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Discrete universality theorems for the Hurwitz zeta-function

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Abstract

In the paper, we obtain a discrete universality theorem on approximation of analytic functions by discrete shifts of the Hurwitz zeta-function under a new hypothesis on the parameter generalizing the transcendence condition.

Keywords: Hurwitz zeta-function; Limit theorem; Space of analytic functions; Universality; Weak convergence of probability measures

1. Introduction

Let $s = \sigma + it$ be a complex variable, and α, $0 < \alpha \leq 1$, be a fixed parameter. The Hurwitz zeta-function $\zeta(s, \alpha)$ is defined, for $\sigma > 1$, by the series

$$\zeta(s, \alpha) = \sum_{m=0}^{\infty} \frac{1}{(m + \alpha)^s},$$

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and is analytically continued to the whole s-plane, except for a simple pole at the point $s = 1$ with residue 1. For $\alpha = 1$, the function $\zeta(s, \alpha)$ reduces to the Riemann zeta-function $\zeta(s)$, and

$$\zeta\left(s, \frac{1}{2}\right) = (2^s - 1)\zeta(s).$$

Except for the cases $\alpha = 1$ and $\alpha = \frac{1}{2}$, the function $\zeta(s, \alpha)$, differently from the function $\zeta(s)$, has no Euler product over primes, and this is reflected in its analytic properties. For example, $\zeta(s) \neq 0$ in the half-plane $\sigma > 1$, while the function $\zeta(s, \alpha)$ with $\alpha \neq 1, \frac{1}{2}$ has infinitely many zeros lying in this half-plane [3,4]. Moreover, if α is transcendental or rational $\neq 1, \frac{1}{2}$, then $\zeta(s, \alpha)$ also has infinitely many zeros in the strip $D = \{s \in \mathbb{C} : \frac{1}{2} < \sigma < 1\}$ [15,9]. We believe that this is also true for algebraic irrational α.

In 1975, Voronin discovered [14] the universality of the Riemann zeta-function. More precisely, he proved that any analytic non-vanishing function uniformly on closed discs of the strip D can be approximated by shifts $\zeta(s + i\tau)$. Various authors improved and generalized the Voronin theorem. Let $\mathcal{K}$ be the class of compact subsets of D with connected complements, and let $H_0(K)$, $K \in \mathcal{K}$, denote the class of continuous non-vanishing functions on K which are analytic in the interior of K. Then the latest version of the Voronin theorem is the following statement, see, for example, [7].

Theorem 1. *Let $K \in \mathcal{K}$, and $f(s) \in H_0(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T}\text{meas}\left\{\tau \in [0, T] : \sup_{s\in K} |\zeta(s + i\tau) - f(s)| < \varepsilon\right\} > 0.$$

Here meas$\{A\}$ denotes the Lebesgue measure of a measurable set $A \subset \mathbb{R}$. Theorem 1 shows that the set of shifts $\zeta(s + i\tau)$, $\tau \in \mathbb{R}$, approximating the function $f(s)$ is infinite and has a positive lower density.

The Hurwitz zeta-function, for some parameters α, is also universal. Let $H(K)$, $K \in \mathcal{K}$, be the class of continuous functions on K which are analytic in the interior of K.

Theorem 2. *Suppose that α is a transcendental or rational number $\neq 1, \frac{1}{2}$. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T}\text{meas}\left\{\tau \in [0, T] : \sup_{s\in K} |\zeta(s + i\tau, \alpha) - f(s)| < \varepsilon\right\} > 0.$$

The theorem for rational α was obtained indirectly by Voronin [15, Theorem 8], Gonek [5] without density, and Bagchi [1, pp. 147–148] with a bit different compact set K. We also note that the Gonek method implies easily the positivity of a lower density. The case of transcendental α was given in [5] without density, and in [9].

In this paper, we consider the so-called discrete universality of the function $\zeta(s, \alpha)$ when the functions from the class $H(K)$ are approximated by shifts $\zeta(s + ikh)$ with a fixed $h > 0$ and $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The following result is known.

Theorem 3. *Suppose that the parameter α, the set K and the function $f(s)$ are as in Theorem 2. If α is rational, then let the fixed number $h > 0$ be arbitrary, while if α is transcendental, then* *let $h > 0$ be such that* $\exp\left\{\frac{2\pi}{h}\right\}$ *is a rational number. Then, for every $\varepsilon > 0$,*

$$\liminf_{N\to\infty} \frac{1}{N+1} \# \left\{ 0 \le k \le N : \sup_{s\in K} |\zeta(s+ikh,\alpha) - f(s)| < \varepsilon \right\} > 0.$$

Theorem 3 in the case of rational α with slightly different hypotheses on the set K was obtained in [1, Corollary 5.3.7], and by a different method in [13]. The case of transcendental α follows from [10].

Our aim is to generalize the case of transcendental α. Let, as usual, $\mathbb{Q}$ be the set of all rational numbers, $\mathbb{Q}_1^+$ be the set of positive rational numbers $\neq 1$, and define, for $q \in \mathbb{Q}_1^+$, the set

$$L(\alpha, q) = \{(\log(m+\alpha) : m \in \mathbb{N}_0), \log q\}.$$

Theorem 4. *Suppose that the set $L(\alpha, q)$, for every $q \in \mathbb{Q}_1^+$, is linearly independent over the field $\mathbb{Q}$, and K and $f(s)$ are as in Theorem 2. Then, for every $h > 0$ such that the number* $\exp\left\{\frac{2\pi}{h}\right\}$ *is a rational, the same assertion as in Theorem 3 is true.*

Clearly, if α is transcendental, then the set $L(\alpha, q)$ is linearly independent over $\mathbb{Q}$. Indeed, suppose that there exist the numbers $k_1, \ldots, k_r, k \in \mathbb{Z} \setminus \{0\}$ such that

$$k_1 \log(m_1+\alpha) + \cdots + k_r \log(m_r+\alpha) + k \log q = 0.$$

Then

$$(m_1+\alpha)^{k_1} \ldots (m_r+\alpha)^{k_r} q^k = 1,$$

and this contradicts the transcendence of α.

If α is algebraic irrational, then it is known [3] that at least 51% of elements of the set $L(\alpha) = \{\log(m+\alpha) : m \in \mathbb{N}_0\}$ are linearly independent over $\mathbb{Q}$. Thus, it is conceivable that, for some algebraic irrational α, the set $L(\alpha, q)$ is also linearly independent over $\mathbb{Q}$. Unfortunately, examples of algebraic irrational α with linearly independent sets $L(\alpha)$ and $L(\alpha, q)$ are not known.

Next, we give a generalization of Theorem 4 for the composite function. Denote by $H(D)$ the space of analytic functions on D endowed with the topology of uniform convergence on compacta.

Theorem 5. *Suppose that the set $L(\alpha, q)$, for every $q \in \mathbb{Q}_1^+$, is linearly independent over $\mathbb{Q}$, $F : H(D) \to H(D)$ is a continuous operator such that, for each polynomial $p = p(s)$, the preimage $F^{-1}\{p\}$ is not empty. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then, for any $h > 0$ such that the number $\exp\{\frac{2\pi}{h}\}$ is rational and every $\varepsilon > 0$,*

$$\liminf_{N\to\infty} \frac{1}{N+1} \# \left\{ 0 \le k \le N : \sup_{s\in K} |F(\zeta(s+ikh,\alpha)) - f(s)| < \varepsilon \right\} > 0.$$

For example, it is easy to see that the operator $F(g) = c_1 g' + \cdots + c_r g^{(r)}$, $g \in H(D)$, $c_1, \ldots, c_r \in \mathbb{C} \setminus \{0\}$, satisfies the hypotheses of Theorem 5.

Proofs of Theorems 4 and 5 are based on discrete limit theorems for weakly convergent probability measures in the space $H(D)$.

2. Limit theorems

Denote by $\mathcal{B}(X)$ the Borel σ-field of a space X. This section is devoted to the weak convergence of the measures

$$P_N(A) \overset{def}{=} \frac{1}{N+1}\#\{0 \le k \le N : \zeta(s+ikh,\alpha) \in A\}, A \in \mathcal{B}(H(D)),$$

and

$$P_{N,F}(A) \overset{def}{=} \frac{1}{N+1}\#\{0 \le k \le N : F(\zeta(s+ikh,\alpha)) \in A\}, \ A \in \mathcal{B}(H(D)),$$

as $N \to \infty$. The weak convergence of $P_{N,F}$ will be derived from that of P_N. It is possible to prove a limit theorem for P_N without any restrictions on the numbers α and h. However, in this case, it is not possible to identify a support of the limit measure, and the class of approximated functions in the universality theorem remains not explicitly defined.

We start with a limit theorem on the torus

$$\Omega = \prod_{m=0}^{\infty} \gamma_m,$$

where $\gamma_m = \{s \in \mathbb{C} : |s| = 1\}$ for all $m \in \mathbb{N}_0$. By the Tikhonov theorem, the torus Ω with the product topology and pointwise multiplication is a compact topological Abelian group. Therefore, on $(\Omega, \mathcal{B}(\Omega))$, the probability Haar measure m_H can be defined, and we obtain the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. Denote by $\omega(m)$ the projection of $\omega \in \Omega$ to the coordinate space γ_m, $m \in \mathbb{N}_0$. Then we have that $\{\omega(m) : m \in \mathbb{N}_0\}$ is a sequence of independent complex-valued random variables on the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$.

Let, for $A \in \mathcal{B}(\Omega)$,

$$Q_N(A) \overset{def}{=} \frac{1}{N+1}\#\{0 \le k \le N : ((m+\alpha)^{-ikh} : m \in \mathbb{N}_0) \in A\}.$$

Lemma 1. *Suppose that the set $L(\alpha, q)$, for every $q \in \mathbb{Q}_1^+$, is linearly independent over $\mathbb{Q}$. Then, for any $h > 0$ such that $\exp\{\frac{2\pi}{h}\}$ is rational, the measure Q_N converges weakly to the Haar measure m_H as $N \to \infty$.*

Proof. The dual group of Ω is isomorphic to

$$\mathcal{D} \overset{def}{=} \bigoplus_{m=0}^{\infty} \mathbb{Z}_m,$$

where $\mathbb{Z}_m = \mathbb{Z}$ for all $m \in \mathbb{N}_0$. An element $\underline{k} = (k_0, k_1, \ldots) \in \mathcal{D}$ acts on Ω by the formula

$$\underline{k} \to \omega^{\underline{k}} = \prod_{m=0}^{\infty} \omega^{k_m}(m),$$

where only a finite number of integers k_m are distinct from zero. Therefore, the Fourier transform $g_N(\underline{k})$ of the measure Q_N is

$$\int_{\Omega} \prod_{m=0}^{\infty} \omega^{k_m}(m) dQ_N.$$

Thus, by the definition of Q_N,

$$g_N(\underline{k}) = \frac{1}{N+1} \sum_{k=0}^{N} \prod_{m=0}^{\infty} (m+\alpha)^{-ikk_m h}$$
$$= \frac{1}{N+1} \sum_{k=0}^{N} \exp\left\{-ikh \sum_{m=0}^{\infty} k_m \log(m+\alpha)\right\}, \tag{1}$$

where only a finite number of integers k_m are distinct from zero. The linear independence of $L(\alpha, q)$ implies that of $L(\alpha)$. Therefore,

$$\sum_{m=0}^{\infty} k_m \log(m+\alpha) = 0$$

if and only if $\underline{k} = \underline{0}$. Moreover,

$$\exp\left\{-ih \sum_{m=0}^{\infty} k_m \log(m+\alpha)\right\} \neq 1 \tag{2}$$

for $\underline{k} \neq \underline{0}$. Indeed, if the above inequality is not true, then

$$\exp\left\{\sum_{m=0}^{\infty} k_m \log(m+\alpha)\right\} = \exp\left\{\frac{2\pi a}{h}\right\}$$

with some $a \in \mathbb{Z} \setminus \{0\}$ is a rational number $q \neq 1$ in view of the rationality of $\exp\left\{\frac{2\pi}{h}\right\}$. Thus, if $q > 1$, then

$$\sum_{m=0}^{\infty} k_m \log(m+\alpha) - \log q = 0,$$

where only a finite number of integers k_m are not zero. However, this contradicts the linear independence of the set $L(\alpha, q)$. If $q < 1$, a similar arguments hold with $\log(1/q)$.

Taking into account (2), we find from (1) that

$$g_N(\underline{k}) = \begin{cases} 1 & \text{if} \quad \underline{k} = \underline{0}, \\ \dfrac{1 - \exp\{-i(N+1)h \sum_{m=0}^{\infty} k_m \log(m+\alpha)\}}{(N+1)\left(1 - \exp\left\{-ih \sum_{m=0}^{\infty} k_m \log(m+\alpha)\right\}\right)} & \text{if } \underline{k} \neq \underline{0}. \end{cases}$$

Hence,

$$\lim_{N\to\infty} g_N(\underline{k}) = \begin{cases} 1 & \text{if } \underline{k} = \underline{0}, \\ 0 & \text{if } \underline{k} \neq \underline{0}. \end{cases}$$

Thus, by a continuity theorem for probability measures on compact groups, Theorem 1.4.2 of [6], we obtain that Q_N converges weakly to m_H as $N \to \infty$.

For a fixed number $\sigma_1 > \frac{1}{2}$, and $m \in \mathbb{N}_0$, $n \in \mathbb{N}$, let

$$v_n(m,\alpha) = \exp\left\{-\left(\frac{m+\alpha}{n+\alpha}\right)^{\sigma_1}\right\},$$

and define

$$\zeta_n(s,\alpha) = \sum_{m=0}^{\infty} \frac{v_n(m,\alpha)}{(m+\alpha)^s}.$$

Then it is well known, see, for examples [9, Chapter 5], that the latter series is absolutely convergent for s with $\sigma > \frac{1}{2}$. Moreover, for $\omega \in \Omega$, let

$$\zeta_n(s,\alpha,\omega) = \sum_{m=0}^{\infty} \frac{\omega(m) v_n(m,\alpha)}{(m+\alpha)^s}.$$

Then the series for $\zeta_n(s,\alpha,\omega)$ also converges absolutely for $\sigma > \frac{1}{2}$. Define two probability measures

$$P_{N,n}(A) = \frac{1}{N+1} \#\{0 \le k \le N : \zeta_n(s+ikh,\alpha) \in A\}, \quad A \in \mathcal{B}(H(D)),$$

and

$$\widehat{P}_{N,n}(A) = \frac{1}{N+1} \#\{0 \le k \le N : \zeta_n(s+ikh,\alpha,\widehat{\omega}) \in A\}, \quad A \in \mathcal{B}(H(D)),$$

where $\widehat{\omega}$ is a fixed element of Ω.

Lemma 2. *Suppose that the set $L(\alpha,q)$, for every $q \in \mathbb{Q}_1^+$, is linearly independent over $\mathbb{Q}$. Then, for every $h > 0$ such that the number $\exp\{\frac{2\pi}{h}\}$ is rational, $P_{N,n}$ and $\widehat{P}_{N,n}$ both converge weakly to the same probability measure P_n on $(H(D), \mathcal{B}(H(D)))$ as $N \to \infty$.*

Proof. We apply standard arguments (see also [10]). Define $u_n : \Omega \to H(D)$ by the formula

$$u_n(\omega) = \zeta_n(s,\alpha,\omega).$$

The absolute convergence of the series for $\zeta_n(s,\alpha,\omega)$ implies the continuity of u_n. Moreover, we have that $P_{N,n} = Q_N u_n^{-1}$, where Q_N is the measure from Lemma 1. Then above remarks, Lemma 1 and Theorem 5.1 of [2] show that $P_{N,n}$ converges weakly to $m_H u_n^{-1}$ as $N \to \infty$.

Let $\widehat{u}_n : \Omega \to H(D)$ be given by the formula

$$\widehat{u}_n(\omega) = \zeta_n(s,\alpha,\widehat{\omega}\omega).$$

Then, similarly as above, we find that $\widehat{P}_{N,n}$ converges weakly to $m_H \widehat{u}_n^{-1}$ as $N \to \infty$. Let $u : \Omega \to \Omega$ be given by $u(\omega) = \widehat{\omega}\omega$. Then we have that $\widehat{u}_n = u_n(u)$. This and the invariance of the Haar measure m_H lead to the equality $m_H \widehat{u}_n^{-1} = m_H u_n^{-1}$. Thus, the measures $P_{N,n}$ and $\widehat{P}_{N,n}$ both converge weakly to $P_n = m_H u_n^{-1}$ as $N \to \infty$.

Now we will approximate $\zeta(s,\alpha)$ by $\zeta_n(s,\alpha)$ in the mean. For this, we need a metric on $H(D)$ which induces the topology of uniform convergence on compacta. Let $\{K_l : l \in \mathbb{N}\}$ be a sequence of compact subsets of D such that

$$D = \bigcup_{l=1}^{\infty} K_l,$$

$K_l \subset K_{l+1}$ for all $l \in \mathbb{N}$, and if $K \subset D$ is compact, then $K \subset K_l$ for some $l \in \mathbb{N}$. For $g_1, g_2 \in H(D)$, define

$$\varrho(g_1, g_2) = \sum_{l=1}^{\infty} 2^{-l} \frac{\sup_{s \in K_l} |g_1(s) - g_2(s)|}{1 + \sup_{s \in K_l} |g_1(s) - g_2(s)|}.$$

Then ϱ is a metric on $H(D)$ inducing its topology.

Lemma 3. *The relation*

$$\lim_{n \to \infty} \limsup_{N \to \infty} \frac{1}{N+1} \sum_{l=0}^{N} \varrho(\zeta(s + ilh, \alpha), \zeta_n(s + ilh, \alpha)) = 0$$

holds for all $h > 0$ and α.

Proof. It is known, see, for example, [9, Theorem 3.3.1] and [10], that, for $\frac{1}{2} < \sigma < 1$,

$$\int_0^T |\zeta(\sigma + it, \alpha)|^2 dt \ll T, \qquad \int_0^T |\zeta'(\sigma + it, \alpha)|^2 dt \ll T.$$

From this, using the Gallagher lemma, Lemma 1.4 of [12], we find that, for $\frac{1}{2} < \sigma < 1$,

$$\sum_{l=0}^{N} |\zeta(\sigma + it + ilh, \alpha)|^2 \ll N(1 + |t|).$$

Taking into account the latter estimate and repeating the proof of Theorem 4.1 from [10], we obtain that, for every compact subset $K \subset D$,

$$\lim_{n \to \infty} \limsup_{N \to \infty} \frac{1}{N+1} \sum_{l=0}^{N} \sup_{s \in K} |\zeta(s + ilh, \alpha) - \zeta_n(s + ilh, \alpha)| = 0.$$

This together with the definition of the metric ϱ proves the lemma.

The case of the approximation of $\zeta(s, \alpha, \omega)$ by $\zeta_n(s, \alpha, \omega)$ is more complicated, and we need a result of the ergodic theory.

Let $a_\tau = \{(m + \alpha)^{-i\tau} : m \in \mathbb{N}_0\}$ for $\tau \in \mathbb{R}$, and let the one-parameter family $\{\varphi_\tau : \tau \in \mathbb{R}\}$ of transformations on Ω be defined by $\varphi_\tau(\omega) = a_\tau \omega$ for $\omega \in \Omega$. Then we have that $\{\varphi_\tau : \tau \in \mathbb{R}\}$ is a one-parameter group of measurable measure preserving transformations on Ω.

Lemma 4. *Suppose that the set $L(\alpha)$ is linearly independent over $\mathbb{Q}$. Then the group $\{\varphi_\tau : \tau \in \mathbb{R}\}$ is ergodic.*

Proof. The lemma is a particular case of Lemma 10 from [8].

Lemma 5. *Suppose that the set $L(\alpha)$ is linearly independent over $\mathbb{Q}$. Then, for almost all $\omega \in \Omega$,*

$$\lim_{n \to \infty} \limsup_{N \to \infty} \frac{1}{N+1} \sum_{l=0}^{N} \varrho(\zeta(s + ilh, \alpha, \omega), \zeta_n(s + ilh, \alpha, \omega)) = 0.$$

Proof. Using Lemma 4 and repeating the proof of Lemma 5.2.10 from [9], we obtain that, for $\frac{1}{2} < \sigma < 1$,

$$\int_0^T |\zeta(\sigma + it, \alpha, \omega)|^2 dt \ll T,$$

for almost all $\omega \in \Omega$. This estimate together with the Gallagher lemma, see also [10], implies, for $\frac{1}{2} < \sigma < 1$ and almost all $\omega \in \Omega$, the estimate

$$\sum_{l=0}^{N} |\zeta(\sigma + it + ilh, \alpha, \omega)|^2 dt \ll N(1 + |t|).$$

Thus, the further part of the proof runs in the same way as that of Lemma 3.

Now we are ready to prove the weak convergence for the measure P_N. By the proof of Lemma 5.2.1 from [9], we have that the series

$$\zeta(s, \alpha, \omega) = \sum_{m=0}^{\infty} \frac{\omega(m)}{(m + \alpha)^s},$$

for almost all $\omega \in \Omega$, converges uniformly on compact subsets of the strip D, and therefore, defines an $H(D)$-valued random element on the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. Denote by Ω_1 the subset of Ω such that $\zeta(s, \alpha, \omega)$ is a function from $H(D)$ for $\omega \in \Omega_1$. Then $m_H(\Omega_1) = 1$. Together with P_N, we consider the measure

$$\widehat{P}_N(A) \stackrel{def}{=} \frac{1}{N+1} \#\{0 \le k \le N : \zeta(s + ikh, \alpha, \omega) \in A\}, \ A \in \mathcal{B}(H(D)),$$

where $\omega \in \Omega_1$.

Lemma 6. *Suppose that the set $L(\alpha, q)$, for every $q \in \mathbb{Q}_1^+$, is linearly independent over $\mathbb{Q}$. Then, for every $h > 0$ such that the number* $\exp\left\{\frac{2\pi}{h}\right\}$ *is rational, the measures P_N and $\widehat{P}_N$ both converge to the same probability measure P as $N \to \infty$.*

Proof. We argue similarly to the proof of Theorem 6.2 from [10]. By Lemma 2, the measure $P_{N,n}$ converges weakly to P_n as $N \to \infty$. Let $X_n(s)$ be the $H(D)$-valued random element with the distribution P_n. Moreover, let θ_N be a discrete random variable defined on a certain probability space $(\Omega_0, \mathcal{B}(\Omega_0), \mathbb{P})$ and having the distribution $\mathbb{P}(\theta_N = lh) = \frac{1}{N+1}$, $l = 0, \ldots, N$. Define an $H(D)$-valued random element $X_{N,n}(s)$ by

$$X_{N,n}(s) = \zeta_n(s + i\theta_N, \alpha).$$

Then we have by Lemma 2 that

$$X_{N,n}(s) \xrightarrow[N \to \infty]{\mathcal{D}} X_n(s), \tag{3}$$

where $\xrightarrow{\mathcal{D}}$ denotes the convergence in distribution.

Let K_m be a compact set in the definition of the metric ϱ. Then, using the estimate

$$\sum_{l=0}^{N} |\zeta(\sigma + ilh, \alpha)|^2 \ll N$$

which is valid for $\frac{1}{2} < \sigma < 1$, we deduce by contour integration that

$$\sum_{l=0}^{N} \sup_{s \in K_m} |\zeta(s + ilh, \alpha)| \ll_m N.$$

This and Lemma 3 imply that

$$\sup_{n\in\mathbb{N}} \limsup_{N\to\infty} \frac{1}{N+1} \sum_{l=0}^{N} \sup_{s\in K_m} |\zeta_n(s+ilh,\alpha)| \le R_m < \infty.$$

From this, using relation (3), we obtain by a standard method, see, for example, [10, Lemma 6.2], that the family of probability measures $\{P_n : n \in \mathbb{N}\}$ is tight. Therefore, by the Prokhorov theorem, see [2, Theorem 6.1], this family is relatively compact. Hence, there exists a subsequence $\{P_{n_k}\} \subset \{P_n\}$ such that P_{n_k} converges weakly to a certain probability measure P on $(H(D), \mathcal{B}(H(D)))$ as $k \to \infty$. In other words, in hybrid notation,

$$X_{n_k}(s) \xrightarrow[k\to\infty]{\mathcal{D}} P. \tag{4}$$

Define one more $H(D)$-valued random element

$$X_N(s) = \zeta(s + i\theta_N, \alpha).$$

Then, in view of Lemma 3, we find that, for every $\varepsilon > 0$,

$$\begin{aligned}
&\lim_{n\to\infty} \limsup_{N\to\infty} \mathbb{P}(\varrho(X_N(s), X_{N,n}(s)) \ge \varepsilon) \\
&\quad = \lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{N+1} \#\{0 \le l \le N : \varrho(\zeta(s+ilh,\alpha), \zeta_n(s+ilh,\alpha)) \ge \varepsilon\} \\
&\quad \le \lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{(N+1)\varepsilon} \sum_{l=0}^{N} \varrho(\zeta(s+ilh,\alpha), \zeta_n(s+ilh,\alpha)) = 0.
\end{aligned}$$

This, (3), (4) and Theorem 4.2 of [2] show that

$$X_N(s) \xrightarrow[N\to\infty]{\mathcal{D}} P. \tag{5}$$

Thus, the measure P_N converges weakly to P as $N \to \infty$. Moreover, relation (5) shows that the measure P does not depend on the sequence $\{P_{n_k}\}$. Thus, the relative compactness of $\{P_n\}$ implies the relation

$$X_n \xrightarrow[n\to\infty]{\mathcal{D}} P. \tag{6}$$

It remains to consider the weak convergence of the measure $\widehat{P}_N$. Define the $H(D)$-valued random elements

$$\widehat{X}_{N,n}(s) = \zeta_n(s + i\theta_N, \alpha, \omega)$$

and

$$\widehat{X}_N(s) = \zeta(s + i\theta_N, \alpha, \omega).$$

Then, repeating the above arguments for these random elements and using (6) and Lemma 5, we find that $\widehat{P}_N$ also converges weakly to P as $N \to \infty$.

Now we state the main result of the section.

Theorem 6. *Suppose that the set $L(\alpha, q)$, for every $q \in \mathbb{Q}_1^+$, is linearly independent over $\mathbb{Q}$. Then, for every $h > 0$ such that the number $\exp\left\{\frac{2\pi}{h}\right\}$ is rational, the measure P_N converges to the distribution P_ζ of the random element $\zeta(s, \alpha, \omega)$ as $N \to \infty$.*

Proof. By Lemma 6, we have that there exists a probability measure P on $(H(D), \mathcal{B}(H(D)))$ such that P_N converges weakly to P as $N \to \infty$. Thus, it suffices to show that $P = P_\zeta$.

Let $a_{h,\alpha} = \{(m+\alpha)^{-ih} : m \in \mathbb{N}_0\}$, and for $\omega \in \Omega$, let $\varphi_{h,\alpha}(\omega) = a_{h,\alpha}\omega$. Then $\varphi_{h,\alpha}$ is a measurable measure preserving transformation on the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. A set $A \in \mathcal{B}(\Omega)$ is invariant with respect to $\varphi_{h,\alpha}$ if the sets A and $A_\varphi = \varphi_{h,\alpha}(A)$ can differ one from another only by a set of zero m_H-measure. The transformation $\varphi_{h,\alpha}$ is ergodic if its σ-field of invariant sets consists only of sets having m_H-measure equal to 1 or 0.

We will prove that the transformation $\varphi_{h,\alpha}$ is ergodic. Let χ be a non-trivial character of Ω. Then, as in the proof of Lemma 1, we have that

$$\chi(\omega) = \prod_{m=0}^{\infty} \omega^{k_m}(m), \qquad \omega \in \Omega,$$

where only a finite number if integers k_m are distinct from zero. Clearly, $a_{h,\alpha} \in \Omega$. Therefore,

$$\chi(a_{h,\alpha}) = \prod_{m=0}^{\infty} (m+\alpha)^{-ik_m h} = \exp\left\{-ih \sum_{m=0}^{\infty} k_m \log(m+\alpha)\right\}.$$

Thus, in view of (2), we have that

$$\chi(a_{h,\alpha}) \neq 1. \tag{7}$$

Let $A \in \mathcal{B}(\Omega)$ be an invariant set of the transformation $\varphi_{h,\alpha}$, and I_A be its indicator function. Then

$$I_A(a_{h,\alpha}\omega) = I_A(\omega)$$

for almost all $\omega \in \Omega$. Hence, for the Fourier transform $\widehat{I_A}(\chi)$ of I_A, we have that

$$\widehat{I_A}(\chi) = \int_\Omega \chi(\omega) I_A(\omega) m_H(d\omega) = \int_\Omega \chi(a_{h,\alpha}\omega) I_A(a_{h,\alpha}\omega) m_H(d\omega)$$
$$\chi(a_{h,\alpha}) \int_\Omega \chi(\omega) I_A(\omega) m_H(d\omega) = \chi(a_{h,\alpha}) \widehat{I_A}(\chi).$$

Therefore, by (7), for a non-trivial character χ,

$$\widehat{I_A}(\chi) = 0. \tag{8}$$

Now let χ_0 be the trivial character of Ω, and $\widehat{I_A}(\chi_0) = u$. Then, using the relations

$$\int_\Omega \chi(\omega) m_H(d\omega) = \begin{cases} 0 & \text{if } \chi \neq \chi_0, \\ 1 & \text{if } \chi = \chi_0, \end{cases}$$

and (8), we find that, for any character χ of Ω,

$$\widehat{I_A}(\chi) = u \int_\Omega \chi(\omega) m_H(d\omega) = u\widehat{1}(\chi) = \widehat{u}(\chi). \tag{9}$$

The function $I_A(\omega)$ is uniquely determined by the Fourier transform $\widehat{I_A}(\chi)$. Thus, by (9), we have that $I_A(\omega) = u$ for almost all $\omega \in \Omega$, i.e., $u = 0$ or $u = 1$. Therefore, $m_H(A) = 0$ or $m_H(A) = 1$. This means that the transformation $\varphi_{h,\alpha}$ is ergodic.

Let A be a fixed continuity set of the measure P in Lemma 6. Then, by Lemma 6 and Theorem 2.1 of [2],

$$\lim_{N\to\infty} \frac{1}{N+1} \#\{0 \le k \le N : (\zeta(s+ikh, \alpha, \omega)) \in A\} = P(A). \tag{10}$$

On $(\Omega, \mathcal{B}(\Omega), m_H)$, define a random variable θ by

$$\theta(\omega) = \begin{cases} 1 & \text{if } \zeta(s, \alpha, \omega) \in A, \\ 0 & \text{if } \zeta(s, \alpha, \omega) \notin A. \end{cases}$$

The expectation of θ is

$$\mathbb{E}(\theta) = \int_\Omega \theta dm_H = m_H(\omega \in \Omega : \zeta(s, \alpha, \omega) \in A) = P_\zeta(A). \tag{11}$$

Since the transformation $\varphi_{h,\alpha}$ is ergodic, by the classical Birkhoff–Khintchine theorem,

$$\lim_{N\to\infty} \frac{1}{N+1} \sum_{m=0}^{N} \theta(\varphi_{h,\alpha}^m(\omega)) = \mathbb{E}(\theta) \tag{12}$$

for almost all $\omega \in \Omega$. However, by the definitions of θ and $\varphi_{h,\alpha}$,

$$\frac{1}{N+1} \sum_{m=0}^{N} \theta(\varphi_{h,\alpha}^m(\omega)) = \frac{1}{N+1} \#\{0 \le k \le N : \zeta(s + ikh, \alpha, \omega) \in A\}.$$

Therefore, in view of (11) and (12),

$$\lim_{N\to\infty} \frac{1}{N+1} \#\{0 \le k \le N : \zeta(s + ikh, \alpha, \omega) \in A\} = P_\zeta(A)$$

for almost all ω. This and (10) show that $P(A) = P_\zeta(A)$ for any continuity set A of the measure P. Hence, $P(A) = P_\zeta(A)$ for all $A \in \mathcal{B}(H(D))$. The theorem is proved.

Theorem 6 implies a limit theorem for the composite function $F(\zeta(s, \alpha))$.

Theorem 7. *Suppose that the set $L(\alpha, q)$, for every $q \in \mathbb{Q}_1^+$, is linearly independent over $\mathbb{Q}$ and that $F : H(D) \to H(D)$ is a continuous operator. Then, for every $h > 0$ such that the number $\exp\left\{\frac{2\pi}{h}\right\}$ is rational, the measure $P_{N,F}$ converges weakly to the measure $P_\zeta F^{-1}$ as $N \to \infty$.*

Proof. We remind that the measure $P_\zeta F^{-1}$ is defined by the formula

$$P_\zeta F^{-1}(A) = P_\zeta(F^{-1}A), \quad A \in \mathcal{B}(H(D)).$$

Clearly, we have that $P_{N,F} = P_N F^{-1}$. This, the continuity of F, Theorems 6 and 5.1 of [2] show that the measure $P_{N,F}$ converges weakly to $P_\zeta F^{-1}$ as $N \to \infty$.

3. Supports

A probabilistic approach of the proof of universality for zeta-functions also includes explicit forms of the supports of the limit measures in limit theorems in the space of analytic functions. In this section, we discuss the supports of the measures P_ζ and $P_\zeta F^{-1}$.

Let P be a probability measure on $(H(D), \mathcal{B}(H(D)))$. We recall that the support of P is the minimal closed set $S \subset H(D)$ such that $P(S) = 1$. The set S consists of all elements $g \in H(D)$ such that, for every open neighbourhood G of g, the inequality $P(G) > 0$ is satisfied.

Theorem 8. *Under the hypotheses of Theorem 6, the support of the measure P_ζ is the whole of $H(D)$.*

Proof. The measure P_ζ does not depend on h. Moreover, since the set $L(\alpha)$ is linearly independent over $\mathbb{Q}$, in [8] it was observed that the support of P_ζ is the whole of $H(D)$.

Theorem 9. *Let α, F and h satisfy the hypotheses of Theorem 5. Then the support of the measure $P_\zeta F^{-1}$ is the whole of $H(D)$.*

Proof. Let g be an arbitrary element of $H(D)$, and G be any open neighbourhood of g. Since F is continuous, the set $F^{-1}G$ is open, too. By the Mergelyan theorem on approximation of analytic functions by polynomials [11], see also [16], and the definition of the metric ϱ, there exists a polynomial $p = p(s) \in G$. Since, by a hypothesis of the theorem, the preimage $F^{-1}\{p\}$ is non-empty, we have that $F^{-1}G$ is an open neighbourhood of some element of the space $H(D)$. Thus, in view of Theorem 8, $P_\zeta(F^{-1}G) > 0$. Therefore,

$$P_\zeta F^{-1}(G) = P_\zeta(F^{-1}G) > 0$$

for every open neighbourhood G of arbitrary element $g \in H(D)$. Thus, the support of $P_\zeta F^{-1}$ is the whole of $H(D)$.

4. Proof of universality theorems

Theorems 4 and 5 are consequences of Theorem 6, 8 and 7, 9, respectively.

Proof of Theorem 4. By the mentioned above Mergelyan theorem, there exists a polynomial $p(s)$ such that

$$\sup_{s\in K} |f(s) - p(s)| < \frac{\varepsilon}{2}. \tag{13}$$

Define

$$G = \left\{ g \in H(D) : \sup_{s\in K} |g(s) - p(s)| < \frac{\varepsilon}{2} \right\}.$$

Then G is an open neighbourhood of the polynomial $p(s)$ which, by Theorem 8, is an element of the support of the measure P_ζ. Thus, $P_\zeta(G) > 0$. Therefore, using an equivalent of the weak convergence of probability measures in terms of open sets, [2, Theorem 2.1], we have by Theorem 6 that

$$\liminf_{N\to\infty} \frac{1}{N+1} \#\{0 \le k \le N : \zeta(s + ikh, \alpha) \in G\} \ge P_\zeta(G) > 0.$$

Thus, by the definition of G,

$$\liminf_{N\to\infty} \frac{1}{N+1} \# \left\{ 0 \le k \le N : \sup_{s\in K} |\zeta(s + ikh, \alpha) - p(s)| < \frac{\varepsilon}{2} \right\} > 0.$$

This together with inequality (13) implies the inequality

$$\begin{aligned} &\liminf_{N\to\infty} \frac{1}{N+1} \#\{0 \le k \le N : \sup_{s\in K} |\zeta(s + ikh, \alpha) - f(s)| < \varepsilon\} \\ &\quad \ge \liminf_{N\to\infty} \frac{1}{N+1} \# \left\{ 0 \le k \le N : \sup_{s\in K} |\zeta(s + ikh, \alpha) - p(s)| < \frac{\varepsilon}{2} \right\} > 0. \end{aligned}$$

The theorem is proved.

Proof of Theorem 5. We use Theorems 7 and 9 in place of Theorems 6 and 8, and repeat the proof of Theorem 4.

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