Math. Z. 182, 149–158 (1983)

Mathematische Zeitschrift

The Bieberbach Conjecture for Restricted Initial Coefficients*

Daoud Bshouty
Department of Mathematics, Technion, Haifa, Israel

1. Introduction

Let S denote the class of functions $f(z)=z+a_2z^2+\ldots$, analytic and univalent in the unit disk. The Bieberbach conjecture states that for functions $f(z)$ in S, $|a_n|\leqq n$. Many methods of univalent function theory have been developed to tackle this conjecture, the latest of which, due to FitzGerald, was found to give the best global estimate [6, 9]

$$|a_n|\leqq\left(\frac{1{,}659{,}164{,}137}{681{,}080{,}400}\right)^{1/14} n<1.0657\,n. \tag{1}$$

FitzGerald's idea of exponentiating the Grunsky inequalities, led him to inequalities involving the absolute values of the coefficients (Main lemma). Upon the investigation of these inequalities, various results were obtained (see [2–5]), in particular if either $|a_2|<1.55$ [3] or $|a|<2.04$ [2] then $|a_n|<n$ for each n. For these results only 2×2 submatrices of the main lemma were considered. In what follows, we first improve the estimate on $|a_3|$ by a modification of the method in [2, 3] up to 2.15. We then proceed to improve the results using 3×3 submatrices of the main lemma, thus obtaining the final estimates: If $|a_2|<1.596$ then $|a_n|<n$. If $|a_3|<2.15$ then $|a_n|<n$. If $|a_3|<2.51$ then $|a_n|<n$, n sufficiently large independent of $f(z)$.

2. Main Lemma and Inequalities

The following lemma of FitzGerald [6] is given in another formulation [11, Section 3.4].

Main Lemma (FitzGerald). *Let $f(z)$ be in S, then for any natural number l, and arbitrary real numbers $x_1, x_2, \ldots, x_l$, we have*

* Based in part on the author's doctoral dissertation in the Technion, Haifa

$$\left(\sum_{p=1}^{l} |a_p|^2 x_p\right)^2 \leqq \sum_{p=1}^{l} \sum_{q=1}^{l} \left(\sum_{k=1}^{p+q-1} \beta_k(p,q)\,|a_k|^2\right) x_p x_q$$

where $\beta_k(p,q)=\beta_k(q,p)$ *and for* $p\leqq q$

$$\beta_k(p,q)=\begin{cases} p-|k-q| & \textit{for } |k-q|<p \\ 0 & \textit{otherwise.} \end{cases}$$

Denoting

$$a_{p,q}=\sum_{k=1}^{p+q-1} \beta_k(p,q)\, b_k^2 - b_p^2 b_q^2, \qquad \text{where } b_k=|a_k| \tag{2}$$

we have $a_{p,q}=a_{q,p}$ and the lemma takes the form

$$Q(x)=\sum_{p=1}^{l}\sum_{q=1}^{l} a_{p,q} x_p x_q \geqq 0,$$

which means that $(a_{p,q})_1^l$ is a real symmetric positive semi-definite matrix. A well known property of such a matrix is that all its principal determinants are non-negative (cf. [7, p. 307]). We immediately deduce that for $n>2$

$$\begin{aligned} &a_{2,2}\geqq 0 \qquad a_{2,2}a_{n,n}-a_{2,n}^2\geqq 0\\ &a_{3,3}\geqq 0 \qquad a_{3,3}a_{n,n}-a_{3,n}^2\geqq 0\\ &a_{2,2}a_{3,3}-a_{2,3}^2\geqq 0, \qquad \det(a_{p,q})_{2,3,n}\geqq 0. \end{aligned} \tag{3}$$

Moreover, $a_{p,q}$ are given explicitly by (2), namely

$$\begin{aligned} a_{2,2}&=1+2b_2^2+b_3^2-b_2^4\equiv A\\ a_{3,3}&=1+2b_2^2+3b_3^2+2b_4^2+b_5^2-b_3^4\equiv C\\ a_{2,3}&=b_2^2+2b_3^2+b_4^2-b_2^2b_3^2\equiv B\\ a_{2,n}&=b_{n-1}^2+2b_n^2+b_{n+1}^2-b_2^2b_n^2\equiv D_n\\ a_{3,n}&=b_{n-2}^2+2b_{n-1}^2+3b_n^2+2b_{n+1}^2+b_{n+2}^2-b_3^2b_n^2\equiv E_n\\ a_{n,n}&=\sum_{k=1}^{n} kb_k^2+\sum_{k=n+1}^{2n-1}(2n-k)\,b_k^2-b_n^4\equiv F_n-b_n^4. \end{aligned} \tag{4}$$

Inequalities (3) reduce respectively to

$$b_n^4\leqq -\frac{[(4-b_2^2)\,b_n^2+t_n]^2}{1+2b_2^2+b_3^2-b_2^4}+F_n, \qquad \text{if } A>0 \tag{5}$$

$$b_n^4\leqq -\frac{[(9-b_3^2)\,b_n^2+f_n]^2}{1+2b_2^2+3b_3^2+2b_4^2+b_5^2-b_3^4}+F_n, \qquad \text{if } C>0 \tag{6}$$

$$b_n^4\leqq -\frac{2BD_nE_n-AE_n^2-CD_n^2}{AC-B^2}+F_n, \qquad \text{if } AC-B^2>0 \tag{7}$$

where

$$\begin{aligned} t_n&=b_{n-1}^2-2b_n^2+b_{n+1}^2\\ f_n&=b_{n-2}^2+2b_{n-1}^2-6b_n^2+2b_{n+1}^2+b_{n+2}^2. \end{aligned}$$

From (5) and (6), one already sees the need of good lower bounds for t_n, f_n, and upper bounds for F_n. Appropriate bounds for t_n, f_n are taken from the main lemma.

In [3, 2] the following inequalities are checked

$$t_n^2 \leqq 6+4(b_2^2+b_3^2+\ldots+b_{n-1}^2)+2b_n^2+t_{2n}$$
$$f_n^2 \leqq 46+52b_2^2+42b_3^2+40(b_4^2+\ldots+b_{n-2}^2)+38b_{n-1}^2+20b_n^2+2b_{n+1}^2+h_n$$

in term, where

$$h_n=b_{2n-3}^2+6b_{2n-2}^2+3b_{2n-1}^2-20b_{2n}^2+3b_{2n+1}^2+6b_{2n+2}^2+b_{2n+3}^2.$$

The bounds for f_n in [2] could be significantly improved. To do so we proceed as in [2, 3] to find bounds for h_n. Lengthy but elementary calculations finally give

$$\begin{aligned} h_n^2 \leqq\ & 492+840b_2^2+738b_3^2+628b_4^2+602b_5^2+600(b_6^2+\ldots+b_{2n-3}^2) \\ & +598b_{2n-2}^2+500b_{2n-1}^2+300b_{2n}^2+100b_{2n+1}^2+2b_{2n+2}^2+g_n, \end{aligned} \tag{9}$$

where

$$\begin{aligned} g_n=\ & b_{4n-5}^2+b_{4n+5}^2+14(b_{4n-4}^2+b_{4n+4}^2)+69(b_{4n-3}^2+b_{4n+3}^2) \\ & +120(b_{4n-2}^2+b_{4n+2}^2)-54(b_{2n-1}^2+b_{4n+1}^2)-300b_{4n}^2 \\ \equiv\ & h_{2n-2}+h_{2n+2}+8(h_{2n-1}+h_{2n+1})+18h_{2n}. \end{aligned} \tag{10}$$

3. Improved Estimates for b_2, b_3

Theorem 1. *There exist two non-decreasing sequences $\{T_k\}_1^\infty$, $\{M_k\}_1^\infty$, such that for any $f(z)$ in S:*

(a) *T_n is largest number $\leqq 2$ such that if $b_2<T_n$ then $b_k<k$ for all $k \geqq n$.*

(b) *M_n is the largest number $\leqq 2$ such that if $b_2<M_n$ then $b_k<k$ for all $k \geqq n$.*

In particular $T_7>1.584$, $M_7>2.15$.

Proof. a) The proof goes along the same lines as in [3] except for the use of a new bound for I_k, now using (1) instead of (17) in [3].

b) The proof goes along the same lines as in [2] except for the following improved bounds for f_n, b_4, b_5 given below.

Lemma 1. *A decreasing bound δ_n for f_n/n^2 exists for which $\delta_7<1.546$.*

Proof. First, a crude estimate for g_n is achieved using Milin's estimate [11, p. 81]

$$|b_k-b_{k+1}|<4.18.$$

We write

$$\begin{aligned} g_n=\ & (b_{4n-5}^2-b_{4n-4}^2)+15(b_{4n-4}^2-b_{4n-3}^2)+84(b_{4n-3}^2\, b_{4n-2}^2)+204(b_{4n-2}^2-b_{4n-1}^2) \\ & +150(b_{4n-1}^2-b_{4n}^2)+150(b_{4n+1}^2-b_{4n}^2)+204(b_{4n+2}^2-b_{4n+1}^2) \\ & +84(b_{4n+3}^2-b_{4n+2}^2)+15(b_{4n+4}^2-b_{4n+3}^2)+(b_{4n+5}^2-b_{4n+4}^2). \end{aligned}$$

By the assumption of the theorem, and for $n \geqq 7$ we have

$$|g_n| \leqq 4.18\{(8n-9)+15(8n-7)+84(8n-5)+204(8n-3)+150(8n-1)$$
$$+150(8n+1)+204(8n+3)+84(8n+5)+15(8n+7)+(8n+9)\}$$
$$\leqq 4.18(454)(16n) \leqq 619.67n^3.$$

Now using (9) and (1), a crude estimate for h_n is

$$h_n^2 \leqq 600\left(\sum_{i=1}^{2n} b_i^2\right)+|g_n|$$
$$\leqq 100(1.0657)^2\, 2n(2n+1)(4n+1)+619.67n^3$$
$$\leqq 2{,}664n^3, \qquad \text{i.e. } |h_n|<51.62n^{3/2}. \tag{11}$$

A better estimate for g_n is now available by inserting this into (10). One is led to

$$|g_n| \leqq 2{,}654.9n^{3/2}<143.4n^3. \tag{12}$$

Using (12) one can improve the estimate in (11) recursively. This procedure, applied twice, gives rise to

$$|h_n| \leqq 46.62n^{3/2}.$$

Finally (see [2, Equation 3.17])

$$|f_n|^2 \leqq (40/3)(1.0657)^2 n^3+\{2-2(1.0657)^2\}n^2+[(32/3)(1.0657)^2+4]\cdot n$$
$$-3{,}462(1.0657)^2+3{,}714+46.62n^{3/2} \equiv n^4\delta_n^2$$

for which δ_n is monotonic decreasing in n, and moreover $\delta_7<1.546$.

We next look for good upper bounds for b_4 and b_5 in terms of b_3. We have no intention to look for sharp bounds, but for good local estimates in the restricted region of b_3 to be considered. The following lemma illustrates such a result.

Lemma 2. *For* $f(z)=z+a_2z^2+\ldots$ *in* S *we have*

$$b_4 \leqq 2.4787-b_2^2/36+b_2b_3/3 \tag{13}$$

$$b_5 \leqq 4.090+b_3^2/6+b_2^4/18-b_2^2/4. \tag{14}$$

Proof. When considering good estimates of coefficients, it is quite natural to look for a representation of these coefficients in terms of "smaller" coefficients which can be well estimated. Such a representation is available by the so-called logarithmic coefficients.

With the notation

$$\log\frac{f(z)}{z}=\sum_{k=1}^{\infty}\gamma_k z^k$$

one has the following identities:

$$a_2=2\gamma_1; \quad a_3=2(\gamma_2+\gamma_1^2); \quad a_4=2\gamma_3+4\gamma_1\gamma_2+4\gamma_1^3/3;$$
$$a_5=2\gamma_4+2\gamma_2^2+4\gamma_1\gamma_3+4\gamma_1^2\gamma_2+2\gamma_1^4/3.$$

We conclude that:

$$a_4 = 2\gamma_3 + 8\gamma_1\gamma_2/3 + 2\gamma_1 a_3/3$$
$$a_5 = 2\gamma_4 + a_3^2/6 + 4\gamma_2^2/3 + 4\gamma_1\gamma_3 + 8\gamma_1^2\gamma_2/3,$$

and thus, denoting $\beta_i = |\gamma_i|$ for $i = 1, \ldots, 4$, we have respectively

$$b_4 \leqq 2\beta_3 + 4b_2\beta_2/3 + b_2 b_3/3 \leqq (1/3 + 3\beta_3^2) + (2\beta_2^2 + 2b_2^2/9) + b_2 b_3/3 \qquad (15)$$

$$b_5 \leqq 2\beta_4 + 4\beta_2^2/3 + 2b_2\beta_3 + 2b_2^2\beta_2/3 + b_3^2/6$$
$$\leqq ((1/4) + 4\beta_4^2) + 4\beta_2^2/3 + (b_2^2/4 + 4\beta_3^2) + (b_2^4/18 + 2\beta_2^2) + b_3^2/6 \qquad (16)$$

as a consequence of the Schwarz inequality.

Now, recalling Milin's estimate [10, 1, p. 51]

$$\sum_{k=1}^{n} k\beta_k^2 \leqq \sum_{k=1}^{n} \frac{1}{k} + 0.312 \qquad (n = 2, 3, \ldots) \qquad (17)$$

for $n=3$, then (13) follows immediately from (15). To deduce (14) we need the inequality [8]

$$\beta_1^2 + 4\beta_2^2/3 + \beta_3^2 \leqq 13/9.$$

From (16) we get

$$b_4 \leqq b_3^2/6 + 1/4 + 13/9 + 4\beta_4^2 + 3\beta_3^2 + 2\beta_2^2 + b_2^4/18, \qquad (18)$$

and finally using (17) for $n=4$, the result follows.

Note that the right handside of (13), (14), attains its maximum, as a function of b_2, when b_2 is maximal (for instance when $b_2 > 1.5$ and $b_3 > 1$). Consequently upper bounds for b_4 and b_5 in terms of b_3 are obtained under the substitution of $(b_3+1)^{1/2}$ for b_2.

4. Improvement of Earlier Results

A further step for the improvement of the estimates in Theorem 1 is to consider the inequality (7) for $n \geqq 7$. More precisely the positive semi-definitness of

$$\begin{pmatrix} A & B & D_n \\ B & C & E_n \\ D_n & E_n & F_n - b_n^4 \end{pmatrix}$$

This property is preserved under the increase of any diagonal element or the division of a row and same column by a fixed number.

We now assume, as before, that $b_n > n$, $b_k < k$ for $k > n$. Dividing the last row and column by b_n^2, and using the estimate

$$F_n/b_n^4 - 1 < 0.1632 \qquad (n > 7)$$

we conclude that

$$\begin{pmatrix} A & B & D \\ B & C' & E \\ D & E & 0.1632 \end{pmatrix}$$

is positive semi-definite, where

$$\begin{aligned} D &= 4 - b_2^2 + \alpha, & |\alpha| &= \left|\frac{t_n}{b_n^2}\right| \leqq 0.478, \\ E &= 9 - b_3^2 + 2\alpha + \beta, & |2\alpha+\beta| &= \left|\frac{f_n}{b_n^2}\right| \leqq 1.546 \end{aligned} \tag{19}$$

and $C' \geqq C$ to be fixed later.

Next, it is shown that

$$\begin{pmatrix} A & B & D \\ B & C' & E \\ D & E & 0.1632 \end{pmatrix} \equiv H(b_2^2, b_3^2, b_4^2, \alpha, \beta) \tag{20}$$

is monotonic decreasing in α if we just restrict the region of variability of the coefficients b_i.

Let us then confine ourselves to the region $1.58 \leqq b_2 \leqq 1.65$, then by a result of Fekete and Szegö (see Eq. (26)), $b_3 < 2.62$ and by Theorem 1 we need only consider the case where $b_3 > 2.15$. Another result of Ahlfors shows that $b_4 < 3.54$ (see [2, p. 15]) and if we consider similar methods as of Theorem 1, one can see that it would be enough to consider the case where $b_4 > 2.89$. Finally, a result of Aharonov [1, Lemma 2.4] shows that $b_5 < 4.78$, so that we may choose for C' the following

$$C \leqq C' \equiv 1 + 4.78^2 + 2b_2^2 + 3b_3^2 + 2b_4^2 - b_3^4.$$

Similar methods as of Lemma 1 shows that

$$|2\alpha - \beta| < 0.71, \qquad |\beta| < 0.69. \tag{21}$$

(19) and (21) describes the region of variability of α, β.

Now we consider

$$\frac{1}{2}\frac{\partial H}{\partial \alpha} = BE - C'D - 2(AE - BD) \equiv G(b_2^2, b_3^2, b_4^2, \alpha, \beta)$$

which satisfies

$$\frac{\partial G}{\partial \alpha} = (2B - C') + 2(B - 2A) < -45.28 + 2b_4^2 < 0$$

$$\frac{\partial G}{\partial (b_4^2)} = E > 0$$

so that

$$G(b_2^2, b_3^2, b_4^2, \alpha, \beta) \leqq G(b_2^2, b_3^2, 3.54^2, (0.71 + \beta)/2, \beta) \equiv J.$$

One verifies that $\partial^2 J/\partial(b_2^2)^2 = 4(9-2\alpha-\beta)>0$, $\partial^2 J/\partial(b_3^2)^2 = 2(4+\alpha)>0$ so that the maximum of J as a function of b_2^2, b_3^2 is attained at one of the four corners of the region of variability of b_2, b_3. Moreover J is linear in β so that its maximum is attained at either $\beta = \pm 0.69$, and hence

$$\frac{\partial H}{\partial \alpha} \leqq G(1.59^2, 2.62^2, 3.54^2, -70, -0.69) < 0.$$

We conclude that (considering the region of variability of α, β)

$$0 \leqq H(b_2^2, b_3^2, b_4^2, \alpha, \beta)$$
$$\leqq \begin{cases} H_1 \equiv H(b_2^2, b_3^2, b_4^2, (-0.71+\beta)/2, \beta); & -0.246 < \beta < 0.69 \\ H_2 \equiv H(b_2^2, b_3^2, b_4^2, -0.478, \beta); & -0.59 \leqq \beta \leqq -0.246 \\ H_3 \equiv H(b_2^2, b_3^2, b_4^2, (-\beta-1.546)/2, \beta); & -0.69 \leqq \beta \leqq -0.59. \end{cases}$$

Now

$$\frac{\partial H_1}{\partial \beta} = (BE - C'D) - 4(AE - BD) \equiv G_1(b_2^2, b_3^2, b_4^2, \beta)$$

satisfies

$$\frac{\partial G_1}{\partial \beta} < 0, \quad \frac{\partial G_1}{\partial (b_4^2)} > 0, \quad \frac{\partial^2 G_1}{\partial (b_2^2)^2} > 0 \quad \text{and} \quad \frac{\partial^2 G_1}{\partial (b_3^2)^2} > 0$$

so that

$$\frac{\partial H_1}{\partial \beta} \leqq G_1(1.59^2, 2.62^2, 3.54^2, -0.69) < 0. \tag{22}$$

Next

$$\frac{H_2}{\partial \beta} = -2(AE - BD) \equiv G_2(b_2^2, b_3^2, b_4^2, \beta)$$

satisfies

$$\frac{\partial G_2}{\partial \beta} < 0, \quad \frac{\partial G_2}{\partial (b_4^2)} > 0 \quad \text{and} \quad \frac{\partial G_2}{\partial (b_3^2)} < 0$$

so that

$$\frac{\partial H_2}{\partial \beta} > G_2(b_2^2, 2.62^2, 2.89^2, -0.246) \equiv K_2(b_2^2),$$

and since

$$\frac{\partial K_2}{\partial (b_2^2)} < 0$$

we finally get

$$\frac{\partial H_2}{\partial \beta} > G_2(1.65^2, 2.62^2, 2.89^2, -0.246) > 0. \tag{23}$$

Also

$$\frac{\partial H_3}{\partial \beta} = CD - BE \equiv G_3(b_2^2, b_3^2, b_4^2, \beta)$$

satisfies

$$\frac{\partial G_3}{\partial \beta}<0, \quad \frac{\partial G_3}{\partial (b_2^2)}>0, \quad \frac{\partial^2 G_3}{\partial (b_3^2)^2}<0$$

and G_3 is linear in b_4^2 so that

$$\frac{\partial H_3}{\partial \beta}>G_3(1.59^2, 2.15^2, 3.54^2, -0.59) \tag{24}$$

Finally (22), (23) and (24) show that

$$0 \leqq H(b_2^2, b_3^2, b_4^2, -0.478, -0.246) \equiv S(b_2^2, b_3^2, b_4^2).$$

Similar techniques show that S is decreasing in b_3^2 so that

$$H(b_2^2(\lambda), v^2(\lambda), b_4^2, -0.478, -0.246) \geqq 0 \tag{25}$$

where

$$b_2(\lambda)=\frac{2}{1-\lambda} \exp\left(\frac{-\lambda}{1-\lambda}\right), \quad 0<\lambda<1$$

and

$$v(\lambda)=1+\lambda b_2^2(\lambda)+2 \exp\left(\frac{-2\lambda}{1-\lambda}\right) \tag{26}$$

is the maximum value of b_3 for fixed b_2 [11, p. 120]. We fix λ and run over all values of b_4^2 [2, p. 15]

$$2.89^2 \leqq b_4^2 \leqq \frac{16}{15}+\frac{339}{135}b_2^2+\frac{23}{18}b_2^4-\frac{35}{144}b_2^6=\mu^2(b_2), \tag{26}$$

it is then seen that for all values of b_4 this inequality is violated unless $b_2>1.596$. Thus $T_7>1.596$ as claimed in the introduction.

The use of inequality (7) for $n \geqq 7$ did not greatly improve the estimate of T_7. Similarly the use of the inequality does not greatly improve the estimate of M_7. On the other hand the inequality does lead to significant results for large n. In what follows we prove that

$$M^*=\lim_{n\to\infty} M_n \geqq 2.526.$$

For the equivalent T^* a better result already exists in [4].

We introduce the following notations

$$S[\alpha]=\{f \text{ in } S;\ b_2(f)\leqq \alpha\}, \qquad d_\alpha=\limsup_{n\to\infty}\left(\sup_{f\in S[\alpha]}\frac{b_n(f)}{n}\right). \tag{27}$$

Fix α, and let $c>d_\alpha$ be given, then there exists $n_0=n_0(c)$ such that for $n>n_0$

$$b_n(f)/n<c \qquad \text{for every } f(z) \text{ in } S[\alpha]. \tag{28}$$

Since $\sum_{k=1}^{2n-1} \beta_k(n,n)\cdot k^2=(7n^4-n^2)/6<7n^4/6$, (28) and (7) give

$$b_n^4 \leqq -\frac{AE_n^2+CD_n^2-2BD_nE_n}{AC-B^2}+\sum_{k=1}^{n_0}\beta_k(n,n)b_k^2+7c^2n^4/6, \tag{29}$$

for every $f(z)$ in $S[\alpha]$ and $n>n_0$.

$S[\alpha]$ is normal and compact. We deduce the existence of $f_k(z)=z+d_{2,k}z^2+\ldots$ in $S[\alpha]$ such that

$$\sup_{f\in S[\alpha]} \frac{b_n(f)}{n}=\frac{d_{n,n}}{n}. \tag{30}$$

Applying (29) to $f_n(z)$ and dividing both sides by n^4 we conclude

$$\left(\frac{d_{n,n}}{n}\right)^4 \leqq -\frac{A\mathscr{E}_n^2+CD_n^2-2B\mathscr{D}_n\mathscr{E}_n}{AC-B^2}+V/n^4+7c^2/6.$$

where $\mathscr{D}_n=(4-b_2^2)(d_{n,n}/n^2)+O(n^{-1/2})$, $\mathscr{E}_n=(9-b_3^2)(d_{n,n}/n)^2+O(n^{-1/2})$.

Passing to the limit, bearing in mind the notations in (27) and (30) we deduce

$$d_\alpha^4 \leqq -\frac{(A\mathscr{E}^2+C\mathscr{D}^2-2B\mathscr{D}\mathscr{E})d_\alpha^4}{AC-B^2}+7c^2/6, \quad \text{where } \mathscr{D}=4-b_2^2;\ \mathscr{E}=9-b_3^2;$$

which holds true for every $c>d_\alpha$, and thus for $c=d_\alpha$. Consequently

$$d_\alpha^2 \leqq -\frac{(A\mathscr{E}^2+C\mathscr{D}^2-2B\mathscr{D}\mathscr{E})d_\alpha^2}{AC-B^2}+7/6$$

and thus $d<1$ if and only if

$$\frac{A\mathscr{E}^2+C\mathscr{D}^2-2B\mathscr{D}\mathscr{E}}{AC-B^2}>\frac{1}{6}.$$

equivalently if and only if

$$\det\begin{pmatrix} A & B & \mathscr{D} \\ B & C & \mathscr{E} \\ \mathscr{D} & \mathscr{E} & 1/6 \end{pmatrix} \leqq 0.$$

Again it is enough to have

$$K(b_2^2, b_3^2, b_4^2) \equiv \det\begin{pmatrix} A & B & D \\ B & C' & \mathscr{E} \\ \mathscr{D} & \mathscr{E} & 1/6 \end{pmatrix} \leqq 0$$

for

$$C \leqq C' \equiv 1+2b_2^2+3b_3^2+2b_4^2+25-b_3^2.$$

One now verifies that for $b_3<2.60$, $b_2<(b_3+1)^{1/2}$ and $b_4<\mu(b_2)$

$$\frac{\partial K}{\partial(b_4^2)}>0$$

so that it would be enough to have

$$K(b_2^2, b_3^2, \kappa^2) \leqq 0 \tag{31}$$

where κ denotes the maximum of b_4 for fixed b_2 and b_3. κ is certainly restricted by $\mu(b_2)$ yet for larger values of b_2 it is restricted by the condition

$$AC' - B^2 \geqq 0.$$

Taking this into account, we fix a value for $b_3 > 2.4$ (see [5]) and run over all values of b_2 ranging between ($T^* > 1.78$ [4])

$$1.78 < b_2 < (b_{3+1})^{1/2}$$

and finally get that (31) holds for $b_3 < 2.526$.

We conclude with the following theorem which sums up the best known results.

Theorem 2. *With the notations of Theorem* 1 *we have the following*

$$T_7 > 1.596, \qquad T^* = \lim_{n\to\infty} T_n \geqq 1.78$$

$$M_7 > 2.15, \qquad M^* = \lim_{n\to\infty} M_n \geqq 2.526.$$

References

1. Aharonov, D.: Special Topics in Univalent Functions. Lecture Notes. University of Maryland 1971
2. Alves, M.: Bieberbach's Conjecture with $|a_3|$ restricted. Not. Commun. Math. **87**, 20 (1979)
3. Bshouty, D.H.: The Bieberbach conjecture for univalent functions with small second coefficients. Math. Z. **149**, 183–187 (1976)
4. Bshouty, D., Hengartner, W.: Asymptotic FitzGerald inequalities. Comment. Math. Helv. **53**, 228–238 (1978)
5. Ehrig, G.: Coefficient estimates concerning the Bieberbach conjecture. Math. Z. **140**, 111–126 (1974)
6. FitzGerald, G.H.: Quadratic inequalities and coefficient estimates for schlicht functions. Arch. Rational. Mech. Anal. **46**, 356–368 (1972)
7. Gantmacher, F.: The Theory of Matrices. New York: Chelsea 1960
8. Grinshpan, A.Z.: The logarithmic coefficients of functions in the class S. Sibirsk. Mat. Ž. **13**, 1145–1157 (1972) [Russian]. Engl. Transl.: Sibirian. Math. J. **13**, 793–801 (1973)
9. Horowitz, D.A.: A refinement for coefficient estimates of univalent functions. Preprint
10. Milin, I.M.: On the coefficients of univalent functions. Dokl. Akad. Nauk. S.S.S.R. **176**, 1015–1018 (1967) [Russian]. Engl. Transl.: Soviet Math. Dokl. **8**, 1255–1258 (1967)
11. Pomerenke, Ch.: Univalent Functions. Göttingen: Vandenhoek und Ruprecht 1975

Received July 28, 1981; received in final form September 25, 1982

Note Added in Proof

D. Hamilton has improved the bound for T^* to 1.8