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Generalization of a class of nonlinear averaging integral operators

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Let $H(\mathrm{U})$ be the space of all analytic functions in the unit disk U, and let $\operatorname{co} E$ denote the convex hull of the set $E \subset \mathbb{C}$. If $K \subset H(\mathrm{U})$ then the operator $\mathrm{I} : K \to H(\mathrm{U})$ is said to be an *averaging operator* if

$$\mathrm{I}[f](0) = f(0) \quad \text{and} \quad \mathrm{I}[f](\mathrm{U}) \subset \operatorname{co} f(\mathrm{U}), \quad \text{for all} \quad f \in K .$$

For a function $h \in \mathcal{A} \subset H(\mathrm{U})$ we will determine simple sufficient conditions on h such that

$$f(z) \prec k(z) \quad \Longrightarrow \quad \mathrm{I}_{h;\beta,\gamma}[f](z) \prec k(z) ,$$

for all $f \in \mathcal{M}'_{1/\beta}$, where

$$\mathrm{I}_{h;\beta,\gamma}[f](z) = \left[\frac{\gamma}{h^{\gamma}(z)} \int_0^z f^{\beta}(t) h^{\gamma-1}(t) h'(t)\, \mathrm{d}t \right]^{1/\beta}$$

and $\mathcal{M}'_{1/\beta}$ represents the class of $1/\beta$-convex functions (not necessarily normalized).

As an application, we will give sufficient conditions on h to insure that the operators $\mathrm{I}_{h;\beta,\gamma}$ are averaging operators on certain subsets of $H(\mathrm{U})$, in order to generalize the result of [5]. In addition, some particular cases of this result obtained for appropriate choices of the function h will also be given.

1 Introduction

Let $H(\mathrm{U})$ be the space of all analytic functions in the unit disk $\mathrm{U} = \{z \in \mathbb{C} : |z| < 1\}$ and for a set $E \subset \mathbb{C}$ we denote by $\operatorname{co} E$ the convex hull of E.

In [9] and [4] the authors introduced the concept of *averaging operator* on a set $K \subset H(\mathrm{U})$, as an operator that satisfies

$$\mathrm{I}[f](0) = f(0) \quad \text{and} \quad \mathrm{I}[f](\mathrm{U}) \subset \operatorname{co} f(\mathrm{U}) , \quad \text{for all} \quad f \in K .$$

In [5] the same authors found conditions on β and $\gamma \in \mathbb{C}$ such that the operators

$$\mathrm{A}_{\beta,\gamma}[f](z) = \left[\frac{\gamma}{z^{\gamma}} \int_0^z f^{\beta}(t) t^{\gamma-1}\, \mathrm{d}t \right]^{1/\beta} , \quad \operatorname{Re}\gamma > 0$$

are averaging operators on a certain subsets of $H(\mathrm{U})$, generalizing the result of [4] related to the operators $\mathrm{A}_{1,\gamma}$.

Let

$$A = \{f \in H(\mathrm{U}) : f(0) = 0,\ f'(0) = 1\}$$

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be the set of analytic and normalized functions in U and let

$$\mathcal{A} \;=\; \{h \in A : h(z)h'(z) \neq 0,\ 0 < |z| < 1\}\,.$$

For a function $h \in \mathcal{A}$ we define the integral operators $\mathrm{I}_{h;\beta,\gamma} : \mathcal{K}_{h,\beta,\gamma} \to H(\mathrm{U})$ by

$$\mathrm{I}_{h;\beta,\gamma}[f](z) \;=\; \left[\frac{\gamma}{h^{\gamma}(z)}\int_0^z f^{\beta}(t)h^{\gamma-1}(t)h'(t)\,\mathrm{d}t\right]^{1/\beta}\,. \tag{1.1}$$

We will determine simple sufficient conditions on h such that these operators are averaging operators on certain subsets of $H(\mathrm{U})$ by using several results involving differential subordination and subordination chain techniques.

2 Preliminaries

In order to prove our main results, we will need the next definitions and lemmas presented in this section.

Let $f, g \in H(\mathrm{U})$. We say that the function f is *subordinated* to g (in a restricted sense), written $f \prec g$ or $f(z) \prec g(z)$, if g is univalent in U, $f(0) = g(0)$ and $f(\mathrm{U}) \subseteq g(\mathrm{U})$.

Let denote by $\mathcal{Q}$ the set of functions q that are analytic and injective on $\overline{\mathrm{U}} \setminus E(q)$, where

$$E(q) \;=\; \left\{\zeta \in \partial\mathrm{U} : \lim_{z\to\zeta} q(z) = \infty\right\},$$

and such that $q'(\zeta) \neq 0$ for $\zeta \in \partial\mathrm{U} \setminus E(q)$.

Lemma 2.1 ([2]) *Let $q \in \mathcal{Q}$, with $q(0) = a$, and let $p(z) = a + a_n z^n + \dots$ be analytic in* U *with $p(z) \not\equiv a$ and $n \geq 1$. If p is not subordinate to q, then there exist points $z_0 \in \mathrm{U}$ and $\zeta_0 \in \partial\mathrm{U} \setminus E(q)$, and an $m \geq n \geq 1$ for which $p(|z| < |z_0|) \subset q(\mathrm{U})$, and*

(i) $p(z_0) = q(\zeta_0)$,

(ii) $z_0 p'(z_0) = m\zeta_0 q'(\zeta_0)$,

(iii) $\mathrm{Re}\,\frac{z_0 p''(z_0)}{p'(z_0)} + 1 \geq m\mathrm{Re}\left[\frac{\zeta_0 q''(\zeta_0)}{q'(\zeta_0)} + 1\right]$.

As in [8], for $\alpha \in \mathbb{R}$, a function $f \in H(\mathrm{U})$ with $f(0) = 0$ and $f'(0) \neq 0$ is said to be an *α-convex* (not necessarily normalized) *function*, if

$$\mathrm{Re}\left[(1-\alpha)\,\frac{zf'(z)}{f(z)} + \alpha\left(\frac{zf''(z)}{f'(z)} + 1\right)\right] \;>\; 0\,, \quad z \in \mathrm{U}\,,$$

and we denote this class by $\mathcal{M}_\alpha$. Note that all α-convex functions are univalent and starlike, and moreover [7],

$$\mathcal{M}_\alpha \subset \mathcal{M}_\beta \subset \mathcal{M}_0\,, \quad \text{for} \quad 0 \;\leq\; \frac{\beta}{\alpha} \;\leq\; 1\,. \tag{2.1}$$

For $\alpha \in \mathbb{R}$ we denote by

$$\mathcal{M}'_\alpha \;=\; \left\{f \in H(\mathrm{U}) : f'(0) \neq 0,\ \mathrm{Re}\left[(1-\alpha)\,\frac{zf'(z)}{f(z)} + \alpha\left(\frac{zf''(z)}{f'(z)} + 1\right)\right] > 0,\ z \in \mathrm{U}\right\},$$

and then

$$\mathcal{K}' \;\equiv\; \mathcal{M}'_1 = \left\{f \in H(\mathrm{U}) : f'(0) \neq 0,\ \mathrm{Re}\,\frac{zf''(z)}{f'(z)} + 1 > 0,\ z \in \mathrm{U}\right\}$$

represents the class of *convex* (not necessarily normalized) *functions* in U.

The next lemma gives us a necessary and sufficient condition for an operator to be an averaging operator.

Lemma 2.2 ([9], [4, Lemma 2].) *Let $K \subset H(\mathrm{U})$ and consider an operator $\mathrm{I} : K \to H(\mathrm{U})$ that satisfies $\mathrm{I}[f](0) = f(0)$ for all $f \in K$. A necessary and sufficient condition for* I *to be an averaging operator on K is that*

$$f \in K\,, \quad k \ \textit{convex} \quad \textit{and} \quad f(z) \prec k(z) \implies \mathrm{I}[f](z) \prec k(z)\,.$$

Let $c \in \mathbb{C}$ with $\operatorname{Re} c > 0$, and let

$$N = N(c) = \frac{|c|\sqrt{1 + 2\operatorname{Re} c} + \operatorname{Im} c}{\operatorname{Re} c}.$$

If K is the univalent function

$$K(z) = \frac{2Nz}{1 - z^2},$$

then we define the *open door function* R_c by

$$R_c(z) = K\left(\frac{z + b}{1 + \bar{b}z}\right), \quad z \in \mathrm{U}, \tag{2.2}$$

where $b = K^{-1}(c)$.

Note that R_c is univalent in U, $R_c(0) = c$ and $R_c(\mathrm{U}) = K(\mathrm{U})$ is the complex plane slit along the half-lines $\operatorname{Re} w = 0, \operatorname{Im} w \geq N$ and $\operatorname{Re} w = 0, \operatorname{Im} w \leq -N$.

Lemma 2.3 ([6, Lemma 1.2c]) *Let $n \geq 0$ be an integer and let $\gamma \in \mathbb{C}$, with $\operatorname{Re}\gamma > -n$. If $f(z) = \sum_{m \geq n} a_m z^m$ is analytic in* U *and F is defined by*

$$F(z) = \frac{1}{z^\gamma}\int_0^z f(t)t^{\gamma-1}\,\mathrm{d}t,$$

then $F(z) = \sum_{m \geq n} \frac{a_m z^m}{m+\gamma}$ is analytic in U.

Lemma 2.4 ([3, Theorem 2], [4, Theorem 2].) *Let k be convex (univalent) in* U *and let $A \geq 0$. Suppose $M > \frac{4}{|h'(0)|}$ and that B and D are analytic in* U, *with $D(0) = 0$ and*

$$\operatorname{Re} B(z) \geq A + M\,|D(z)|, \quad z \in \mathrm{U}.$$

If p is analytic in U *with $p(0) = k(0)$, and if p satisfies*

$$Az^2p''(z) + B(z)zp'(z) + p(z) + D(z) \prec k(z),$$

then $p(z) \prec k(z)$.

Finally, a function $L : \mathrm{U} \times [0, +\infty) \to \mathbb{C}$ is called to be a *subordination* (or a *Loewner*) *chain* if $L(\cdot; t)$ is analytic and univalent in U for all $t \geq 0$ and $L(z; s) \prec L(z; t)$, when $0 \leq s \leq t$.

Lemma 2.5 ([10, p. 159]) *The function $L(z; t) = a_1(t)z + a_2(t)z^2 + \ldots$, with $a_1(t) \neq 0$ for $t \geq 0$, and $\lim_{t\to\infty} |a_1(t)| = \infty$, is a subordination chain if and only if there exist constants $r \in (0, 1]$ and $M > 0$ such that*

(i) *$L(z; t)$ is analytic in $|z| < r$ for each $t \geq 0$, locally absolutely continuous in $[0, \infty)$ for each $|z| < r$, and satisfies*

$$|L(z; t)| \leq M\,|a_1(t)|, \quad \textit{for} \quad |z| < r \quad \textit{and} \quad t \geq 0,$$

(ii) *there exists a function $p(z, t)$ analytic in* U *for all $t \in [0, \infty)$ and measurable in $[0, \infty)$ for each $z \in$* U, *such that $\operatorname{Re} p(z, t) > 0$ for $z \in$* U, *$t \in [0, \infty)$, and*

$$\frac{\partial L(z; t)}{\partial t} = z\frac{\partial L(z; t)}{\partial z}p(z, t), \quad \textit{for } |z| < r \textit{ and for almost all } t \in [0, \infty).$$

3 Main results

For $h \in \mathcal{A}$ let define the integral operators $\widetilde{\mathrm{I}}_{h;\beta,\gamma} : \widetilde{\mathcal{K}}_{h,\beta,\gamma} \to H(\mathrm{U})$ by

$$\widetilde{\mathrm{I}}_{h;\beta,\gamma}[f](z) = \left[\frac{\beta + \gamma}{h^\gamma(z)}\int_0^z f^\beta(t)h^{\gamma-1}(t)h'(t)\,\mathrm{d}t\right]^{1/\beta}. \tag{3.1}$$

We have

$$\mathrm{I}_{h;\beta,\gamma}[f] \;=\; \left(\frac{\gamma}{\beta+\gamma}\right)^{1/\beta} \widetilde{\mathrm{I}}_{h;\beta,\gamma}[f] \quad \text{if} \quad \gamma \,\neq\, 0\,.$$

In order to determine the subset $\mathcal{K}_{h;\beta,\gamma} \subset H(\mathrm{U})$ where the operator $\mathrm{I}_{h;\beta,\gamma}$ given by (1.1) is well-defined, we need to determine the subset $\widetilde{\mathcal{K}}_{h;\beta,\gamma} \subset H(\mathrm{U})$ such that the integral operator $\widetilde{\mathrm{I}}_{h;\beta,\gamma}$ given by (3.1) will be well-defined.

Lemma 3.1 *Let $\beta, \gamma \in \mathbb{C}$ with $\beta \neq 0$, $\operatorname{Re}(\beta+\gamma) > 0$, let $h \in \mathcal{A}$ and denote*

$$J(\gamma, h)(z) \;=\; (\gamma-1)\,\frac{zh'(z)}{h(z)} + 1 + \frac{zh''(z)}{h'(z)}\,. \tag{3.2}$$

If $R_{\beta+\gamma}$ represents the open door function defined by (2.2) *and if*

$$\widetilde{\mathcal{K}}_{h,;\beta,\gamma} \;=\; \left\{ f \in A : \beta\,\frac{zf'(z)}{f(z)} + J(\gamma,h)(z) \prec R_{\beta+\gamma}(z) \right\}, \quad \textit{for} \quad \beta \,\neq\, 1\,,$$

$$\widetilde{\mathcal{K}}_{h;1,\gamma} \;=\; H(\mathrm{U})\,, \quad \textit{for} \quad \beta \,=\, 1\,, \quad \textit{if in addition } \operatorname{Re}\gamma \,>\, 0\,,$$

then the integral operator $\widetilde{\mathrm{I}}_{h;\beta,\gamma}$ is well-defined.

Proof. The case $\beta \neq 1$ represents Lemma 3.1 of [1]. If $\beta = 1$, denoting $t = wz$ we have

$$\widetilde{\mathrm{I}}_{h;1,\gamma}[f](z) \;=\; (\gamma+1)\left[\frac{z}{h(z)}\right]^{\gamma} \int_0^1 f(wz) \left[\frac{h(wz)}{wz}\right]^{\gamma-1} h'(wz) w^{\gamma-1}\,\mathrm{d}w\,,$$

and according to Lemma 2.3 we obtain our result. □

Using this lemma and the previous remark, we deduce the next result:

Corollary 3.2 *Let $\beta, \gamma \in \mathbb{C}$ with $\beta \neq 0$, $\operatorname{Re}(\beta+\gamma) > 0$ and let $h \in \mathcal{A}$. Then the integral operator $\mathrm{I}_{h;\beta,\gamma}$ given by* (1.1) *is well-defined on the set $\mathcal{K}_{h;\beta,\gamma} = \widetilde{\mathcal{K}}_{h;\beta,\gamma}$, i.e.*

$$\mathcal{K}_{h;\beta,\gamma} \;=\; \left\{ f \in A : \beta\,\frac{zf'(z)}{f(z)} + J(\gamma,h)(z) \prec R_{\beta+\gamma}(z) \right\}, \quad \textit{for} \quad \beta \,\neq\, 1\,,$$

$$\mathcal{K}_{h;1,\gamma} \;=\; H(\mathrm{U})\,, \quad \textit{for} \quad \beta \,=\, 1\,, \quad \textit{if in addition} \quad \operatorname{Re}\gamma \,>\, 0\,,$$

where $J(\gamma, h)$ is given by (3.2)*.*

Remark 3.3 Using Lemma 3.1 of [1], under the assumptions of Lemma 3.1, for $\beta \neq 1$, we have

$$\widetilde{F} \;=\; \widetilde{\mathrm{I}}_{h;\beta,\gamma}[f] \in A\,, \quad \frac{\widetilde{F}(z)}{z} \,\neq\, 0 \quad \text{and} \quad \operatorname{Re}\left[\beta\,\frac{z\widetilde{F}'(z)}{\widetilde{F}(z)} + \gamma\,\frac{zh'(z)}{h(z)}\right] \,>\, 0\,, \quad \text{for all} \quad z \in \mathrm{U}\,,$$

hence

$$\mathrm{I}_{h;\beta,\gamma}[f](z) \;=\; \left(\frac{\gamma}{\beta+\gamma}\right)^{1/\beta} z + \ldots \in H(\mathrm{U})\,, \quad \text{for all} \quad f \in \mathcal{K}_{h;\beta,\gamma} \quad \text{and} \quad \beta \,\neq\, 1\,.$$

Remark 3.4 Under the same assumptions of Lemma 3.1, by applying Lemma 2.3 it is easy to check that

$$\mathrm{I}_{h;\beta,\gamma}[f](0) \;=\; f(0)\,, \quad \text{for all} \quad f \in \mathcal{K}_{h;\beta,\gamma}\,.$$

Theorem 3.5 *Let $\beta, \gamma \in \mathbb{C}$ with $\beta > 0$ and $\operatorname{Re}\gamma > 0$, let $h \in A$ and suppose that*

(i) $$\operatorname{Re}\left[\gamma\,\frac{zh'(z)}{h(z)}\right] \,>\, 0\,, \quad z \in \mathrm{U}\,.$$

Let $k \in \mathcal{M}'_{1/\beta}$ and $f \in \mathcal{K}_{h;\beta,\gamma}$ (*see Corollary* 3.2)*. Then*

$$f(z) \,\prec\, k(z) \quad \Longrightarrow \quad \mathrm{I}_{h;\beta,\gamma}[f](z) \,\prec\, k(z)\,.$$

P r o o f. Since $h \in A$, from (i) we have that h is a γ-*spirallike* function, as defined in [11], and the additional condition $\operatorname{Re}\gamma > 0$ implies that h is univalent in U, hence $h \in \mathcal{A}$. According to Corollary 3.2, the operator $\mathrm{I}_{h;\beta,\gamma}$ is well-defined on the set $\mathcal{K}_{h;\beta,\gamma}$.

Since $f(z) \prec k(z)$ we have $f(0) = k(0)$. If $\beta \neq 1$ then $f \in A$, hence $k(0) = f(0) = 0$ i.e. $k \in \mathcal{M}_{1/\beta}$ and it follows that k is univalent in U. If $\beta = 1$ then $k \in \mathcal{K}' \equiv \mathcal{M}'_1$, hence k is a convex (and univalent) function in U.

If we define $F(z) = \mathrm{I}_{h;\beta,\gamma}[f](z)$, then by Remark 3.4 we have $F(0) = f(0)$ and

$$f(z) \;=\; F(z)\left[\frac{\beta}{\gamma}\frac{zF'(z)}{F(z)}\frac{h(z)}{zh'(z)} + 1\right]^{1/\beta}. \tag{3.3}$$

Thus we need to prove the next implication:

$$F(z)\left[\frac{\beta}{\gamma}\frac{zF'(z)}{F(z)}\frac{h(z)}{zh'(z)} + 1\right]^{1/\beta} \prec\; k(z) \quad\Longrightarrow\quad F(z) \;\prec\; k(z)\,. \tag{3.4}$$

For the special case when $\beta = 1$, the above implication reduces to

$$F(z) + B(z)zF'(z) \;\prec\; k(z) \;\prec\; k(z) \quad\Longrightarrow\quad F(z) \;\prec\; k(z)\,,$$

where $B(z) = \frac{h(z)}{\gamma z h'(z)}$. Using Lemma 2.4 for $A = 0$ and $D(z) \equiv 0$, according to (i) we deduce that the implication (3.4) holds for $\beta = 1$.

Next we will prove our result for the case $\beta \neq 1$. Without loss of generality we can assume that k satisfies the conditions of the theorem on the closed disk $\overline{\mathrm{U}}$ and $k'(\zeta) \neq 0$ for $|\zeta| = 1$. If not, then we replace f, k and h by $f_r(z) = f(rz)$, $k_r(z) = k(rz)$ and $h_r(z) = h(rz)$ where $0 < r < 1$, and then k_r is univalent on $\overline{\mathrm{U}}$. Since $f_r(z) \prec k_r(z)$, we would prove that $F_r(z) = F(rz) = \mathrm{I}_{h_r;\beta,\gamma}[f_r](z) \prec k_r(z)$ for $0 < r < 1$, and by letting $r \to 1^-$ we obtain $F(z) \prec k(z)$.

If we suppose that $F(z) \not\prec k(z)$, then by Lemma 2.1 there exist points $z_0 \in \mathrm{U}$ and $\zeta_0 \in \partial\mathrm{U}$, and a number $m \geq 1$, such that

$$F(z_0) \;=\; k(\zeta_0)\,, \tag{3.5}$$

$$z_0F'(z_0) \;=\; m\zeta_0 k'(\zeta_0)\,. \tag{3.6}$$

In order to prove (3.4) we define the function $L : \mathrm{U} \times [0,\infty) \to \mathbb{C}$ by

$$L(z;t) \;=\; k(z)\left[\frac{\beta}{\gamma}\,t\,\frac{zk'(z)}{k(z)}\frac{h(z_0)}{z_0h'(z_0)} + 1\right]^{1/\beta} \;=\; a_1(t)z + \dots\,,$$

and first we will show that $L(z;t)$ is a subordination chain.

From $\left.\frac{zk'(z)}{k(z)}\right|_{z=0} = 1$ and the assumptions (i) and $\beta > 0$, we have

$$\operatorname{Re}\frac{\beta}{\gamma}\left.\frac{zk'(z)}{k(z)}\right|_{z=0}\frac{h(z_0)}{z_0h'(z_0)} \;>\; 0\,,$$

then $L(z;t)$ is analytic in $|z| < r < 1$, for sufficient small $r > 0$ and for all $t \geq 0$. We also have that $L(z;t)$ is continuously differentiable on $[0,\infty)$ for each $|z| < r < 1$.

A simple calculation shows that

$$a_1(t) \;=\; \frac{\partial L(0;t)}{\partial z} \;=\; k'(0)\left[\frac{\beta}{\gamma}\,t\,\frac{h(z_0)}{z_0h'(z_0)} + 1\right]^{1/\beta},$$

and because $k'(0) \neq 0$, from (i) and $\beta > 0$ we deduce

$$\operatorname{Re}\left[\frac{\beta}{\gamma}\,t\,\frac{h(z_0)}{z_0h'(z_0)} + 1\right] \;\geq\; 1 \;>\; 0\,, \quad \text{for all} \quad t \;\geq\; 0\,,$$

hence $a_1(t) \neq 0$, for all $t \geq 0$. From (i) we have $\frac{\beta}{\gamma}\, t\, \frac{h(z_0)}{z_0 h'(z_0)} \neq 0$, for all $z_0 \in \mathrm{U}$ and for all $t > 0$, hence $\lim_{t\to\infty} |a_1(t)| = \infty$.

Using the definition of the $L(z;t)$ function we obtain

$$\mathrm{Re}\left[z\frac{\partial L/\partial z}{\partial L/\partial t}\right] \;=\; t\beta\,\mathrm{Re}\left[\left(1-\frac{1}{\beta}\right)\frac{zk'(z)}{k(z)} + \frac{1}{\beta}\left(1+\frac{zk''(z)}{k'(z)}\right)\right] + \mathrm{Re}\left[\gamma\,\frac{z_0 h'(z_0)}{h(z_0)}\right].$$

From the above relation, by using the fact that $k \in \mathcal{M}'_{1/\beta}$ and the assumption (i), we deduce that

$$\mathrm{Re}\left[z\,\frac{\partial L/\partial z}{\partial L/\partial t}\right] \;>\; 0\,, \quad \text{for all } z \in \mathrm{U}\,, \quad \text{for all } t \;\geq\; 0\,,$$

and using Lemma 2.5 we conclude that $L(z;t)$ is a subordination chain. In particular, this implies

$$k(z) \;=\; L(z;0) \;\prec\; L(z;t)\,, \quad \text{for all} \quad t \;\geq\; 0\,. \tag{3.7}$$

Using the equality (3.3) and the relations (3.5) and (3.6) we obtain

$$\begin{aligned} f(z_0) \;&=\; F(z_0)\left[\frac{\beta}{\gamma}\frac{z_0F'(z_0)}{F(z_0)}\,\frac{h(z_0)}{z_0h'(z_0)} + 1\right]^{1/\beta} \\ &=\; k(\zeta_0)\left[\frac{\beta}{\gamma}\,m\,\frac{\zeta_0 k'(\zeta_0)}{k(\zeta_0)}\,\frac{h(z_0)}{z_0h'(z_0)} + 1\right]^{1/\beta} \;=\; L(\zeta_0;m)\,, \quad m \;\geq\; 1\,. \end{aligned}$$

From here, according to (3.7) we finally deduce that

$$f(z_0) \;=\; L(\zeta_0;m) \notin k(\mathrm{U})\,.$$

Since this contradicts the assumption $f(z) \prec k(z)$, we conclude that $F(z) \prec k(z)$. □

From the above theorem, for the case $k(z) = f(z)$ we obtain in particular the next result:

Corollary 3.6 *Let $\beta, \gamma \in \mathbb{C}$ with $\beta > 0$ and $\mathrm{Re}\,\gamma > 0$, let $h \in A$ and suppose that*

(i) $$\mathrm{Re}\left[\gamma\,\frac{zh'(z)}{h(z)}\right] \;>\; 0\,, \quad z \in \mathrm{U}\,.$$

If $f \in \mathcal{K}_{h;\beta,\gamma} \cap \mathcal{M}'_{1/\beta}$, then

$$\mathrm{I}_{h;\beta,\gamma}[f](z) \;\prec\; f(z)\,.$$

Theorem 3.7 *Let $\beta, \gamma \in \mathbb{C}$ with $\beta \geq 1$ and $\mathrm{Re}\,\gamma > 0$, let $h \in A$ and suppose that*

(i) $$\mathrm{Re}\left[\gamma\,\frac{zh'(z)}{h(z)}\right] \;>\; 0\,, \quad z \in \mathrm{U}\,.$$

Then the integral operator $\mathrm{I}_{h;\beta,\gamma}$ is an averaging operator on $\mathcal{K}_{h;\beta,\gamma}$.

P r o o f. In order to prove our result we will use Theorem 3.5 and Lemma 2.2.

If $f \in \mathcal{K}_{h;\beta,\gamma}$, then by Remark 3.4 we have $\mathrm{I}_{h;\beta,\gamma}[f](0) = f(0)$. Let k be a convex function such that $f(z) \prec k(z)$.

For the case $\beta = 1$ we have $k \in \mathcal{K}' \equiv \mathcal{M}'_1$ and, applying Theorem 3.5, we obtain that

$$\mathrm{I}_{h;1,\gamma}[f](z) \;\prec\; k(z)\,.$$

Then, by Lemma 2.2, the integral operator $\mathrm{I}_{h;1,\gamma}$ is an averaging operator on $\mathcal{K}_{h;1,\gamma}$.

For the case $\beta > 1$, since $f \in \mathcal{K}_{h;\beta,\gamma}$ and $f(z) \prec k(z)$, we have $k(0) = 0$. According to (2.1) we have $k \in \mathcal{M}_1 \subset \mathcal{M}_{1/\beta} \subset \mathcal{M}'_{1/\beta}$, whenever $\beta > 1$. Applying Theorem 3.5 we deduce

$$\mathrm{I}_{h;\beta,\gamma}[f](z) \;\prec\; k(z)\,, \quad \text{for} \quad \beta \;>\; 1\,,$$

and then, using Lemma 2.2, the integral operator $\mathrm{I}_{h;\beta,\gamma}$ is an averaging operator on $\mathcal{K}_{h;\beta,\gamma}$ for $\beta > 1$. □

Remark 3.8 1. Note that this result generalizes Theorem 1 of [5], which may be obtained for the particular case $h(z) = z$, i.e.:

If $\beta \geq 1$ *and* $\gamma \in \mathbb{C}$ *with* $\operatorname{Re}\gamma > 0$*, then the operator* $\mathrm{I}_{z;\beta,\gamma}$ *is an averaging operator on* $\mathcal{K}_{z;\beta,\gamma}$*.*

We mention that the case $\beta = 1$ of Theorem 3.7 may be found in [6, Theorem 4.4c].

4 Particular cases

In this section we will discuss several particular cases of Theorem 3.7 obtained for appropriate choices of the function h.

1. Taking $h(z) = z + az^2$, $|a| \leq 1$, in Theorem 3.7 then for $z \in \partial\mathrm{U}$ we have $az = |a|\, e^{i\theta}$, $\theta \in [0, 2\pi]$, so we obtain

$$\frac{zh'(z)}{h(z)} = \frac{1 + 2\,|a|\, e^{i\theta}}{1 + |a| e^{i\theta}}\,, \quad \theta \in [0, 2\pi]\,.$$

Then we deduce that the function $\varphi(z) = \frac{zh'(z)}{h(z)}$ maps the unit disk U onto the disk

$$\varphi(\mathrm{U}) = \mathrm{U}\left(\frac{1 - 1\,|a|^2}{1 - |a|^2}; \frac{|a|}{1 - |a|^2}\right).$$

Using this fact, it follows that

$$\operatorname{Re}\frac{zh'(z)}{h(z)} > \frac{1 - 2\,|a|}{1 - |a|} \geq 0\,, \text{ for all } z \in \mathrm{U} \iff |a| \leq \frac{1}{2}\,,$$

and, if $|a| \leq \frac{1}{2}$ then

$$\operatorname{Re}\left[\gamma\,\frac{zh'(z)}{h(z)}\right] > 0\,, \ z \in \mathrm{U} \iff |\arg\gamma| \leq \frac{\pi}{2} - \arcsin\frac{|a|}{1 - 2\,|a|^2}\,.$$

Now, using Theorem 3.7 we obtain:

Example 4.1 Let $\beta, \gamma \in \mathbb{C}$ with $\beta \geq 1$, $\operatorname{Re}\gamma > 0$ and

$$|\arg\gamma| \leq \frac{\pi}{2} - \arcsin\frac{|a|}{1 - 2\,|a|^2}\,,$$

where $a \in \mathbb{C}$ with $|a| \leq \frac{1}{2}$. Then the integral operator $\mathrm{I}_{z+az^2;\beta,\gamma}$ is an averaging operator on $\mathcal{K}_{z+az^2;\beta,\gamma}$, where

$$\mathrm{I}_{z+az^2;\beta,\gamma}[f](z) = \left[\frac{\gamma}{(z + az^2)^\gamma}\int_0^z f^\beta(t)\big(t + at^2\big)^{\gamma-1}(1 + 2at)\,\mathrm{d}t\right]^{1/\beta}$$

and

$$\mathcal{K}_{z+az^2;\beta,\gamma} = \left\{f \in A : \beta\,\frac{zf'(z)}{f(z)} + (\gamma - 1)\,\frac{1 + 2az}{1 + az} + \frac{1 + 4az}{1 + 2az} \prec R_{\beta+\gamma}(z)\right\}\,, \text{ for } \beta \neq 1\,,$$

$$\mathcal{K}_{z+az^2;1,\gamma} = H(\mathrm{U})\,, \text{ for } \beta = 1\,.$$

Remark that this particular case also extends Theorem 1 of [5], that can be obtained for $a = 0$.

2. Taking $h(z) = ze^{\lambda z}$, $\lambda \in \mathbb{C}$, in Theorem 3.7, then a simple calculus shows that the function $\varphi(z) = \frac{zh'(z)}{h(z)}$ maps the unit disk U onto the disk $\varphi(\mathrm{U}) = \mathrm{U}\,(1; |\lambda|)$. Using this fact we obtain that

$$\operatorname{Re}\frac{zh'(z)}{h(z)} > 1 - |\lambda| \geq 0\,, \quad \text{for all} \quad z \in \mathrm{U} \iff |\lambda| \leq 1\,,$$

and, if $|\lambda| \le 1$ then

$$\operatorname{Re}\left[\gamma\,\frac{zh'(z)}{h(z)}\right] > 0\,, \quad z \in \mathrm{U} \quad \Longleftrightarrow \quad |\arg\gamma| \le \frac{\pi}{2} - \arcsin|\lambda|\,.$$

According to Theorem 3.7 we obtain:

Example 4.2 Let $\beta, \gamma \in \mathbb{C}$ with $\beta \ge 1$, $\operatorname{Re}\gamma > 0$ and

$$|\arg\gamma| \le \frac{\pi}{2} - \arcsin|\lambda|\,,$$

where $\lambda \in \mathbb{C}$ with $|\lambda| \le 1$. Then the integral operator $\mathrm{I}_{ze^{\lambda z};\beta,\gamma}$ is an averaging operator on $\mathcal{K}_{ze^{\lambda z};\beta,\gamma}$, where

$$\mathrm{I}_{ze^{\lambda z};\beta,\gamma}[f](z) = \left[\frac{\gamma}{(ze^{\lambda z})^{\gamma}}\int_0^z f^{\beta}(t)t^{\gamma-1}(1+\lambda t)e^{\lambda\gamma t}\,\mathrm{d}t\right]^{1/\beta}$$

and

$$\mathcal{K}_{ze^{\lambda z};\beta,\gamma} = \left\{f \in A : \beta\,\frac{zf'(z)}{f(z)} + \gamma(1+\lambda z) + \frac{\lambda z}{1+\lambda z} \prec R_{\beta+\gamma}(z)\right\}, \quad \text{for } \beta \ne 1\,,$$

$$\mathcal{K}_{ze^{\lambda z};1,\gamma} = H(\mathrm{U}), \quad \text{for } \beta = 1\,.$$

Note that this example also extends Theorem 1 of [5], that can be obtained for the particular case $\lambda = 0$.

3. Taking $h(z) = \frac{z}{(1+z)^{2(1-a)}}$, $a < 1$, in Theorem 3.7 we obtain

$$\frac{zh'(z)}{h(z)} = \frac{1+2(a-1)z}{1+z}\,.$$

Hence, the function $\varphi(z) = \frac{zh'(z)}{h(z)}$ maps the unit disk U onto the half-plane

$$\varphi(\mathrm{U}) = \{w \in \mathbb{C} : \operatorname{Re} w > a\}\,,$$

and, for $a \ge 0$ it follows that

$$\operatorname{Re}\left[\gamma\,\frac{zh'(z)}{h(z)}\right] > 0\,, \quad z \in \mathrm{U} \quad \Longleftrightarrow \quad \gamma > 0\,.$$

Using Theorem 3.7 we have:

Example 4.3 Let $\beta, \gamma \in \mathbb{R}$ with $\beta \ge 1$, $\gamma > 0$ and let $a \in [0,1)$. Then the integral operator $\mathrm{I}_{z/(1+z)^{2(1-a)};\beta,\gamma}$ is an averaging operator on $\mathcal{K}_{z/(1+z)^{2(1-a)};\beta,\gamma}$, where

$$\mathrm{I}_{z/(1+z)^{2(1-a)};\beta,\gamma}[f](z) = \left[\frac{\gamma(1+z)^{2\gamma(1-a)}}{z^{\gamma}}\int_0^z f^{\beta}(t)t^{\gamma-1}\,\frac{1+(2a-1)t}{(1+t)^{2\gamma(1-a)+1}}\,\mathrm{d}t\right]^{1/\beta}$$

and

$$\mathcal{K}_{z/(1+z)^{2(1-a)};\beta,\gamma} = \left\{f \in A : \beta\,\frac{zf'(z)}{f(z)} + \frac{[\gamma(2a-1)-1]z+\gamma}{1+z} + \frac{(2a-1)z}{(2a-1)z+1} \prec R_{\beta+\gamma}(z)\right\}, \quad \text{for } \beta \ne 1\,,$$

$$\mathcal{K}_{z/(1+z)^{2(1-a)};1,\gamma} = H(\mathrm{U})\,, \quad \text{for } \beta = 1\,.$$

4. Taking in Theorem 3.7 $h(z) = z(1-z)^{e^{-i\alpha}(2\rho\cos\alpha - e^{-i\alpha})-1}$, where $\alpha \in \mathbb{R}$ with $|\alpha| \le \frac{\pi}{2}$ and $\rho \in [0,1)$, we get

$$\frac{zh'(z)}{h(z)} = \frac{1+e^{-i\alpha}(e^{-i\alpha}-2\rho\cos\alpha)z}{1-z}\,.$$

Consequently, if $\gamma = |\gamma|\, e^{i\alpha}$ we obtain

$$\operatorname{Re}\left[\gamma\, \frac{zh'(z)}{h(z)}\right] \;=\; |\gamma| \operatorname{Re}\left[e^{i\alpha}\, \frac{zh'(z)}{h(z)}\right] \;>\; |\gamma|\, \rho \cos\alpha \;\geq\; 0\,, \quad z \in \mathrm{U}\,,$$

and then the function h is an α-*spirallike function of order* ρ [10]. From Theorem 3.7 we obtain:

Example 4.4 Let $\beta, \gamma \in \mathbb{C}$ with $\beta \geq 1$, $\gamma = |\gamma|\, e^{i\alpha}$ where $\alpha \in \mathbb{R}$, $|\alpha| < \frac{\pi}{2}$ and let $\rho \in [0,1)$. Then the integral operator $\mathrm{I}_{h;\beta,\gamma}$ is an averaging operator on $\mathcal{K}_{h;\beta,\gamma}$, where $h(z) = z(1-z)^{e^{-i\alpha}(2\rho\cos\alpha - e^{-i\alpha})-1}$.

Remark that, if we take in this example $\alpha = \rho = 0$ we obtain a similar result to that of Example 4.3 for the particular case $a = 0$.

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