A NOTE ON THE SOLUTION OF THE MEXICAN HAT PROBLEM

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ABSTRACT. We prove a technical estimate needed in our recent solution of the completeness question for the non-orthogonal Mexican hat wavelet system, in L^p for $1 < p < 2$ and in the Hardy space H^p for $2/3 < p \leq 1$.

1. Introduction

In our recent paper [1, §8] solving the Mexican hat wavelet completeness problem, we needed that $\Delta_*(\Phi, \Psi) < 1$, where we make the following definitions. Let

$$\Psi(\xi) = (2\pi\xi)^2 \exp(-2\pi^2\xi^2) \qquad \text{and} \qquad \Phi = \kappa/\Psi,$$

with κ being the "double bump" function

$$\kappa(\xi) = \begin{cases} 0, & \xi \in [0, 1/12], \\ \sin^2\big((12\xi - 1)\pi/2\big), & \xi \in [1/12, 1/6], \\ \cos^2\big((6\xi - 1)\pi/2\big), & \xi \in [1/6, 1/3], \\ 0, & \xi \in [1/3, \infty), \\ \kappa(-\xi), & \xi \in (-\infty, 0). \end{cases}$$

(Note Ψ is the Fourier transform of the Mexican hat function $\psi(x) = (1-x^2)e^{-x^2/2}$.) Put

$$\Theta(\xi) = \xi\Phi'(\xi) \qquad \text{and} \qquad \Gamma(\xi) = \xi\Phi(\xi).$$

Define

$$\Delta(\Phi, \Psi) = \sum_{l \neq 0} \Big\| \sum_{j \in \mathbb{Z}} |\Phi(\xi 2^{-j})\Psi(\xi 2^{-j} - l)| \Big\|_{L^\infty(\mathbb{R})}^{1/2} \Big\| \sum_{j \in \mathbb{Z}} |\Phi(\xi 2^{-j} + l)\Psi(\xi 2^{-j})| \Big\|_{L^\infty(\mathbb{R})}^{1/2},$$

and let

$$\Delta_*(\Phi, \Psi) = \Delta(\Phi, \Psi) + 2\Delta(\Theta, \Psi) + 2\Delta(\Gamma, \Psi').$$

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2. **Proof that** $\Delta_*(\Phi, \Psi) < 1$

We will prove $\Delta_*(\Phi, \Psi) < 0.52$. If rigor is not required then the better numerical estimate $\Delta_*(\Phi, \Psi) < 0.03$ can be used. The purpose of this note is simply to demonstrate that a rigorous estimate can be obtained.

First we simplify the expression for Δ.

Lemma 1. *Assume A and B are measurable functions on $\mathbb{R}$. Suppose A is supported in $[-1/3, -1/12] \cup [1/12, 1/3]$, that $|A|$ and $|B|$ are even functions, and that $|B(\xi)|$ is decreasing for $\xi \geq 2/3$. Then*

$$\Delta(A, B) \leq 2\sqrt{2}\|A(\xi)B(1-\xi)\|_{L^\infty[1/12,1/3]} + 2\sqrt{2}\|A\|_{L^\infty[1/12,1/3]} \sum_{l=2}^{\infty} |B(l-1/3)|.$$

Proof. We start by noting

$$|B(l + \xi)| \leq |B(l - \xi)| \qquad \text{whenever } \xi \in [1/12, 1/3], \quad l \in \mathbb{N}, \tag{1}$$

because $l + \xi > l - \xi \geq 1 - 1/3 = 2/3$ and $|B|$ is decreasing on $[2/3, \infty)$.

Now consider $l \neq 0$. The support hypothesis on A implies that

$$\begin{aligned}
&\Big\| \sum_{j\in\mathbb{Z}} |A(\xi 2^{-j})B(\xi 2^{-j} - l)| \Big\|_{L^\infty(\mathbb{R})} \\
&= \big\| |A(\xi)B(\xi - l)| + |A(\xi/2)B(\xi/2 - l)| \big\|_{L^\infty([-1/3,-1/6]\cup[1/6,1/3])} \\
&\leq 2\|A(\xi)B(\xi - l)\|_{L^\infty([-1/3,-1/12]\cup[1/12,1/3])} \\
&\leq 2 \max_{\pm} \|A(\xi)B(|l| \pm \xi)\|_{L^\infty[1/12,1/3]} \qquad \text{by evenness of } |A| \text{ and } |B| \\
&= 2\|A(\xi)B(|l| - \xi)\|_{L^\infty[1/12,1/3]}
\end{aligned} \tag{2}$$

by (1).

Next we claim the sets $\{(\mathrm{supp}(A) - l)2^j\}_{j\in\mathbb{Z}}$ are disjoint. When $l < 0$,

$$\mathrm{supp}(A) - l \subset \Big[|l| - \frac{1}{3}, |l| + \frac{1}{3}\Big],$$

and the left endpoint of this last interval dilates under multiplication by 2 to the right of the right endpoint, because $2(|l| - 1/3) \geq |l| + 1/3$; argue similarly for disjointness when $l > 0$.

The disjointness ensures that

$$\begin{aligned}
\Big\| \sum_{j\in\mathbb{Z}} |A(\xi 2^{-j} + l)B(\xi 2^{-j})| \Big\|_{L^\infty(\mathbb{R})} &= \|A(\xi + l)B(\xi)\|_{L^\infty(\mathrm{supp}(A)-l)} \\
&= \|A(\xi)B(\xi - l)\|_{L^\infty(\mathrm{supp}(A))} \\
&= \|A(\xi)B(|l| - \xi)\|_{L^\infty[1/12,1/3]}
\end{aligned} \tag{3}$$

by evenness of $|A|$ and $|B|$ and estimate (1).

By putting the estimates (2) and (3) into the definition of $\Delta(A,B)$, we conclude that

$$\Delta(A,B) \le 2\sqrt{2}\sum_{l=1}^{\infty}\|A(\xi)B(l-\xi)\|_{L^\infty[1/12,1/3]}.$$

The lemma now follows by splitting off the term with $l=1$ and using that $|B|$ is decreasing on $[2/3,\infty)$. □

Next we state some calculus facts about the function $\Psi(\xi) = (2\pi\xi)^2\exp(-2\pi^2\xi^2)$.

Lemma 2. $|\Psi|$ *and* $|\Psi'|$ *are decreasing for* $\xi \in [2/3,\infty)$. *(Hence* Ψ *and* Ψ' *satisfy the hypotheses on "*B*" in Lemma 1.)*

Lemma 3. *Let* $m,n \in \{0,1,2,3\}$. *Then* $\xi^{-m}(1-\xi)^n e^{4\pi^2\xi}$ *is increasing for* $\xi \in [1/12,1/3]$.

Now we estimate the three terms in $\Delta_*(\Phi,\Psi)$.

Estimation of $\Delta(\Phi,\Psi)$. We have $|\kappa| \le 1$ and

$$\Phi(\xi) = \frac{\kappa(\xi)}{\Psi(\xi)} = \kappa(\xi)(2\pi\xi)^{-2}e^{2\pi^2\xi^2},$$

$$\Psi(1-\xi) = (2\pi)^2e^{-2\pi^2}(1-\xi)^2e^{4\pi^2\xi}e^{-2\pi^2\xi^2}, \tag{4}$$

so that (by using Lemma 3 and evaluating at $\xi = 1/3$)

$$|\Phi(\xi)\Psi(1-\xi)| < 0.006, \qquad \xi \in [1/12,1/3]. \tag{5}$$

Further, for $l \ge 2$ we have

$$|\Psi(l-1/3)| < (2\pi)^2l^2e^{-2\pi^2(l/2)^2} \le (2\pi)^2 2^2 e^{l-2}e^{-\pi^2 l},$$

so that by a geometric series,

$$\sum_{l=2}^{\infty}|\Psi(l-1/3)| < (2\pi)^2 4e^{-2\pi^2}/(1-e^{1-\pi^2}). \tag{6}$$

Combining (6) with the fact that

$$|\Phi(\xi)| < 200(2\pi)^{-2}, \qquad \xi \in [1/12,1/3],$$

gives that

$$\|\Phi\|_{L^\infty[1/12,1/3]}\sum_{l=2}^{\infty}|\Psi(l-1/3)| < 0.000003.$$

Substituting this last estimate and (5) into Lemma 1 shows that

$$\Delta(\Phi,\Psi) < 0.02. \tag{7}$$

Estimation of $\Delta(\Theta, \Psi)$. By definition of $\Phi = \kappa/\Psi$, we have

$$|\Theta(\xi)| = |\xi\Phi'(\xi)|$$

$$\leq (2\pi)^{-2} e^{2\pi^2\xi^2} \begin{cases} 6\pi\xi^{-1} + 2\xi^{-2} & \text{when } \xi \in [1/12, 1/6] \\ 3\pi\xi^{-1} + \big(4\pi^2(1/3)^2 - 2\big)\xi^{-2} & \text{when } \xi \in [1/6, 1/3] \end{cases} \tag{8}$$

$$< (2\pi)^{-2} \cdot 600.$$

Multiplying this last estimate by (6) shows

$$\|\Theta\|_{L^\infty[1/12,1/3]} \sum_{l=2}^{\infty} |\Psi(l - 1/3)| < 0.000007. \tag{9}$$

Using (4), (8) and Lemma 3 gives that

$$|\Theta(\xi)\Psi(1-\xi)| < 0.031, \qquad \xi \in [1/12, 1/3]. \tag{10}$$

Substituting (9) and (10) into Lemma 1 shows that

$$\Delta(\Theta, \Psi) < 0.09. \tag{11}$$

Estimation of $\Delta(\Gamma, \Psi')$. Recall the definition

$$\Gamma(\xi) = \xi\Phi(\xi) = \kappa(\xi)(2\pi)^{-2}\xi^{-1}e^{2\pi^2\xi^2}.$$

From

$$\Psi'(\xi) = 2(2\pi)^2(\xi - 2\pi^2\xi^3)e^{-2\pi^2\xi^2}$$

we find for $\xi < 1$ that

$$|\Psi'(1-\xi)| \leq 2(2\pi)^2 e^{-2\pi^2}\big((1-\xi) + 2\pi^2(1-\xi)^3\big)e^{4\pi^2\xi}e^{-2\pi^2\xi^2}.$$

Hence (by Lemma 3 and evaluating at $\xi = 1/3$)

$$|\Gamma(\xi)\Psi'(1-\xi)| < 0.055, \qquad \xi \in [1/12, 1/3]. \tag{12}$$

Next,

$$|\Psi'(\xi)| \leq (2\pi)^4\xi^3 e^{-2\pi^2\xi^2}, \qquad \xi \geq 1.$$

Hence for $l \geq 2$,

$$|\Psi'(l - 1/3)| \leq (2\pi)^4 l^3 e^{-2\pi^2(l/2)^2} \leq (2\pi)^4 3^3 e^{l-3} e^{-\pi^2 l},$$

so that by a geometric series,

$$\sum_{l=2}^{\infty} |\Psi'(l - 1/3)| \leq 27(2\pi)^4 e^{-1-2\pi^2}/(1 - e^{1-\pi^2}).$$

Combining this last estimate with the fact that

$$|\Gamma(\xi)| < 30(2\pi)^{-2}, \qquad \xi \in [1/12, 1/3],$$

gives that

$$\|\Gamma\|_{L^\infty[1/12,1/3]} \sum_{l=2}^{\infty} |\Psi'(l - 1/3)| < 0.00004. \tag{13}$$

Substituting (12) and (13) into Lemma 1 shows that

$$\Delta(\Gamma, \Psi') < 0.16. \tag{14}$$

Estimation of $\Delta_*(\Phi, \Psi)(\Phi, \Psi)$. We obtain that

$$\Delta_*(\Phi, \Psi) = \Delta(\Phi, \Psi) + 2\Delta(\Theta, \Psi) + 2\Delta(\Gamma, \Psi') < 0.52,$$

by summing estimates (7), (11) and (14). The proof is complete.

REFERENCES

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