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Positive integral operators in unbounded domains

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Abstract

We study positive integral operators $\mathcal{K}$ in $L^2(\mathbb{R})$ with continuous kernel $k(x, y)$. We show that if $k(x,x) \in L^1(\mathbb{R})$ the operator is compact and Hilbert–Schmidt. If in addition $k(x,x) \to 0$ as $|x| \to \infty$, k is represented by an absolutely and uniformly convergent bilinear series of uniformly continuous eigenfunctions and $\mathcal{K}$ is trace class. Replacing the first assumption by the stronger $k^{1/2}(x,x) \in L^1(\mathbb{R})$ then $k \in L^1(\mathbb{R}^2)$ and the bilinear series converges also in L^1. Sharp norm bounds are obtained and Mercer's theorem is derived as a special case.

1. Introduction

Given an interval $I \subset \mathbb{R}$, a linear operator $\mathcal{K}: L^2(I) \to L^2(I)$ is said to be integral if there exists an almost everywhere measurable function $k(x, y)$ on $I \times I$ such that for all $\phi \in L^2(I)$,

$$\phi \mapsto \mathcal{K}(\phi) = \int_I k(x, y)\phi(y)\,dy,$$

almost everywhere. The function $k(x, y)$ is called the kernel of $\mathcal{K}$. If in addition $\mathcal{K}$ satisfies the condition

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$$\int_I \int_I k(x,y)\phi(y)\overline{\phi(x)}\,dx\,dy \geqslant 0$$

for all $\phi \in L^2(I)$, then it is a positive operator. We shall be concerned in this paper mainly with positive operators on unbounded domains, treating without loss of generality the case $I = \mathbb{R}$. We note, however, that (a version of) our results contain the compact interval as a special case; see Section 3.

In general, integral operators in unbounded domains are not compact. However, if an operator is compact, the spectrum of $\mathcal{K}$ consists in a sequence $\{\mu_n\}_{n\geqslant 0}$ of eigenvalues with finite multiplicity accumulating only at zero. Furthermore, if $\mathcal{K}$ is positive the eigenvalues are positive and standard Hilbert space methods show that the kernel $k(x,y)$ satisfies the bilinear series expansion

$$k(x,y) = \sum_{n\geqslant 0} \mu_n \phi_n(x)\overline{\phi_n(y)}, \tag{1}$$

where the $\{\phi_n\}_{n\geqslant 0}$ are an L^2-orthonormal set of eigenfunctions spanning the range of $\mathcal{K}$ and equality is to be understood in the sense of convergence in L^2.

A classical result of Mercer (see, e.g., [4]) strengthens the convergence properties of the bilinear expansion (1) when the interval I is compact and the kernel k is continuous. In this case, the eigenfunctions ϕ_n are continuous and the corresponding bilinear series (1) is absolutely and uniformly convergent, implying in particular that the corresponding operator is trace class.

As described above, the corresponding situation is more delicate for unbounded domains. Novitskii's results [3] may be seen as an extension of Mercer's theorem to a more general class of operators: compact normal integral operators in $L^2(\mathbb{R})$ whose spectra lie in a sector of angle less than π with vertex at 0 (which include as a special case compact positive integral operators). Novitskii finds sufficient conditions on the kernels $k(x,y)$ of such operators so that they may be represented by a bilinear series analogous to (1), where the eigenfunctions are continuous and convergence of the series is absolute and uniform. Those conditions, besides the obviously necessary ones of compactness of $\mathcal{K}$ (which must be imposed a priori) and continuity of $k(x,y)$, are

(a) $\lim_{x\to x_0} \int_{-\infty}^{+\infty} |k(x_0,y) - k(x,y)|^2\,dy = 0$ for all $x_0 \in \mathbb{R}$;
(b) $k(x,x)$ is uniformly continuous and $k(x,x) \in L^1(\mathbb{R})$.

The purpose of this paper is to show that for positive integral operators in $L^2(\mathbb{R})$ some of these hypotheses are redundant. If the kernel $k(x,y)$ is continuous, the condition $k(x,x) \in L^1(\mathbb{R})$ is sufficient to ensure that the corresponding integral operator is Hilbert–Schmidt, and therefore automatically compact, with operator norm $\|\mathcal{K}\| = \|k\|_{L^2} \leqslant \int_{-\infty}^{+\infty} k(x,x)\,dx$. Under the further assumption that $k(x,x) \to 0$ as $|x| \to \infty$, the eigenfunctions ϕ_n of $\mathcal{K}$ are uniformly continuous, the bilinear series 1 is absolutely and uniformly convergent and the operator $\mathcal{K}$ is trace class with $\operatorname{tr} \mathcal{K} = \int_{-\infty}^{+\infty} k(x,x)\,dx = \sum_{n\geqslant 0} \mu_n$. Replacing the first hypothesis by the stronger assumption that $k^{1/2}(x,x) \in L^1(\mathbb{R})$, it follows that $k(x,y)$ is in $L^1(\mathbb{R}^2)$, the eigenfunctions are in $L^1(\mathbb{R})$ and the bilinear series for k converges in the L^1

norm. These results are sharp. It is also shown that the hypotheses underlying these results may be interpreted as conditions on the rate of decay of the kernel along the diagonal.

2. Positive integral operators in $\mathbb{R}$

We summarize some basic properties of continuous kernels of positive operators in the lemma below. These follow easily from well-known facts about 'positive definite matrices' in the sense of Moore [1] together with continuity of the kernel; the proof is omitted.

Lemma 1. *Let $k(x, y)$ be continuous and the kernel of an $L^2(\mathbb{R})$ positive operator. Then*:

(i) *$k(x, x)$ is real and $\geqslant 0$ for all $x \in \mathbb{R}$;*
(ii) *for all $x, y \in \mathbb{R}$, $k(x, y) = \overline{k(y, x)}$;*
(iii) *for all $x, y \in \mathbb{R}$, $|k(x, y)|^2 \leqslant k(x, x)k(y, y)$.*

If k is not continuous, (i) and (iii) do not hold in general, while (ii) only holds a.e. However, for our purposes we shall only be interested in the case where k is continuous.

Lemma 2. *Let $\mathcal{K} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be a positive integral operator with continuous kernel $k(x, y)$ such that $k(x, x) \in L^1(\mathbb{R})$. Then $k \in L^2(\mathbb{R}^2)$, and thus $\mathcal{K}$ is a Hilbert–Schmidt operator. Furthermore,*

$$\|\mathcal{K}\| = \|k\|_{L^2} \leqslant \int_{-\infty}^{+\infty} k(x, x)\, dx.$$

Proof. Since by Lemma 1 we have $|k(x, y)|^2 \leqslant k(x, x)k(y, y)$, it follows that

$$\iint_{\mathbb{R}^2} \bigl|k(x, y)\bigr|^2\, dx\, dy \leqslant \iint_{\mathbb{R}^2} k(x, x)k(y, y)\, dx\, dy = \left(\int_{-\infty}^{+\infty} k(x, x)\, dx\right)^2. \tag{2}$$

Thus $k \in L^2(\mathbb{R}^2)$ and by Lemma 1 $k(x, y) = \overline{k(y, x)}$. Hence $\mathcal{K}$ is Hilbert–Schmidt and, by 2,

$$\|\mathcal{K}\| = \|k\|_{L^2} \leqslant \int_{-\infty}^{+\infty} k(x, x)\, dx. \qquad \Box$$

Observe that in the proof above $\int_{-\infty}^{+\infty} k(x, x)\, dx$ is the $L^1(\mathbb{R})$ norm of $k(x, x)$ since $k(x, x) \geqslant 0$.

From Lemma 2 it follows that $\mathcal{K}$, being Hilbert–Schmidt, is automatically compact. Since it is also positive (and self-adjoint, although this is already implied by positivity), the spectrum of $\mathcal{K}$ is as described above and a bilinear expansion (1) holds in L^2, where for convenience we assume as usual that the eigenvalues satisfy $\mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_n \cdots \geqslant 0$ and are repeated according to multiplicity.

In fact, with an extra uniformity assumption much stronger statements are valid for $\mathcal{K}$, as seen in Theorem 1 below. We denote by C_0 the space of continuous functions vanishing at infinity equipped with the sup norm.

Before stating our main results we isolate a simple lemma.

Lemma 3. *Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous and in $L^1(\mathbb{R})$. Then f is uniformly continuous if and only if $f(x) \to 0$ as $|x| \to \infty$.*

Proof. The 'if' part is trivial: a continuous function vanishing at infinity is uniformly continuous. Conversely, if there exists $\delta > 0$ and a sequence $x_n \to \pm\infty$ with $x_n > \delta$, then by uniform continuity the set $\{x \in \mathbb{R}\colon |f(x)| > \delta/2\}$ has infinite measure, contradicting $f \in L^1(\mathbb{R})$. □

Theorem 1. *Let $\mathcal{K} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be a positive integral operator with continuous kernel $k(x, y)$ satisfying $k(x, x) \in L^1(\mathbb{R})$ and $k(x, x) \to 0$ as $|x| \to \infty$. Let $M = \max_{x\in\mathbb{R}} k(x, x)$. Then*:

(i) *For all $\phi \in L^2(\mathbb{R})$, $\mathcal{K}\phi(x)$ is continuous and*

$$\|\mathcal{K}\phi\|_{C_0} \leqslant 4M \int_{-\infty}^{+\infty} k(x, x)\, dx\, \|\phi\|_{L^2}.$$

(ii) *Eigenfunctions ϕ_n associated to nonzero eigenvalues μ_n of $\mathcal{K}$ are uniformly continuous and satisfy*

$$\|\phi_n\|_{C_0} \leqslant \frac{4M}{\mu_n} \int_{-\infty}^{+\infty} k(x, x)\, dx\, \|\phi_n\|_{L^2}.$$

(iii) *$k(x, y)$ is represented by a bilinear series* (1), *where the ϕ_n are uniformly continuous, L^2-orthonormal eigenfunctions of $\mathcal{K}$, μ_n are the associated eigenvalues and the series is absolutely and uniformly convergent.*

(iv) *The operator $\mathcal{K} : L^2(\mathbb{R}) \to C_0$ is continuous with operator norm bounded by $4M \int_{-\infty}^{+\infty} k(x, x)\, dx$.*

(v) *The operator $\mathcal{K} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is trace class with*

$$\operatorname{tr} \mathcal{K} = \int_{-\infty}^{+\infty} k(x, x)\, dx = \sum_{n\geqslant 0} \mu_n.$$

Proof. First of all we note that $\lim_{x\to x_0} \int_{-\infty}^{+\infty} |k(x_0, y) - k(x, y)|^2\, dy = 0$ for all $x_0 \in \mathbb{R}$ implies that, for all $\phi \in L^2(\mathbb{R})$, $\mathcal{K}(\phi)(x)$ is a continuous function. In fact,

$$\begin{aligned}\left|\mathcal{K}(\phi)(x_0)-\mathcal{K}(\phi)(x)\right| &= \left|\int_{-\infty}^{+\infty} k(x_0,y)\phi(y)\,dy-\int_{-\infty}^{+\infty} k(x,y)\phi(y)\,dy\right| \\ &\leqslant \int_{-\infty}^{+\infty} \left|\bigl(k(x_0,y)-k(x,y)\bigr)\right|\left|\phi(y)\right|dy \\ &\leqslant \left[\int_{-\infty}^{+\infty} \left|k(x_0,y)-k(x,y)\right|^2 dy\right]^{1/2} \|\phi\|_{L^2}, \qquad (3)\end{aligned}$$

from which the statement follows by taking limits.

We now show that continuity of k together with the hypotheses $k(x,x)\in L^1(\mathbb{R})$ and $k(x,x)\to 0$ as $|x|\to\infty$ imply

$$\lim_{x\to x_0}\int_{-\infty}^{+\infty} \left|k(x_0,y)-k(x,y)\right|^2 dy = 0 \quad \text{for all } x_0\in\mathbb{R}.$$

Given $x_0\in\mathbb{R}$ we estimate the integral $\int_{-\infty}^{+\infty}|k(x_0,y)-k(x,y)|^2\,dy$ in the following way. By Lemma 1 we have that for all $x,y\in\mathbb{R}$, $|k(x,y)|^2\leqslant k(x,x)\,k(y,y)$. This implies that, for all $x\in\mathbb{R}$,

$$\int_{-\infty}^{+\infty}\left|k(x,y)\right|^2 dy \leqslant k(x,x)\int_{-\infty}^{+\infty} k(y,y)\,dy;$$

observe in particular that $k(x,y)\in L^2(\mathbb{R})$ along any vertical (or indeed, by symmetry with respect to the diagonal, horizontal) line. Set $eM=\max_{x\in\mathbb{R}} k(x,x)$ (which exists since $k(x,x)$ is continuous and vanishes at infinity) and denote by $I_L = \left]-\infty,-L\right]\cup\left[L,+\infty\right[$. Since $k(y,y)$ is continuous, positive and converges to 0 as $|y|\to\infty$, it is possible, given $\epsilon>0$, to choose L sufficiently large that

$$\int_{I_L} k(y,y)\,dy < \frac{\epsilon}{5M}.$$

We now separate the integral into two parts,

$$\begin{aligned}\int_{-\infty}^{+\infty}\left|k(x_0,y)-k(x,y)\right|^2 dy &= \int_{-L}^{+L}\left|k(x_0,y)-k(x,y)\right|^2 dy \\ &\quad + \int_{I_L}\left|k(x_0,y)-k(x,y)\right|^2 dy,\end{aligned}$$

bounding each one separately.

For the first integral, by uniform continuity of k in compact sets there exists an x-neighborhood $V_\delta(x_0) = \,]x_0 - \delta, x_0 + \delta[$ such that $|k(x_0, y) - k(x, y)|^2 < \epsilon/10L$ for all $y \in [-L, L]$ and all $x \in V_\delta(x_0)$, so that

$$\int_{-L}^{L} \left|k(x_0, y) - k(x, y)\right|^2 dy < \epsilon/5 \quad \text{for all } x \in V_\delta(x_0). \tag{4}$$

For the second integral, we note that for all $x_0, x, y \in \mathbb{R}$,

$$\begin{aligned}\left|k(x_0, y) - k(x, y)\right|^2 &\leqslant \left|k(x_0, y)\right|^2 + \left|k(x, y)\right|^2 + 2\left|k(x_0, y)\right|\left|k(x, y)\right| \\ &\leqslant k(x_0, x_0)k(y, y) + k(x, x)k(y, y) \\ &\quad + 2k(x_0, x_0)^{1/2}k(x, x)^{1/2}k(y, y) \\ &\leqslant 4Mk(y, y).\end{aligned} \tag{5}$$

Integrating in I_L, we thus find that for all $x_0, x \in \mathbb{R}$,

$$\int_{I_L} \left|k(x_0, y) - k(x, y)\right|^2 dy \leqslant 4M \int_{I_L} k(y, y)\, dy < \frac{4\epsilon}{5} \tag{6}$$

by our choice of L. Adding the estimates given by (4) and (6) we find that for all $x_0 \in \mathbb{R}$ and all $\epsilon > 0$ there is $\delta > 0$ such that

$$\int_{-\infty}^{+\infty} \left|k(x_0, y) - k(x, y)\right|^2 dy < \epsilon$$

whenever $x \in V_\delta(x_0)$. This proves continuity at arbitrary x_0 of the integral $\int_{-\infty}^{+\infty} |k(x_0, y) - k(x, y)|^2 dy$. By the first paragraph this implies continuity of $\mathcal{K}(\phi)(x)$ for all $\phi \in L^2(\mathbb{R})$.

We now note that by (5) we have

$$\int_{-\infty}^{+\infty} \left|k(x_0, y) - k(x, y)\right|^2 dy \leqslant 4M \int_{-\infty}^{+\infty} k(y, y)\, dy \quad \forall x, x_0 \in \mathbb{R}$$

so that the integral is uniformly bounded. Taking suprema in (3) finishes the proof of (i), proving at the same time (iv).

To prove (ii), observe that if ϕ_n is an eigenfunction of $\mathcal{K}$ associated with a nonzero eigenvalue μ_n, then $\mathcal{K}(\phi_n) = \mu_n\phi_n$; by part (i) ϕ_n is continuous with uniform norm satisfying

$$\|\phi_n\|_{C_0} \leqslant \frac{4M}{\mu_n} \int_{-\infty}^{+\infty} k(x, x)\, dx\, \|\phi_n\|_{L^2}.$$

Uniform continuity of ϕ_n will be proved below.

To prove (iii), note that the hypotheses on k imply that $k(x,x)$ is uniformly continuous and $k(x,x) \in L^1(\mathbb{R})$. Since we proved above that they also imply

$$\lim_{x\to 0} \int_{-\infty}^{+\infty} \left|k(x_0,y) - k(x,y)\right|^2 dy = 0,$$

we may apply Novitskii's theorem [3] and conclude that the bilinear series (1) converges absolutely and uniformly.

Uniform continuity of ϕ_n now follows from uniform convergence of this expansion and the fact that $k(x,x) \geqslant \sum_{n=1}^{N} \mu_n |\phi_n(x)|^2$. Since $k(x,x) \to 0$ as $|x| \to \infty$ it follows that for every n, $|\phi_n(x)| \to 0$ as $|x| \to \infty$. Since ϕ_n is continuous, ϕ_n is uniformly continuous.

We now prove statement (v). Since the convergence of the series in (1) is uniform, termwise integration is permissible, yielding

$$\int_{-\infty}^{+\infty} k(x,x)\,dx = \sum_{n\geqslant 0} \mu_n. \tag{7}$$

On the other hand, from the eigenfunctions ϕ_n in (1) we may construct a complete orthonormal basis of $L^2(\mathbb{R})$ by adjoining orthonormal eigenfunctions associated with 0 (in fact, for this purpose we may without loss of generality suppose that the $\{\phi_n\}_{n\geqslant 0}$ already include these since they do not contribute to the sum). Computing the trace of $\mathcal{K}$ in this basis yields

$$\operatorname{tr}\mathcal{K} = \sum_{n\geqslant 0} \langle \mathcal{K}\phi_n, \phi_n\rangle = \sum_{n\geqslant 0} \mu_n \langle \phi_n, \phi_n\rangle = \sum_{n\geqslant 0} \mu_n$$

which is convergent by (7). Thus $\mathcal{K}$ is trace class with $\operatorname{tr}\mathcal{K} = \int_{-\infty}^{+\infty} k(x,x)\,dx = \sum_{n\geqslant 0} \mu_n$. □

We can formulate a slightly weaker version of Theorem 1 suitable for applications and which shows that Theorem 1 can be interpreted in terms of the rate of decay of the kernel along the diagonal.

Corollary 1. *Let $\mathcal{K}: L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be a positive integral operator with continuous kernel $k(x,y)$ satisfying $k(x,x) = O(1/x^{1+\epsilon})$ for some $\epsilon > 0$. Then all the statements of Theorem* 1 *hold.*

Proof. $k(x,x) = O(1/x^{1+\epsilon})$ implies the hypotheses of Theorem 1. □

Remark 1. The fact that the condition $k(x,x) \to 0$ as $|x| \to \infty$, or equivalently uniform continuity of $k(x,x)$, is necessary for uniform convergence of the bilinear series in Theorem 1 may be seen by the following counterexample. Choose $\mu_n = 1/n^2$. It is not hard to construct a family of continuous functions $\{\phi_n(x)\}$ such that $\|\phi_n\|_{L^2} = 1$ for all $n \in \mathbb{N}$, the support of ϕ_n is contained in $I_n = [n, n+1]$, $\max_{I_n} \phi_n > n^2$ and $\sum_{n\geqslant 0} \mu_n |\phi_n(x)|^2 \in L^1(\mathbb{R})$. Then the associated operator satisfies all other conditions in

Theorem 1 but the series (1) does not converge uniformly since each finite sum is bounded but the infinite sum is not.

In the following theorem we use the positive function $k^{1/2}(x,x)$, which is well defined and continuous since $k(x,x) \geqslant 0$ and is continuous for all $x \in \mathbb{R}$.

Theorem 2. *Let $\mathcal{K}: L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be a positive integral operator with continuous kernel $k(x,y)$ satisfying $k^{1/2}(x,x) \in L^1(\mathbb{R})$ and $k(x,x) \to 0$ as $|x| \to \infty$. Then, in addition to the statements of Theorem* 1:

(i) *For all $\phi \in L^2(\mathbb{R})$, $\mathcal{K}\phi(x) \in L^1(\mathbb{R})$ and*

$$\|\mathcal{K}\phi\|_{L^1} \leqslant \int_{-\infty}^{+\infty} k^{1/2}(x,x)\,dx \left[\int_{-\infty}^{+\infty} k(x,x)\,dx\right]^{1/2} \|\phi\|_{L^2}.$$

(ii) *Eigenfunctions ϕ_n associated to nonzero eigenvalues μ_n of $\mathcal{K}$ are in $L^1(\mathbb{R})$ and*

$$\|\phi_n\|_{L^1} \leqslant \frac{1}{\mu_n} \int_{-\infty}^{+\infty} k^{1/2}(x,x)\,dx \left[\int_{-\infty}^{+\infty} k(x,x)\,dx\right]^{1/2} \|\phi_n\|_{L^2}.$$

(iii) *$k(x,y) \in L^1(\mathbb{R}^2)$ with L^1 norm bounded by*

$$\|k\|_{L^1(\mathbb{R}^2)} \leqslant \left[\int_{-\infty}^{+\infty} k^{1/2}(x,x)\,dx\right]^2,$$

and the bilinear series (1) *converges to k in the L^1 norm.*

(iv) *The operator $\mathcal{K}: L^2(\mathbb{R}) \to L^1(\mathbb{R})$ is a continuous map with operator norm bounded by $\int_{-\infty}^{+\infty} k^{1/2}(x,x)\,dx\,[\int_{-\infty}^{+\infty} k(x,x)\,dx]^{1/2}$.*

Proof. Observe first of all that the hypotheses on k imply those of Theorem 1: as shown above, $k^{1/2}(x,x) \in L^1(\mathbb{R})$ and $k(x,x) \to 0$ as $|x| \to \infty$ imply that $k^{1/2}(x,x)$ is uniformly continuous. Since $k^{1/2}(x,x) \in L^1(\mathbb{R})$, it follows that $k^{1/2}(x,x) = o(1/x)$ as $|x| \to \infty$, and therefore $k(x,x) = o(1/x^2)$ as $|x| \to \infty$. Thus $k(x,x) \in L^1(\mathbb{R})$ and all the results of Theorem 1 automatically hold.

To show (i), given $\phi \in L^2(\mathbb{R})$ we estimate the L^1 norm of $\mathcal{K}\phi$ as follows:

$$\begin{aligned}
\|\mathcal{K}\phi\|_{L^1} &= \int_{-\infty}^{+\infty} \left| \int_{-\infty}^{+\infty} k(x,y)\phi(y)\,dy \right| dx \leqslant \int_{-\infty}^{+\infty} \left[\int_{-\infty}^{+\infty} \left|k(x,y)\right|\left|\phi(y)\right| dy \right] dx \\
&\leqslant \int_{-\infty}^{+\infty} \left[\int_{-\infty}^{+\infty} \left|k(x,y)\right|^2 dy \right]^{1/2} \left[\int_{-\infty}^{+\infty} \left|\phi(y)\right|^2 dy \right]^{1/2} dx \\
&\leqslant \int_{-\infty}^{+\infty} \left[\int_{-\infty}^{+\infty} k(x,x)k(y,y)\,dy \right]^{1/2} \left[\int_{-\infty}^{+\infty} \left|\phi(y)\right|^2 dy \right]^{1/2} dx
\end{aligned}$$

$$= \int_{-\infty}^{+\infty} k^{1/2}(x,x)\,dx \left[\int_{-\infty}^{+\infty} k(y,y)\,dy \right]^{1/2} \|\phi\|_{L^2},$$

proving statement (i).

This also shows that $\mathcal{K}$ considered as an operator from $L^2(\mathbb{R})$ into $L^1(\mathbb{R})$ is continuous with operator norm bounded by $\int_{-\infty}^{+\infty} k^{1/2}(x,x)\,dx\,[\int_{-\infty}^{+\infty} k(y,y)\,dy]^{1/2}$, which proves (iv).

To prove (ii), observe that if ϕ_n is an eigenfunction of $\mathcal{K}$ associated with a nonzero eigenvalue μ_n, then $\mathcal{K}(\phi_n) = \mu_n \phi_n$ and by the previous paragraph $\phi_n \in L^1(\mathbb{R})$ with L^1 norm satisfying

$$\|\phi_n\|_{L^1} \leqslant \frac{1}{\mu_n} \int_{-\infty}^{+\infty} k^{1/2}(x,x)\,dx \left[\int_{-\infty}^{+\infty} k(y,y)\,dy \right]^{1/2} \|\phi_n\|_{L^2}. \tag{8}$$

To prove (iii), we calculate the $L^1(\mathbb{R}^2)$ norm of $k(x,y)$,

$$\begin{aligned} \|k\|_{L^1(\mathbb{R}^2)} &= \iint_{\mathbb{R}^2} \big|k(x,y)\big|\,dx\,dy \leqslant \iint_{\mathbb{R}^2} k^{1/2}(x,x)k^{1/2}(y,y)\,dx\,dy \\ &= \int_{-\infty}^{+\infty} k^{1/2}(x,x)\,dx \int_{-\infty}^{+\infty} k^{1/2}(y,y)\,dy = \left[\int_{-\infty}^{+\infty} k^{1/2}(x,x)\,dx \right]^2 < +\infty. \end{aligned}$$

Thus $k \in L^1(\mathbb{R}^2)$. But, in fact, more can be said using the bilinear expansion of k:

$$\begin{aligned} \iint_{\mathbb{R}^2} \big|k(x,y)\big|\,dx\,dy &= \iint_{\mathbb{R}^2} \Big| \sum_{n \geqslant 0} \mu_n\, \phi_n(x)\, \overline{\phi_n(y)} \Big|\,dx\,dy \\ &\leqslant \iint_{\mathbb{R}^2} \sum_{n \geqslant 0} \mu_n \big|\phi_n(x)\big| \big|\phi_n(y)\big|\,dx\,dy \\ &\leqslant \iint_{\mathbb{R}^2} \Big[\sum_{n \geqslant 0} \mu_n \big|\phi_n(x)\big|^2 \Big]^{1/2} \Big[\sum_{m \geqslant 0} \mu_m \big|\phi_m(y)\big|^2 \Big]^{1/2} dx\,dy \\ &= \int_{-\infty}^{+\infty} \Big[\sum_{n \geqslant 0} \mu_n \big|\phi_n(x)\big|^2 \Big]^{1/2} dx \int_{-\infty}^{+\infty} \Big[\sum_{m \geqslant 0} \mu_m \big|\phi_m(y)\big|^2 \Big]^{1/2} dy \\ &= \left(\int_{-\infty}^{+\infty} \Big[\sum_{n \geqslant 0} \mu_n \big|\phi_n(x)\big|^2 \Big]^{1/2} dx \right)^2. \end{aligned}$$

Since $[\sum_{n \geqslant 0} \mu_n |\phi_n(x)|^2]^{1/2}$ is monotonously convergent to $k^{1/2}(x,x) \in L^1(\mathbb{R})$, it follows from Lebesgue's theorem that

$$\int_{-\infty}^{+\infty} \Big[\sum_{n \geqslant 0} \mu_n \big|\phi_n(x)\big|^2 \Big]^{1/2} dx = \int_{-\infty}^{+\infty} k^{1/2}(x,x)\,dx,$$

thus proving that the bilinear series converges in the L^1 norm and producing the same bound as above for $\|k\|_{L^1}$. By Theorem 1 its limit is of course $k(x,y)$ for all $x, y \in \mathbb{R}$. □

Observe that in the proof above $\int_{-\infty}^{+\infty} k^{1/2}(x,x)\,dx$ is the $L^1(\mathbb{R})$ norm of $k^{1/2}(x,x)$ since $k^{1/2}(x,x) \geqslant 0$.

As with Theorem 1, we can formulate a slightly weaker version of this result in terms of the rate of decay of the kernel along the diagonal.

Corollary 2. *Let $\mathcal{K}: L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be a positive integral operator with continuous kernel $k(x,y)$ satisfying $k(x,x) = O(1/x^{2+\epsilon})$ for some $\epsilon > 0$. Then all the statements of Theorems* 1 *and* 2 *hold.*

Proof. The condition $k(x,x) = O(1/x^{2+\epsilon})$ implies trivially $k(x,x) \to 0$ as $|x| \to \infty$ and $k(x,x) \in L^1(\mathbb{R})$ while $k^{1/2}(x,x) = O(1/x^{1+\epsilon/2})$ implies $k^{1/2}(x,x) \in L^1(\mathbb{R})$. □

Remark 2. Theorem 2 is sharp, as the following counterexample shows. Let $\phi(x) = \sin x/x$ and consider the positive operator with kernel $k(x,y) = \phi(x)\phi(y)$. This operator is of finite rank; indeed it has a single simple nonzero eigenvalue π, with associated normalized eigenfunction $(1/\sqrt{\pi})\phi(x)$. Obviously $k(x,x) = \sin^2 x/x^2$ is in $L^1(\mathbb{R})$ and $k(x,x) \to 0$ as $|x| \to \infty$. On the other hand, $k^{1/2}(x,x) \in L^p(\mathbb{R})$ for every $p > 1$ but $k^{1/2}(x,x) \notin L^1(\mathbb{R})$. In this case all the statements of Theorem 2 are false: $\mathcal{K}\phi(x) = \pi\phi(x) \notin L^1(\mathbb{R})$, so that neither the image of L^2 is contained in L^1 nor the eigenfunctions associated with nonzero eigenvalues are in L^1. Furthermore, $k(x,y)$ itself is not in $L^1(\mathbb{R}^2)$, and therefore the bilinear series is not L^1-convergent.

Remark 3. The norm bound obtained for $k(x,y)$ in Theorem 2 is optimal, as the following example shows. Given a continuous $\phi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ with $\phi(x) \to 0$ as $|x| \to \infty$, consider the (rank one) integral operator with kernel $k(x,y) = \phi(x)\overline{\phi(y)}$. We have $k(x,x) = |\phi(x)|^2$, so that $\int_{-\infty}^{+\infty} k^{1/2}(x,x)\,dx = \|\phi\|_{L^1}$. On the other hand, $\|k\|_{L^1} = \int_{-\infty}^{+\infty} |\phi(x)|\,dx \int_{-\infty}^{+\infty} |\phi(y)|\,dy = \|\phi\|_{L^1}^2$, so that the bound is attained and cannot in general be improved.

3. Bounded and unbounded domains

Although we treated explicitly the case of $\mathbb{R}$, it is clear that Theorems 1 and 2 apply with the obvious adjustments to integral positive operators on $L^2(I)$, where I is an unbounded interval in $\mathbb{R}$, that is, $I =]-\infty, b]$ or $I = [a, +\infty[$. In fact, we can slightly reformulate the results so that they apply to bounded or unbounded closed intervals.

As noted in Lemma 3, under the standing hypotheses $k(x,x)$ continuous and $k(x,x) \in L^1(\mathbb{R})$, uniform continuity of $k(x,x)$ is equivalent to $k(x,x) \to 0$ as $|x| \to \infty$. Remark 1 shows that this condition is essential for the results and cannot be dispensed with.

This means that Theorems 1 and 2 might have been formulated replacing the condition $k(x,x) \to 0$ as $|x| \to \infty$ by the equivalent one of uniform continuity of $k(x,x)$. The version we present is more suitable for applications since it deals with conditions which are immediate to check; however, a version of Theorems 1 and 2 in terms of uniform continuity is mathematically more elegant since it applies without reference to boundedness of the interval I.

We thus reformulate Theorems 1 and 2 below. In what follows, $I \subset \mathbb{R}$ will be a closed, bounded or unbounded, interval. To shorten, in the formulation below we abbreviate to 'the corresponding statements' the statements of Theorems 1 and 2 replacing the spaces $C(\mathbb{R})$, $L^1(\mathbb{R})$ and $L^2(\mathbb{R})$ by $C(I)$, $L^1(I)$ and $L^2(I)$, respectively, and integration in $\mathbb{R}$ by integration in I.

Theorem 1′. *Let $I \subset \mathbb{R}$ be a closed interval, $\mathcal{K} : L^2(\mathbb{R}) \to L^2(I)$ be a positive integral operator with continuous kernel $k(x, y)$ satisfying $k(x, x) \in L^1(I)$ and $k(x, x)$ uniformly continuous in I. Let $M = \max_{x\in I} k(x, x)$. Then the corresponding statements in Theorem* 1 *hold.*

Theorem 2′. *Let $I \subset \mathbb{R}$ be a closed interval, $\mathcal{K} : L^2(I) \to L^2(I)$ be a positive integral operator with continuous kernel $k(x, y)$ satisfying $k^{1/2}(x, x) \in L^1(I)$ and $k(x, x)$ uniformly continuous in I. Then, in addition to the statements of Theorem* 1′, *the corresponding statements in Theorem* 2 *hold.*

Theorems 1′ and 2′ hold irrespective of whether I is bounded or unbounded, and coincide with Theorems 1 and 2 when $I = \mathbb{R}$.

It is also worth considering the special case where I is a compact interval $[a, b]$. In this case, continuity of the kernel $k(x, y)$ in $[a, b]^2$ immediately implies all the other hypotheses: $k(x, x)$ is uniformly continuous, $k(x, x) \in L^1([a, b])$ and $k^{1/2}(x, x) \in L^1([a, b])$. Theorems 1′ and 2′ automatically hold simultaneously in this setting; compactness of $[a, b]$ makes them 'collapse' into a single result. We sum up below the situation for compact intervals, stressing that most of these results are well known from the literature.

Corollary 3. *Let $\mathcal{K} : L^2([a, b]) \to L^2([a, b])$ be a positive integral operator with continuous kernel $k(x, y)$. Let $M = \max_{x\in[a,b]} k(x, x)$. Then*:

(i) *For all $\phi \in L^2([a, b])$, $\mathcal{K}\phi(x)$ is continuous and*

$$\|\mathcal{K}\phi\|_{C_0} \leqslant 4M \int_a^b k(x, x)\, dx\, \|\phi\|_{L^2},$$

$$\|\mathcal{K}\phi\|_{L^1} \leqslant \int_a^b k^{1/2}(x, x)\, dx \left[\int_a^b k(x, x)\, dx\right]^{1/2} \|\phi\|_{L^2}.$$

(ii) *Eigenfunctions ϕ_n associated to nonzero eigenvalues μ_n of $\mathcal{K}$ satisfy*

$$\|\phi_n\|_{C_0} \leqslant \frac{4M}{\mu_n} \int_a^b k(x, x)\, dx\, \|\phi_n\|_{L^2},$$

$$\|\phi_n\|_{L^1} \leqslant \frac{1}{\mu_n} \int_a^b k^{1/2}(x,x)\,dx \left[\int_a^b k(x,x)\,dx \right]^{1/2} \|\phi_n\|_{L^2}.$$

(iii) *$k(x,y)$ is represented by a bilinear series* (1), *where the ϕ_n are continuous, L^2-orthonormal eigenfunctions of $\mathcal{K}$, μ_n are the associated eigenvalues and the series converges absolutely, uniformly, in L^1 and in L^2. Moreover*

$$\|k\|_{L^1([a,b]^2)} \leqslant \left[\int_a^b k^{1/2}(x,x)\,dx \right]^2.$$

(iv) *The operator $\mathcal{K}: L^2([a,b]) \to L^2([a,b])$ is trace class with*

$$\operatorname{tr}\mathcal{K} = \int_a^b k(x,x)\,dx = \sum_{n\geqslant 0} \mu_n.$$

(v) *The operator $\mathcal{K}: L^2([a,b]) \to C([a,b])$ is continuous with operator norm bounded by $4M \int_a^b k(x,x)\,dx$.*

(vi) *The operator $\mathcal{K}: L^2([a,b]) \to L^1([a,b])$ is continuous with operator norm bounded by $\int_a^b k^{1/2}(x,x)\,dx\,[\int_a^b k(x,x)\,dx]^{1/2}$.*

Remark 4. Statement (iii) in Corollary 3 is Mercer's theorem, which can therefore be considered as one of the statements of a special case of Theorems 1′ and 2′. Conversely, Theorems 1 and 2 may be thought of as an extension of Mercer's theorem to unbounded domains, which is sometimes needed in some applications to engineering (see, e.g., [2]).

Remark 5. The bound for $\|\mathcal{K}\phi\|_{L^1}$ in Corollary 3 is sharp. Considering the kernel $k(x,y) = xy$ in $[0,1]^2$, the associated integral operator has a single nonzero eigenvalue $1/3$ with associated eigenfunction $\phi(x) = x$. The bound in Corollary 3 gives $\|\mathcal{K}\phi\|_{L^1} \leqslant (1/2\sqrt{3})\|\phi\|_{L^2}$, and a simple calculation shows that this bound is attained for the eigenfunction $\phi(x)$. Thus the bound cannot in general be improved.

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