, $F(z)/z \neq 0$, $\forall z \in U$ and

$$\operatorname{Re} \left[\beta \frac{zF'(z)}{F(z)} + \frac{z\phi'(z)}{\phi(z)} + \gamma \right] > 0, \quad z \in U.$$

(All powers in (2.4) are principal ones.)

LEMMA 2.4 [6, Lemma 1.2c]. Let $n \geq 0$ be an integer and let $\gamma \in \mathbf{C}$, with $\operatorname{Re} \gamma > -n$. If $f(z) = \sum_{m \geq n} a_m z^m$ is analytic in U and F is defined by

$$F(z) = \frac{1}{z^\gamma} \int_0^z f(t) t^{\gamma-1} dt,$$

then $F(z) = \sum_{m \geq n} a_m z^m / (m + \gamma)$ is analytic in U .

LEMMA 2.5 [2, Theorem 2], [4, Theorem 2]. Let k be convex (univalent) in U and let $A \geq 0$. Suppose $M > 4/|h'(0)|$ and that B and D are analytic in U , with $D(0) = 0$ and

$$\operatorname{Re} B(z) \geq A + M|D(z)|, \quad z \in U.$$

If p is analytic in U with $p(0) = k(0)$, and if p satisfies

$$Az^2 p''(z) + B(z) z p'(z) + p(z) + D(z) < k(z),$$

then $p(z) < k(z)$.

A function $L : U \times [0, +\infty) \rightarrow \mathbf{C}$ is called a *subordination* (or a *Loewner chain*) if $L(\cdot; t)$ is analytic and univalent in U for all $t \geq 0$, and $L(z; s) < L(z; t)$, when $0 \leq s \leq t$.

LEMMA 2.6 [11, p. 159]. The function $L(z; t) = a_1(t)z + a_2(t)z^2 + \dots$, with $a_1(t) \neq 0$ for $t \geq 0$, and $\lim_{t \rightarrow \infty} |a_1(t)| = \infty$, is a subordination chain if and only if there exist constants $r \in (0, 1]$ and $M > 0$ such that

(i) $L(z; t)$ is analytic in $|z| < r$ for each $t \geq 0$, locally absolutely continuous in $[0, \infty)$ for each $|z| < r$, and satisfies

$$|L(z; t)| \leq M|a_1(t)|, \quad \text{for } |z| < r \text{ and } t \geq 0$$

(ii) there exists a function $p(z, t)$ analytic in U for all $t \in [0, \infty)$ and measurable in $[0, \infty)$ for each $z \in U$, such that $\operatorname{Re} p(z, t) > 0$ for $z \in U$, $t \in [0, \infty)$, and

$$\frac{\partial L(z; t)}{\partial t} = z \frac{\partial L(z; t)}{\partial z} p(z, t), \quad \text{for } |z| < r \text{ and for almost all } t \in [0, \infty).$$

3. Main results

First we need to determine sufficient conditions on the ϕ and φ functions such that the integral operators $I_{\phi, \varphi; \beta, \gamma}$ are well defined.

From the fact that

$$(3.1) \quad \mathbf{I}_{\phi, \varphi; \beta, \gamma}[f](z) = \left(\frac{\gamma}{\beta + \gamma}\right)^{1/\beta} \tilde{F}(z), \quad \text{if } \gamma \neq 0,$$

where

$$\tilde{F}(z) = \left[\frac{\beta + \gamma}{z^\gamma \phi(z)} \int_0^z f^\beta(t) t^{\gamma-1} \varphi(t) dt \right]^{1/\beta},$$

in order to determine the subset $\mathcal{H}_{\phi; \beta, \gamma} \subset H(\mathbf{U})$ so that the operator $\mathbf{I}_{\phi, \varphi; \beta, \gamma}$ given by (1.1) is well defined, we need to find the set $\mathcal{H}_{\phi; \beta, \gamma}$ such that $\tilde{F} \in H(\mathbf{U})$ for all $f \in \mathcal{H}_{\phi; \beta, \gamma}$.

LEMMA 3.1. *Let $\beta, \gamma \in \mathbf{C}$ with $\beta \neq 0$, $\text{Re}(\beta + \gamma) > 0$ and let $\phi, \varphi \in \mathcal{D}$. If $R_{\beta+\gamma}$ represents the open door function defined by (2.3) and if*

$$\mathcal{H}_{\phi; \beta, \gamma} = \left\{ f \in \mathcal{A} : \beta \frac{zf'(z)}{f(z)} + \frac{z\varphi'(z)}{\varphi(z)} + \gamma < R_{\beta+\gamma}(z) \right\}, \quad \text{for } \beta \neq 1,$$

and

$$\mathcal{H}_{\phi; 1, \gamma} = H(\mathbf{U}), \quad \text{for } \beta = 1, \text{ if in addition } \text{Re } \gamma > 0,$$

then the integral operator $\mathbf{I}_{\phi, \varphi; \beta, \gamma}$ is well-defined.

Proof. If $\beta \neq 1$, from (3.1) by using Lemma 2.3 for the case $\alpha = \beta$ and $\delta = \gamma$ we deduce the first part of the result.

If $\beta = 1$, denoting $t = wz$ we have

$$\tilde{F}(z) = \frac{\gamma + 1}{\phi(z)} \int_0^1 f(wz)\varphi(wz)w^{\gamma-1} dw,$$

and from (3.1) according to Lemma 2.4 we obtain the second part of our result. □

Using Lemma 2.3, the previous result and the relation (3.1), respectively Lemma 2.4 and the relation (3.1) we deduce the next two remarks:

Remark 3.1. Under the assumptions of Lemma 3.1, for $\beta \neq 1$, we have

$$\tilde{F} \in \mathcal{A}, \quad \frac{\tilde{F}(z)}{z} \neq 0, \quad z \in U \quad \text{and} \quad \text{Re} \left[\beta \frac{z\tilde{F}'(z)}{\tilde{F}(z)} + \frac{z\phi'(z)}{\phi(z)} + \gamma \right] > 0, \quad z \in U,$$

hence,

$$\mathbf{I}_{\phi, \varphi; \beta, \gamma}[f](z) = \left(\frac{\gamma}{\beta + \gamma}\right)^{1/\beta} z + \dots \in H(\mathbf{U}), \quad \forall f \in \mathcal{H}_{\phi; \beta, \gamma} \text{ and } \beta \neq 1.$$

Remark 3.2. Under the assumptions of Lemma 3.1, we have

$$\mathbf{I}_{\phi, \varphi; \beta, \gamma}[f](0) = f(0), \quad \forall f \in \mathcal{H}_{\phi; \beta, \gamma}.$$

THEOREM 3.1. *Let $\beta, \gamma \in \mathbf{C}$ with $\beta > 0$ and $\operatorname{Re} \gamma > 0$, let $\phi, \varphi \in \mathcal{D}$ and suppose that*

$$(i) \quad \operatorname{Re} \left[\frac{z\phi'(z)}{\phi(z)} + \gamma \right] > 0, \quad z \in \mathbf{U}.$$

Let $k \in \mathcal{M}'_{1/\beta}$ and $f \in \mathcal{H}_{\phi; \beta, \gamma}$. Then

$$\left[\frac{\varphi(z)}{\phi(z) + (1/\gamma)z\phi'(z)} \right]^{1/\beta} f(z) < k(z) \Rightarrow \mathbf{I}_{\phi, \varphi; \beta, \gamma}[f](z) < k(z).$$

Proof. Since $\beta \neq 0$ and $\operatorname{Re}(\beta + \gamma) > 0$, according to Lemma 3.1, the operator $\mathbf{I}_{\phi, \varphi; \beta, \gamma}$ is well-defined on the set $\mathcal{H}_{\phi; \beta, \gamma}$.

From the assumption $[\varphi(z)/(\phi(z) + (1/\gamma)z\phi'(z))]^{1/\beta} f(z) < k(z)$ we have $f(0) = k(0)$. If $\beta \neq 1$ then $f \in \mathcal{A}$, hence $k(0) = f(0) = 0$ i.e. $k \in \mathcal{M}'_{1/\beta}$, so it follows that k is univalent in \mathbf{U} . If $\beta = 1$ then $k \in \mathcal{K}' \equiv \mathcal{M}'_1$, hence k is a convex (and univalent) function in \mathbf{U} .

If we denote by $F(z) = \mathbf{I}_{\phi, \varphi; \beta, \gamma}[f](z)$, then by Remark 3.2 we have $F(0) = f(0)$ and

$$(3.2) \quad \left[\frac{\varphi(z)}{\phi(z) + (1/\gamma)z\phi'(z)} \right]^{1/\beta} f(z) = F(z) \left[\frac{\beta}{\gamma} \frac{zF'(z)}{F(z)} \frac{1}{H(z)} + 1 \right]^{1/\beta},$$

where

$$H(z) = 1 + \frac{1}{\gamma} \frac{z\phi'(z)}{\phi(z)}.$$

Remark that the assumption (i) implies $H(z) \neq 0$ for all $z \in \mathbf{U}$.

Thus, we need to prove the next implication:

$$(3.3) \quad F(z) \left[\frac{\beta}{\gamma} \frac{zF'(z)}{F(z)} \frac{1}{H(z)} + 1 \right]^{1/\beta} < k(z) \Rightarrow F(z) < k(z).$$

For the particular case $\beta = 1$, the implication (3.3) becomes

$$F(z) + \frac{1}{\gamma H(z)} zF'(z) < k(z) \Rightarrow F(z) < k(z).$$

According to Lemma 2.5 for $A = 0$ and $D(z) \equiv 0$, and by using the inequality (i) we deduce that the above implication holds.

Now we will prove our result for the case $\beta \neq 1$. Without loss of generality we can assume that k satisfies the conditions of the theorem on the closed disk $\bar{\mathbf{U}}$ and $k'(\zeta) \neq 0$ for $|\zeta| = 1$. If not, then we replace f , k , ϕ and φ with $f_r(z) = f(rz)$, $k_r(z) = k(rz)$, $\phi_r(z) = \phi(rz)$ and $\varphi_r(z) = \varphi(rz)$ where $0 < r < 1$, and then k_r is univalent on $\bar{\mathbf{U}}$. Since

$$\left[\frac{\varphi_r(z)}{\phi_r(z) + (1/\gamma)z\phi'_r(z)} \right]^{1/\beta} f_r(z) < k_r(z),$$

we would then prove that

$$F_r(z) = F(rz) = \mathbf{I}_{\phi_r, \varphi_r; \beta, \gamma}[f_r](z) \prec k_r(z), \quad \text{for } 0 < r < 1,$$

and by letting $r \rightarrow 1^-$ we obtain $F(z) \prec k(z)$.

If we suppose that the implication (3.3) is not true, i.e. $F(z) \not\prec k(z)$, then from Lemma 2.1 there exist points $z_0 \in \mathbf{U}$ and $\zeta_0 \in \partial\mathbf{U}$, and a number $m \geq 1$, such that

$$(3.4) \quad F(z_0) = k(\zeta_0)$$

and

$$(3.5) \quad z_0 F'(z_0) = m \zeta_0 k'(\zeta_0).$$

To prove the implication (3.3) we define the function $L : \mathbf{U} \times [0, \infty) \rightarrow \mathbf{C}$ by

$$(3.6) \quad L(z; t) = k(z) \left[\frac{\beta}{\gamma} t \frac{zk'(z)}{k(z)} \frac{1}{H(z_0)} + 1 \right]^{1/\beta},$$

and we will show that $L(z; t)$ is a subordination chain.

From the fact that $zk'(z)/k(z)|_{z=0} = 1$ and the assumptions (i) and $\beta > 0$, we have

$$\operatorname{Re} \frac{\beta}{\gamma} \frac{zk'(z)}{k(z)} \Big|_{z=0} \frac{1}{H(z_0)} > 0,$$

hence $L(z; t)$ is analytic in $|z| < r < 1$, for sufficient small $r > 0$ and for all $t \geq 0$. We also have that $L(z; t)$ is continuously differentiable on $[0, \infty)$ for each $|z| < r < 1$.

A simple calculus shows that

$$a_1(t) = \frac{\partial L(0; t)}{\partial z} = k'(0) \left[\frac{\beta}{\gamma} t \frac{1}{H(z_0)} + 1 \right]^{1/\beta},$$

and because $k'(0) \neq 0$, from (i) and $\beta > 0$ we deduce

$$\operatorname{Re} \left[\frac{\beta}{\gamma} t \frac{1}{H(z_0)} + 1 \right] \geq 1 > 0, \quad \forall t \geq 0,$$

hence $a_1(t) \neq 0, \forall t \geq 0$. From (i) we have $(\beta/\gamma)t(1/H(z_0)) \neq 0, \forall z_0 \in \mathbf{U}$ and $\forall t > 0$, so we obtain that $\lim_{t \rightarrow \infty} |a_1(t)| = \infty$.

Using the definition (3.6), by a directly computation we obtain

$$\operatorname{Re} \left[z \frac{\partial L / \partial z}{\partial L / \partial t} \right] = t\beta \operatorname{Re} \left[\left(1 - \frac{1}{\beta} \right) \frac{zk'(z)}{k(z)} + \frac{1}{\beta} \left(1 + \frac{zk''(z)}{k'(z)} \right) \right] + \operatorname{Re}[\gamma H(z_0)].$$

From the above relation, by using the fact that $k \in \mathcal{M}'_{1/\beta}$ and the assumption (i), we deduce that

$$\operatorname{Re} \left[z \frac{\partial L / \partial z}{\partial L / \partial t} \right] > 0, \quad \forall z \in \mathbf{U}, \forall t \geq 0,$$

and according to Lemma 2.6 we conclude that $L(z; t)$ is a subordination chain. This implies in particular:

$$(3.7) \quad k(z) = L(z; 0) \prec L(z; t), \quad \forall t \geq 0.$$

Using the equality (3.2) and the relations (3.4) and (3.5) we obtain

$$\begin{aligned} \left[\frac{\varphi(z_0)}{\phi(z_0) + (1/\gamma)z_0\phi'(z_0)} \right]^{1/\beta} f(z_0) &= F(z_0) \left[\frac{\beta}{\gamma} \frac{z_0 F'(z_0)}{F(z_0)} \frac{1}{H(z_0)} + 1 \right]^{1/\beta} \\ &= k(\zeta_0) \left[\frac{\beta}{\gamma} m \frac{\zeta_0 k'(\zeta_0)}{k(\zeta_0)} \frac{1}{H(z_0)} + 1 \right]^{1/\beta} \\ &= L(\zeta_0; m), \quad m \geq 1, \end{aligned}$$

and then, according to (3.7) we deduce that

$$\left[\frac{\varphi(z_0)}{\phi(z_0) + (1/\gamma)z_0\phi'(z_0)} \right]^{1/\beta} f(z_0) = L(\zeta_0; m) \notin k(\mathbf{U}).$$

This last relation contradicts the assumption $[\varphi(z)/(\phi(z) + (1/\gamma)z\phi'(z))]^{1/\beta} f(z) \prec k(z)$, then we finally conclude that $F(z) \prec k(z)$. \square

4. Particular cases

In this section we will discuss several particular cases of Theorem 3.1 obtained for appropriate choices of the ϕ and φ functions.

1. For a given function $\phi \in \mathcal{D}$, taking $\varphi(z) = \phi(z) + (1/\gamma)z\phi'(z)$ in Theorem 3.1, then $\varphi(0) = 1$ and the assumption (i) of the theorem is equivalent to

$$\operatorname{Re} \left[\gamma \frac{\varphi(z)}{\phi(z)} \right] > 0, \quad z \in \mathbf{U}.$$

Since $\phi(z) \neq 0$, $\forall z \in \mathbf{U}$, from the above inequality it follows that $\varphi \in \mathcal{D}$ and then we obtain:

COROLLARY 4.1. *Let $\beta, \gamma \in \mathbf{C}$ with $\beta > 0$ and $\operatorname{Re} \gamma > 0$, let $\phi \in \mathcal{D}$ and suppose that*

$$(i) \quad \operatorname{Re} \left[\frac{z\phi'(z)}{\phi(z)} + \gamma \right] > 0, \quad z \in \mathbf{U}.$$

Let $k \in \mathcal{M}'_{1/\beta}$ and $f \in \mathcal{K}_{\phi+(1/\gamma)z\phi'; \beta, \gamma}$. Then

$$f(z) \prec k(z) \Rightarrow \mathbf{I}_{\phi, \phi+(1/\gamma)z\phi'; \beta, \gamma}[f](z) \prec k(z),$$

where

$$(4.1) \quad \mathbf{I}_{\phi, \phi+(1/\gamma)z\phi'; \beta, \gamma}[f](z) = \left[\frac{\gamma}{z^\gamma \phi(z)} \int_0^z f^\beta(t) t^{\gamma-1} \left(\phi(t) + \frac{1}{\gamma} t \phi'(t) \right) dt \right]^{1/\beta}.$$

Using this corollary in the special case $k(z) = f(z)$ we obtain the next example:

Example 4.1. Let $\beta, \gamma \in \mathbf{C}$ with $\beta > 0$ and $\operatorname{Re} \gamma > 0$, let $\phi \in \mathcal{D}$ and suppose that

$$(i) \quad \operatorname{Re} \left[\frac{z\phi'(z)}{\phi(z)} + \gamma \right] > 0, \quad z \in \mathbf{U}.$$

If $f \in \mathcal{K}_{\phi+(1/\gamma)z\phi';\beta,\gamma} \cap \mathcal{M}'_{1/\beta}$, then

$$\mathbf{I}_{\phi,\phi+(1/\gamma)z\phi';\beta,\gamma}[f](z) < f(z),$$

where $\mathbf{I}_{\phi,\phi+(1/\gamma)z\phi';\beta,\gamma}[f]$ is given by (4.1).

Our next result deals with a general class of averaging integral operators.

COROLLARY 4.2. Let $\beta, \gamma \in \mathbf{C}$ with $\beta \geq 1$ and $\operatorname{Re} \gamma > 0$, let $\phi \in \mathcal{D}$ and suppose that

$$(i) \quad \operatorname{Re} \left[\frac{z\phi'(z)}{\phi(z)} + \gamma \right] > 0, \quad z \in \mathbf{U}.$$

Then the integral operator $\mathbf{I}_{\phi,\phi+(1/\gamma)z\phi';\beta,\gamma}$ given by (4.1) is an averaging operator on $\mathcal{K}_{\phi+(1/\gamma)z\phi';\beta,\gamma}$.

Proof. If $f \in \mathcal{K}_{\phi+(1/\gamma)z\phi';\beta,\gamma}$, then from Remark 3.2 we have $\mathbf{I}_{\phi,\phi+(1/\gamma)z\phi';\beta,\gamma}[f](0) = f(0)$. Let consider an arbitrary convex function k such that $f(z) < k(z)$.

For the case $\beta = 1$ we have $k \in \mathcal{K}' \equiv \mathcal{M}'_1$ and, according to Corollary 4.1 we deduce that $\mathbf{I}_{\phi,\phi+(1/\gamma)z\phi';1,\gamma}[f](z) < k(z)$.

For the case $\beta > 1$, since $f(z) < k(z)$ and $f \in \mathcal{K}_{\phi+(1/\gamma)z\phi';\beta,\gamma}$ then $k(0) = f(0) = 0$. From (2.1) we have $k \in \mathcal{M}_1 \subset \mathcal{M}_{1/\beta} \subset \mathcal{M}'_{1/\beta}$ for $\beta > 1$, and using Corollary 4.1 we obtain that $\mathbf{I}_{\phi,\phi+(1/\gamma)z\phi';\beta,\gamma}[f](z) < k(z)$ for $\beta > 1$.

Now, from Lemma 2.2 we conclude that in the both two cases the integral operator $\mathbf{I}_{\phi,\phi+(1/\gamma)z\phi';\beta,\gamma}$ is an averaging operator on $\mathcal{K}_{\phi+(1/\gamma)z\phi';\beta,\gamma}$. \square

Remark 4.1. Remark that this corollary generalizes Theorem 1 of [5] that may be obtained for the particular case $\phi(z) = 1$.

2. Taking $\phi(z) = \varphi(z) = e^{\lambda z}$ with $\lambda \in \mathbf{C}$ in Theorem 3.1, then $\phi \equiv \varphi \in \mathcal{D}$ and the condition (i) of the theorem reduces to $\operatorname{Re}(\lambda z + \gamma) > 0$, $\forall z \in \mathbf{U}$. Since this inequality holds whenever $\operatorname{Re} \gamma \geq |\lambda|$, we obtain:

COROLLARY 4.3. Let $\beta, \gamma \in \mathbf{C}$ with $\beta > 0$ and $\operatorname{Re} \gamma > 0$. For a number $\lambda \in \mathbf{C}$, suppose in addition that $\operatorname{Re} \gamma \geq |\lambda|$. Let $k \in \mathcal{M}'_{1/\beta}$ and $f \in \mathcal{K}_{e^{\lambda z};\beta,\gamma}$. Then

$$\left(\frac{\gamma}{\lambda z + \gamma}\right)^{1/\beta} f(z) < k(z) \Rightarrow \mathbf{I}_{e^{\lambda z}, e^{\lambda z}; \beta, \gamma}[f](z) < k(z),$$

where

$$\mathbf{I}_{e^{\lambda z}, e^{\lambda z}; \beta, \gamma}[f](z) = \left[\frac{\gamma}{z^\gamma e^{\lambda z}} \int_0^z f^\beta(t) t^{\gamma-1} e^{\lambda t} dt \right]^{1/\beta}.$$

By the same reasons, if we take $\phi(z) = \varphi(z) = e^{\lambda z}$ with $\lambda \in \mathbf{C}$ in Corollary 4.2, we have:

COROLLARY 4.4. *Let $\beta, \gamma \in \mathbf{C}$ with $\beta > 0$ and $\operatorname{Re} \gamma > 0$. For a number $\lambda \in \mathbf{C}$, suppose in addition that $\operatorname{Re} \gamma \geq |\lambda|$. Then the integral operator $\mathbf{I}_{e^{\lambda z}, e^{\lambda z}; \beta, \gamma}$ given by*

$$\mathbf{I}_{e^{\lambda z}, e^{\lambda z + (\lambda z/\gamma)e^{\lambda z}}; \beta, \gamma}[f](z) = \left[\frac{\gamma}{z^\gamma e^{\lambda z}} \int_0^z f^\beta(t) t^{\gamma-1} \left(e^{\lambda z} + \frac{\lambda z}{\gamma} e^{\lambda z} \right) dt \right]^{1/\beta}$$

is an averaging operator on $\mathcal{H}_{e^{\lambda z}; \beta, \gamma}$.

Remark 4.2. Note that this corollary also extends Theorem 1 of [5] that may be obtained for the special case $\lambda = 0$.

3. Considering $\phi(z) = 1 + \lambda z$, $\lambda \in \mathbf{C}$, then $\phi(0) = 1$ and for $|\lambda| \leq 1$ we have $\phi(z) \neq 0$, $\forall z \in \mathbf{U}$, i.e. $\phi \in \mathcal{D}$.

It is easy to check that the condition (i) of Theorem 3.1 becomes

$$(4.2) \quad \operatorname{Re} \left[\frac{\lambda z}{1 + \lambda z} + \gamma \right] > 0, \quad z \in \mathbf{U}.$$

Letting $\chi(\zeta) = \zeta/(1 + \zeta)$, since $\chi'(0) \neq 0$ and

$$\operatorname{Re} \frac{\zeta \chi''(\zeta)}{\chi'(\zeta)} + 1 = \operatorname{Re} \frac{1 - \zeta}{1 + \zeta} > 0, \quad |\zeta| < 1,$$

the function χ is a convex function in $D = \{\zeta \in \mathbf{C} : |\zeta| < |\lambda|\}$, if $|\lambda| \leq 1$. From the fact that $\chi(\bar{\zeta}) = \bar{\chi}(\zeta)$, $\zeta \in D$, we deduce that the function χ maps the disk D onto the convex domain $\chi(D)$ that is symmetric with respect the real axis. Hence

$$\inf \left\{ \frac{\lambda z}{1 + \lambda z} : z \in \mathbf{U} \right\} = t(-|\lambda|) = \frac{|\lambda|}{|\lambda| - 1}, \quad \text{for } |\lambda| < 1,$$

which shows that the condition (4.2) holds whenever $\operatorname{Re} \gamma \geq |\lambda|/(1 - |\lambda|)$, for $|\lambda| < 1$. From here, by using Theorem 3.1 we have:

COROLLARY 4.5. *Let $\beta, \gamma \in \mathbf{C}$ with $\beta > 0$ and $\operatorname{Re} \gamma > 0$. Let $\varphi \in \mathcal{D}$ and for a number $\lambda \in \mathbf{C}$ with $|\lambda| < 1$, suppose in addition that*

$$\operatorname{Re} \gamma \geq \frac{|\lambda|}{1 - |\lambda|}.$$

Let $k \in \mathcal{M}'_{1/\beta}$ and $f \in \mathcal{H}_{\phi; \beta, \gamma}$. Then

$$\left[\frac{\varphi(z)}{1 + ((\gamma + 1)/\gamma)\lambda z} \right]^{1/\beta} f(z) < k(z) \Rightarrow \mathbf{I}_{1+\lambda z, \varphi; \beta, \gamma}[f](z) < k(z).$$

By the same reasons, if we take $\phi(z) = 1 + \lambda z$ with $|\lambda| < 1$ in Corollary 4.2, we have:

COROLLARY 4.6. *Let $\beta, \gamma \in \mathbf{C}$ with $\beta > 0$ and $\operatorname{Re} \gamma > 0$. For a number $\lambda \in \mathbf{C}$ with $|\lambda| < 1$, suppose in addition that*

$$\operatorname{Re} \gamma \geq \frac{|\lambda|}{1 - |\lambda|}.$$

Then the integral operator $\mathbf{I}_{1+\lambda z, 1+((\gamma+1)/\gamma)\lambda z; \beta, \gamma}$ given by

$$\mathbf{I}_{1+\lambda z, 1+((\gamma+1)/\gamma)\lambda z; \beta, \gamma}[f](z) = \left[\frac{\gamma}{z^\gamma(1 + \lambda z)} \int_0^z f^\beta(t) t^{\gamma-1} \left(1 + \frac{\gamma+1}{\gamma} \lambda t \right) dt \right]^{1/\beta}$$

is an averaging operator on $\mathcal{H}_{1+((\gamma+1)/\gamma)\lambda z; \beta, \gamma}$.

Remark 4.3. This corollary also extends Theorem 1 of [5] that can be obtained for the particular case $\lambda = 0$.

4. If we take $\phi(z) = (1 + z)^{2a}$ with $a \leq 0$ in Theorem 3.1, then the condition (i) of the theorem reduces to

$$\operatorname{Re} \left[\frac{2az}{1 + z} + \gamma \right] > 0, \quad z \in \mathbf{U}.$$

If $a < 0$, the function $\chi(z) = 2az/(1 + z)$ maps the unit disk \mathbf{U} onto the half-plane $\Delta = \{w \in \mathbf{C} : \operatorname{Re} w > a\}$, and we deduce that the above inequality holds if and only if $\operatorname{Re} \gamma \geq -a$.

If $a = 0$, the same inequality holds for all $\gamma \in \mathbf{C}$ with $\operatorname{Re} \gamma > 0$, hence we obtain the next result:

COROLLARY 4.7. *Let $\beta, \gamma \in \mathbf{C}$ with $\beta > 0$ and $\operatorname{Re} \gamma > 0$. Let $\varphi \in \mathcal{D}$ and for a number $a \leq 0$ suppose in addition that*

$$\operatorname{Re} \gamma \geq -a.$$

Let $k \in \mathcal{M}'_{1/\beta}$ and $f \in \mathcal{H}_{\phi; \beta, \gamma}$. Then

$$\left[\frac{\varphi(z)}{(1 + z)^{2a-1} (1 + ((2a + \gamma)/\gamma)z)} \right]^{1/\beta} f(z) < k(z) \Rightarrow \mathbf{I}_{(1+z)^{2a}, \varphi; \beta, \gamma}[f](z) < k(z).$$

Similarly, by taking $\phi(z) = (1+z)^{2a}$ with $a \leq 0$ in Corollary 4.2 we have:

COROLLARY 4.8. *Let $\beta, \gamma \in \mathbf{C}$ with $\beta > 0$ and $\operatorname{Re} \gamma > 0$. For a number $a \leq 0$ suppose in addition that*

$$\operatorname{Re} \gamma \geq -a.$$

Then the integral operator $\mathbf{I}_{(1+z)^{2a}, (1+z)^{2a-1}(1+((2a+\gamma)/\gamma)z); \beta, \gamma}$ given by

$$\begin{aligned} & \mathbf{I}_{(1+z)^{2a}, (1+z)^{2a-1}(1+((2a+\gamma)/\gamma)z); \beta, \gamma}[f](z) \\ &= \left[\frac{\gamma}{z^\gamma (1+z)^{2a}} \int_0^z f^\beta(t) t^{\gamma-1} (1+t)^{2a-1} \left(1 + \frac{2a+\gamma}{\gamma} t\right) dt \right]^{1/\beta} \end{aligned}$$

is an averaging operator on $\mathcal{K}_{(1+z)^{2a-1}(1+((2a+\gamma)/\gamma)z); \beta, \gamma}$.

Remark 4.4. For the particular case $a = 0$ we remark that this corollary represents Theorem 1 of [5].

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