Computational Methods and Function Theory
Volume 8 (2008), No. 1, 261–275

On the Boundary Behaviour of Univalent Harmonic Mappings onto Convex Domains

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(Communicated by J. Milne Anderson and Philip J. Rippon)

Dedicated to Walter Hayman on his 80th birthday

Abstract. In this note we discuss the boundary behavior of a univalent harmonic mapping f from the unit disk U “onto” a bounded convex domain Ω in the sense of Hengartner and Schober, whose second dilatation function a is an inner function. This problem was raised by Laugesen in [10].

Keywords. Harmonic mapping, analytic dilatation, convex domains, elliptic differential equation.

2000 MSC. Primary 26C10; Secondary 30C15.

1. Preliminaries

A harmonic orientation-preserving mapping f defined on the open unit disk $U = \{z\colon |z| < 1\}$ is a solution of the system of linear elliptic partial differential equations

$$\overline{f_{\overline{z}}}(z) = a(z) f_z(z), \tag{1}$$

where the function a, called the (*second*) *dilatation* of f, belongs to the class $\mathcal{B}$ of holomorphic functions from U to U.

For the special case where $|a| < k < 1$ in U, it is classical that the *Riemann Mapping Theorem* (*RMT*) for the system in (1) holds; namely, for a given bounded simply connected domain Ω and a fixed $w_0 \in \Omega$, there is a univalent solution f of (1) satisfying $f(0) = w_0$ and $f_z(0) > 0$, which maps U onto Ω. In addition, if Ω is a Jordan domain, then f extends to a homeomorphism from $\overline{U}$ onto $\overline{\Omega}$. However, this may not be the case if $\|a(z)\|_\infty = 1$ in which case the following theorem holds [9, Thm. 4.2 and Thm. 4.3].

Received February 6, 2007, in revised form March 28, 2007.
Published online July 25, 2007.
The first author thanks the Promotion of Research Fund at the Technion for its support.

ISSN 1617-9447/$ 2.50

Theorem A (Hengartner and Schober [9]). *Let* Ω *be a bounded simply connected domain whose boundary* $\partial\Omega$ *is locally connected. Suppose that* $a \in \mathcal{B}$ *and* w_0 *is a fixed point of* Ω. *Then there exists a univalent solution* f *of* (1) *having the following properties:*

(a) $f(0) = w_0$, $f_z(0) > 0$ *and* $f(U) \subset \Omega$;
(b) *there is a countable set* $E \subset \partial U$ *such that the unrestricted limits* $f^*(e^{it}) = \lim_{z \to e^{it}} f(z)$ *exist on* $\partial U \setminus E$ *and they are on* $\partial\Omega$;
(c) *the functions*

$$f^*_-(e^{it}) = \operatorname{ess\,lim}_{s \uparrow t} f^*(e^{is}) \qquad \text{and} \qquad f^*_+(e^{it}) = \operatorname{ess\,lim}_{s \downarrow t} f^*(e^{is})$$

exist on ∂U, *belong to* $\partial\Omega$ *and are equal on* $\partial U \setminus E$;
(d) *the cluster set of* f *at* $e^{it} \in E$ *is the straight line segment joining* $f^*_-(e^{it})$ *to* $f^*_+(e^{it})$.

The mapping f is termed a *Generalized Riemann Mapping* (*GRM*) from U onto Ω.

Only in rare cases is the uniqueness of f known. Note that the boundary function f^* is continuous at every point in $\partial U \setminus E$ and has a jump discontinuity at every point in E. The term *jump* will be used in order to describe the behavior of f^* at every point of E. Also, note that the cluster set of f on an interval $J \subset \partial U$ induces a positively directed boundary arc in $\partial f(U)$; this arc is denoted, with abuse of notation, by $f^*(J)$.

Let Ω be a bounded Jordan domain of $\mathbb{C}$. We say that a point $\omega_0 \in \partial\Omega$ is a *point of convexity* (*with respect to* Ω) if there is a line segment L containing ω_0 as an interior point such that $L \setminus \{\omega_0\}$ lies in the exterior of Ω. Conversely, we say that $\omega_0 \in \partial\Omega$ is a *point of concavity* (*with respect to* Ω) if there is a line segment L containing ω_0 as an interior point such that $L \setminus \{\omega_0\}$ lies in the interior of Ω. Also, we say that a subarc C of $\partial\Omega$ is *concave* (*with respect to* Ω) if every point of C is either a point of concavity or an interior point of a line segment in C.

Note that the points of convexity of the boundary of a bounded convex domain Ω are its extreme points, whereas it has no points of concavity. If Ω is a Euclidean triangle, then the sides of $\partial\Omega$ are concave arcs.

Further information about the behavior of f^* on intervals where $|a| \equiv 1$ is needed and is summarized as follows; see [2, Thm. 2.2 and Thm. 2.13].

Theorem B (Bshouty and Hengartner [2]). *Let* f *be a GRM from the unit disc* U *onto a Jordan domain* Ω *whose dilatation* $a \in \mathcal{B}$ *admits an analytic extension across an open interval* $J = \{e^{it} : \gamma < t < \delta\}$, $\gamma < \delta < \gamma + 2\pi$, *such that* $|a| \equiv 1$ *on* J. *Then*

(a) $\operatorname{Im}\Big(\sqrt{a(e^{it})}\, df^*(e^{it})\Big) = 0$, *a.e.* df^* *on* J.

Furthermore, if a *does not extend analytically across any subinterval of* ∂U *properly containing* J, *and if for a fixed* $\omega \in J$, $q = f^*(\omega)$ *exists, and* $\alpha(q)$ *is the size of the opening angle at* q *as seen from the inside of* Ω, *then the largest open interval* $J_1 \subset J$, *possibly a singleton, such that* $f^*(J_1) \equiv q$ *satisfies the following:*

(b) $\Delta_{J_1} \arg \sqrt{a(e^{it})} < 2\pi$;
(c) *if* $0 < \alpha(q) < \pi$, *then* $\Delta_{J_1} \arg \sqrt{a(e^{it})} = \alpha(q)$;
(d) *if* $\alpha(q) = \pi$, *then either* (c) *holds or* J_1 *is a singleton.*

Then with J *not necessarily satisfying the above maximal property we have*

(e) *if* Ω *is a convex domain with* $\Delta_J \arg \sqrt{a(z)} > \pi$, *then* f^* *attains at least one jump in* J.

Throughout the paper, the notation $\arg \sqrt{a(z)}$ and $\Delta_J \arg \sqrt{a(z)}$ denote a continuous single-valued branch of $\arg \sqrt{a}$ and its net variation over the aforementioned circular arc J, respectively.

Theorem B refers to results in [2] as follows. Part (a) is equation (2.9), part (b) follows from Corollary 2.8, and parts (c), (d) and (e) comprise essentially Theorem 2.13.

This work uses Blaschke products and, more generally, inner functions. Let $\{\zeta_n\}$ be an infinite sequence of points in U with a finite number m of zeros, say $\zeta_1, \zeta_2, \cdots, \zeta_m$. Then the *Blaschke product* associated with $\{\zeta_n\}$ is defined as the infinite product

$$B(z) = e^{i\alpha} z^m \prod_{n=m+1}^{\infty} \frac{|\zeta_n|}{\zeta_n} \frac{\zeta_n - z}{1 - \bar{\zeta}_n z}. \tag{2}$$

It is well known that B belongs to $\mathcal{B}$ if and only if $\sum(1 - |\zeta_n|)$ converges.

An *inner function* is a function $h \in \mathcal{B}$ for which the radial limits exist and have modulus 1 a.e. A striking relationship between Blaschke products and inner functions is stated as follows.

Theorem C (Frostman [7]). *If* h *is an inner function, then for all* $k \in U$ *but possibly a set of zero capacity, the function*

$$h_k(z) = \frac{h(z) - k}{1 - \bar{k} h(z)}$$

is a Blaschke product.

The results of this paper are closely related to the boundary behavior of Blaschke products. The following theorem on the radial limits of a Blaschke product at a boundary point provides a relationship.

Theorem D (Frostman [8]). *A necessary and sufficient condition for a Blaschke product of the form* (2) *and all of its sub-products to have radial limits of modulus* 1 *at a point* $e^{i\theta}$ *is that*

$$\sum_{n=1}^{\infty} \frac{1-|\zeta_n|}{|e^{i\theta}-\zeta_n|} < \infty.$$

In what follows, we call an interval $J = \{e^{it} : \gamma < t < \delta\}$, where $\gamma < \delta < \gamma + 2\pi$, a *right* and *left interval* or *neighborhood* of $e^{i\gamma}$ and $e^{i\delta}$ respectively.

2. Main results and proofs

The problem raised by Laugesen in [10] is to study the boundary behavior of a GRM f from U onto a bounded convex domain Ω whose dilatation a is an inner function and $w_0 = 0$. This problem becomes more relevant knowing that such a mapping is unique; see [5] and [4].

We contend that there exists an affine transformation M such that

(i) $M \circ f$ is a GRM from U onto another bounded convex domain Ω' whose dilatation is a Blaschke product, and
(ii) f and $M \circ f$ share the same boundary behavior at every point $e^{i\theta} \in \partial U$ whether it involves continuity, jump discontinuity, or constancy on some interval of ∂U terminating at $e^{i\theta}$ (a case of interest in its own right).

To achieve this, we use Theorem C to obtain a value $k \in U$ such that the function $h_k \circ a(z) = (a(z) - k)/(1 - \overline{k}a(z))$ is a Blaschke product, then define the affine transformation $M(w) = w - k\overline{w}$. Given this, it can be easily verified that conditions (i) and (ii) above hold; in particular, the dilatation of $M \circ f$ is

$$\frac{\overline{(M \circ f)_{\overline{z}}}}{(M \circ f)_z} = h_k \circ a(z).$$

We conclude that the study of the above-mentioned problem can be reduced to the case where the dilatation a of f is a Blaschke product. Another conclusion answers a problem of Laugesen [10, p. 42] formulated here as follows: "must the boundary function f^* of a GRM f from U onto a bounded convex domain have a jump if the dilatation of f is an infinite Blaschke product?" Indeed, following the statement of his problem, Laugesen noted that if f is the harmonic extension of an orientation-preserving homeomorphism f^* from the unit circle onto itself satisfying $(f^*)'(e^{i\theta}) = 0$ a.e., then f is a GRM from U onto itself, with $f(U) = U$, whose dilatation is an inner function. Thus, a negative answer to the problem follows by simply post-composing f by an appropriate affine transformation M to obtain a GRM $M \circ f$ from U onto an elliptic domain U' with $M \circ f(U) = U'$, whose dilatation is an infinite Blaschke product. Observe that M may be chosen so that U' is as close to U as desired.

The first result of this paper is of general nature and provides a sufficient condition for a GRM f from U onto a Jordan domain Ω to be continuous at a boundary point.

Theorem 1. *Let f be a GRM from the unit disk U onto a Jordan domain Ω with rectifiable boundary, let the dilatation a of f be a Blaschke product of form* (2), *and let $e^{i\theta_0} \in \partial U$.*

(a) *If $\lim_{r\to 1^-} a(re^{i\theta_0})$ does not exist, or if otherwise*

$$\lim_{r\to 1^-} a(re^{i\theta_0}) = \alpha, \qquad |\alpha| \neq 1, \tag{3}$$

then f^* is continuous at $e^{i\theta_0}$.

(b) *If*

$$\sum_{m+1}^{\infty} \frac{1-|\zeta_n|}{|e^{i\theta_0}-\zeta_n|} = \infty, \tag{4}$$

then f^* is non-constant on any right or left interval of $e^{i\theta_0}$.

We require the following lemma for the proof of Theorem 1.

Lemma 1. *Let a be a Blaschke product of form* (2) *whose zeros accumulate at* 1, *and let $J = \{e^{it}\colon\ -\gamma < t < 0\}$, where $0 < \gamma < \pi$.*

(a) *If $\operatorname{Im}\zeta_n > 0$ for all n, then*

$$\Delta_J \arg\sqrt{a(z)} = \frac{1}{2}\sum_1^{\infty} \Delta_J \arg\frac{z-\zeta_n}{1-\overline{\zeta_n}z} \sim \sum_1^{\infty}\frac{1-|\zeta_n|}{|1-\zeta_n|}, \tag{5}$$

where $\sim$ means that both sides of (5) *converge or diverge simultaneously.*

(b) *If infinitely many ζ_n satisfy $\operatorname{Im}\zeta_n \le 0$, then $\Delta_J \arg\sqrt{a(z)} = \infty$.*

Proof. Let $\phi_n = \Delta_J \arg(z-\zeta_n)$, λ_n be the point of intersection of the unit circle with the straight line passing through $e^{-i\gamma}$ and ζ_n, and μ_n be the size of the angle whose vertex is 1 and which subtends the line segment $[\lambda_n, \zeta_n]$; see Figure 1. Then

$$\Delta_J \arg\frac{z-\zeta_n}{1-\overline{\zeta_n}z} = \Delta_J \arg\big((z-\zeta_n)^2\big) - \Delta_J \arg z = 2\left(\phi_n - \frac{\gamma}{2}\right) = 2\mu_n. \tag{6}$$

(a) Since $\operatorname{Im}\zeta_n > 0$ and $0 < \gamma < \pi$, we have $0 < \mu_n < \pi/2$. Observe that

$$|\lambda_n - \zeta_n||e^{-i\gamma} - \zeta_n| = 1 - |\zeta_n|^2.$$

Then, by the sine rule,

$$\frac{\sin(\gamma/2)}{|1-\zeta_n|} = \frac{\sin\mu_n}{|\lambda_n-\zeta_n|} = \frac{|e^{-i\gamma}-\zeta_n|\sin\mu_n}{1-|\zeta_n|^2}.$$

Thus

$$\sin\mu_n = \frac{(1+|\zeta_n|)\sin(\gamma/2)}{|e^{-i\gamma}-\zeta_n|}\frac{1-|\zeta_n|}{|1-\zeta_n|}.$$

Since the first fraction on the right-hand side of the equality is bounded away from zero and bounded above, and because of (6), the proof of (a) is complete.

(b) Observe that if Im $\zeta_n \le 0$ and ζ_n is sufficiently close to 1, then μ_n is larger than $\pi/2$, see Figure 1, and, by (6), the proof of (b) is complete. ∎

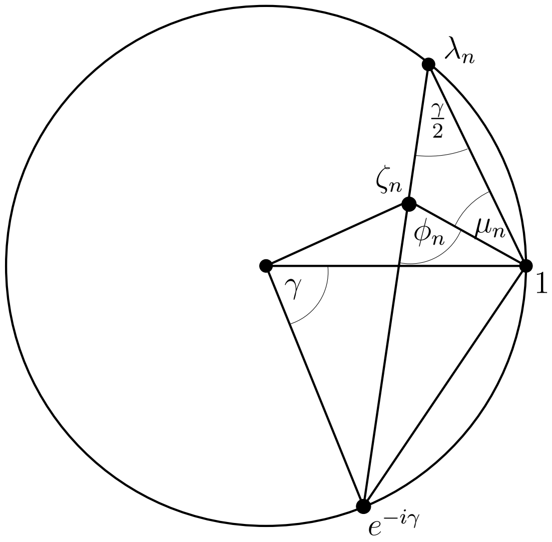

FIGURE 1. Proof of Lemma 1.

Proof of Theorem 1. We may assume that $\theta_0 = 0$.

(a) Assume to the contrary that f^* has a jump at 1 of size $df^*(1) \neq 0$. Assuming that $f^*(e^{it})$, $0 \le t \le 2\pi$, is right-continuous on $[0, 2\pi]$, then, as in [10, p. 47], we have

$$a(r) = \frac{(1-r)g'(r)}{(1-r)h'(r)} = \left(\int_{-\pi}^{\pi} \frac{(1-r)\overline{df^*(e^{it})}}{e^{it}-r}\right) \Big/ \left(\int_{-\pi}^{\pi} \frac{(1-r)df^*(e^{it})}{e^{it}-r}\right).$$

But $\partial\Omega$ is rectifiable and, by Theorem A, the value $\int_{-\pi}^{\pi} |df^*(e^{it})|$ is at most the length of $\partial\Omega$. Hence, by the Lebesgue Dominated Convergence Theorem,

$$\lim_{r\to 1} a(r) = \frac{\overline{df^*(1)}}{df^*(1)}.$$

Consequently, the radial limit of a at 1 exists and has absolute value 1. This contradicts (3) and (a) follows.

(b) We show that f^* is non-constant on any left or right interval of 1. Assume that f^* is constant on a left interval $J = \{e^{it}\colon -\delta < t < 0\}$ of 1 for some positive

δ. Then f admits a harmonic extension beyond J and, consequently, a admits an analytic extension also beyond J such that $|a| \equiv 1$ on J. We infer, by Theorem B (b), that $\Delta_J \arg \sqrt{a(e^{it})} \leq 2\pi$.

We construct now a subsequence $\{\xi_n\}$ of $\{\zeta_n\}$ such that $\xi_n \to 1$ and

$$\sum \frac{1-|\xi_n|}{|1-\xi_n|} = \infty. \tag{7}$$

Indeed, since $\sum_{m+1}^{\infty}(1-|\zeta_n|) < \infty$, there exists a subsequence $\{\zeta_{1k}\}_{k=1}^{\infty}$ of $\{\zeta_n\}$ such that each $|\zeta_{1k} - 1| < 1$ and

$$\sum_{k=1}^{\infty} \frac{1-|\zeta_{1k}|}{|1-\zeta_{1k}|} = \infty.$$

This implies that there exists a positive integer N_1 such that

$$\sum_{k=1}^{N_1} \frac{1-|\zeta_{1k}|}{|1-\zeta_{1k}|} \geq 1.$$

By the same argument above, there exists a subsequence $\{\zeta_{2k}\}_{k=1}^{\infty}$ of $\{\zeta_{1k}\}_{k=N_1+1}^{\infty}$ such that each $|\zeta_{2k} - 1| < 1/2$ and

$$\sum_{k=1}^{\infty} \frac{1-|\zeta_{2k}|}{|1-\zeta_{2k}|} = \infty.$$

This implies that there exists a positive integer N_2 such that

$$\sum_{k=1}^{N_2} \frac{1-|\zeta_{2k}|}{|1-\zeta_{2k}|} \geq 1.$$

Repeating the above procedure indefinitely, the desired sequence $\{\xi_n\}$ will be the following:

$$\zeta_{11}, \zeta_{12}, \ldots, \zeta_{1N_1}, \zeta_{21}, \zeta_{22}, \ldots, \zeta_{2N_2}, \ldots, \zeta_{k1}, \zeta_{k2}, \ldots, \zeta_{kN_k}, \ldots.$$

We may assume that $0 < \delta < \pi$. It is immediate that there exists a subsequence $\{\xi_{n_k}\}_{k=1}^{\infty}$ of $\{\xi_n\}$ whose terms satisfy only one of the inequalities $\operatorname{Im} \xi_{n_k} > 0$ or $\operatorname{Im} \xi_{n_k} \leq 0$. In either case, if

$$a_1(z) = \prod_{n=1}^{\infty} \frac{|\xi_{n_k}|}{\xi_{n_k}} \frac{\xi_{n_k} - z}{1 - \bar{\xi}_{n_k}},$$

then, by Lemma 1, $\Delta_J \arg \sqrt{a_1(e^{it})} = \infty$. But

$$\Delta_J \arg \sqrt{a_1(e^{it})} \leq \Delta_J \arg \sqrt{a(e^{it})}.$$

Hence, $\Delta_J \arg \sqrt{a(e^{it})} = \infty$ and we have a contradiction. Therefore, the assumption that f^* is constant on a left interval of 1 is absurd.

The proof that f^* is non-constant on any right interval of 1 follows essentially in the same manner. This completes the proof of Theorem 1. ■

Frostman [8] showed that (4) by itself is not enough to prevent a Blaschke product from having radial limits of modulus 1, but this together with a restriction on the arguments of the zeros can accomplish this. In particular, if the zeros lie on one side of a ray, then the radial limits along the ray will not exist (cf. also [6, p. 3]). Therefore, we have the following corollary.

Corollary 1. *Let f be a GRM from U onto a bounded convex domain Ω whose dilatation a is a Blaschke product of form* (2)*, where $m = 0$, $\operatorname{Im}\zeta_n > 0$ for all n, and* (4) *holds for $\theta_0 = 0$. Then f^* is continuous at* 1*.*

Proof. Reorder the sequence $\{\zeta_n\}$ so that the sequence $\{\operatorname{Re}\zeta_n\}$ increases to 1. Fix r, $0 < r < 1$. Let $e^{-i\gamma_n}$ be the point of intersection of the unit circle with the line segment from ζ_n to r; see Figure 2. Denote by J_n, $n = 1, 2, \ldots$, the minor arc of the unit circle terminating at $e^{-i\gamma_n}$ and 1. Observe that $J_1 \subset J_n$ for sufficiently large n. In this case we have:

$$\begin{aligned}\Delta_{[r,1)} \arg \frac{z-\zeta_n}{1-\overline{\zeta_n}z} &= \Delta_{[r,1)} \arg(z-\zeta_n)^2 \\ &= 2\phi_n > 2\mu > \Delta_{J_n} \arg \frac{z-\zeta_n}{1-\overline{\zeta_n}z} \\ &\geq \Delta_{J_1} \arg \frac{z-\zeta_n}{1-\overline{\zeta_n}z}.\end{aligned}$$

In particular, by the proof of Lemma 1 (a) and the fact that (4) holds for $\theta_0 = 0$ we have $\Delta_{[r,1)} \arg \sqrt{a(z)} = \infty$. Thus $\lim_{r\to 1^-} a(r)$ fails to exist and, by Theorem 1 (a), f^* is continuous at 1. ∎

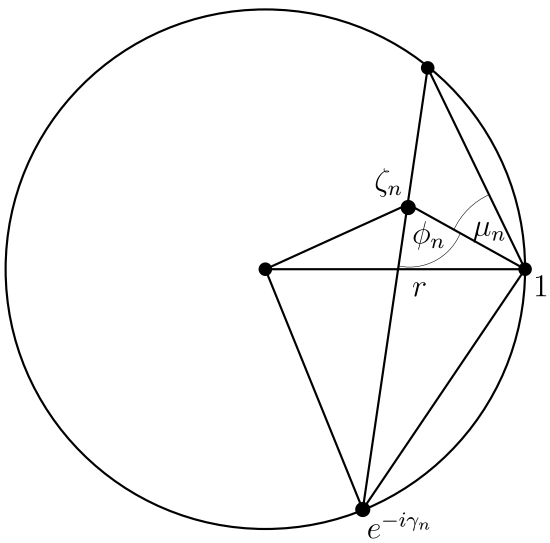

FIGURE 2. Proof of Corollary 1.

The second result of the paper is stated in the next theorem.

Theorem 2. *Let f be a GRM from the unit disk U onto a Jordan domain Ω with rectifiable boundary whose dilatation a is a Blaschke product of form (2). Let there be a left (right) interval J of θ_0 across which a continues analytically with $|a| \equiv 1$ on J.*

(a) *A sufficient condition for f^* to be non-constant on any left (right) interval of θ_0 is one of the following:*
   (i) *for every left (right) interval $K \subset J$ of θ_0, $\Delta_K \arg \sqrt{a(z)} = \infty$;*
   (ii) *there exists a subsequence $\{\xi_n\}$ of $\{\zeta_n\}$ converging to $e^{i\theta_0}$ such that*

$$\sum_{\arg \xi_n \le \theta_0} 1 + \sum_{\arg \xi_n \ge \theta_0} \frac{1-|\xi_n|}{|e^{i\theta_0} - \xi_n|} = \infty, \tag{8}$$
$$\left(\sum_{\arg \xi_n \ge \theta_0} 1 + \sum_{\arg \xi_n \le \theta_0} \frac{1-|\xi_n|}{|e^{i\theta_0} - \xi_n|} = \infty \right).$$

(b) *If Ω is a bounded convex domain, then each of the above sufficient conditions is also necessary for f^* to be non-constant on any left (right) interval of θ_0.*

For the proof of Theorem 2, we require the following two lemmas.

Lemma 2. *Let f be a GRM from the unit disk U onto a bounded convex domain Ω whose dilatation $a \in \mathcal{B}$ continues analytically across an open arc $J = \{e^{it} \colon \gamma < t < \delta\}$, where $\gamma < \delta < \gamma + 2\pi$, such that $|a| \equiv 1$ on J. Then $f^*(J)$ is either a singleton or a concave curve with at most countably many points of convexity which do not accumulate in $f^*(J)$.*

Proof. The fact that $f^*(J)$ is a singleton or a concave curve comprises [2, Cor. 2.8]. Suppose that there exists a sequence $\{e^{it_n}\}_{n=1}^{\infty}$ of points in J, such that $e^{it_n} \to e^{it_0} \in J$ and each $q_n = f^*(e^{it_n})$ exists and is point of convexity of Ω. Let $I \subset J$ be a compact arc that contains all the points e^{it_n}. Since a is analytic on I, $\Delta_I \arg \sqrt{a(z)} < \infty$. On the other hand, $q_n \to q_0 \in \partial\Omega$ and, by the convexity of Ω, $\alpha(q_n) \to \pi$. Since, by Theorem B(c,d),

$$\Delta_I \arg \sqrt{a(z)} > \sum_n \alpha(q_n),$$

and consequently $\Delta_I \arg \sqrt{a(z)} = \infty$ which yields a contradiction. This completes the proof of Lemma 2. ■

Lemma 3. *Let f be a GRM from the unit disk U onto a bounded convex domain Ω whose dilatation $a \in \mathcal{B}$ continues analytically across an open arc $J = \{e^{it} \colon \gamma < t < \delta\}$, where $\gamma < \delta < \gamma + 2\pi$, such that $|a| \equiv 1$ on J. Assume that f^* has n jumps on J.*

(a) *If n is finite, then*

$$(n-3)\pi < \Delta_J \arg \sqrt{a(e^{it})} < (n+1)\pi. \tag{9}$$

(b) *If n is infinite, then* $\Delta_J \arg \sqrt{a(e^{it})} = \infty$.

Proof. (a) We consider two different cases according to the number of jumps n of f^* on J.

The case $n = 1$. There exists $t_0, \gamma < t_0 < \delta$, such that $f^*(J_1) \equiv q_1 \in \partial\Omega$ and $f^*(J_2) \equiv q_2 \in \partial\Omega$, where $J_1 = \{e^{it} \colon \gamma < t < t_0\}$, $J_2 = \{e^{it} \colon t_0 < t < \delta\}$, $q_1 \neq q_2$, and the line segment $[q_1, q_2]$ is a boundary arc of $f(U) \subset \Omega$. Then, by Theorem B(c,d),

$$0 < \Delta_{J_1} \arg \sqrt{a(e^{it})} \leq \pi \qquad \text{and} \qquad 0 < \Delta_{J_2} \arg \sqrt{a(e^{it})} \leq \pi,$$

and

$$0 < \Delta_J \arg \sqrt{a(e^{it})} = \Delta_{J_1} \arg \sqrt{a(e^{it})} + \Delta_{J_2} \arg \sqrt{a(e^{it})} \leq 2\pi,$$

which proves (9).

The case $2 \leq n < \infty$. Since Ω is convex, there exists a subdivision

$$\gamma = t_0 < t_1 < \ldots < t_{n+1} = \delta$$

of $[\gamma, \delta]$ such that the following hold:

- If $J_i = \{e^{it} \colon t_{i-1} < t < t_i\}$, $1 \leq i \leq n+1$, then $f^*(J_i) \equiv q_i \in \partial\Omega$.
- The points q_i are distinct and their order of appearance on the positively-traversed $\partial\Omega$ is $q_1, q_2, \ldots, q_{n+1}$.
- $f^*(J) = \bigcup_{i=1}^{n} [q_i, q_{i+1}]$ is a simple $n-$sided polygonal line.

Again, by the convexity of Ω,

$$0 < \Delta_{J_1} \arg \sqrt{a(e^{it})} \leq \pi \qquad \text{and} \qquad 0 < \Delta_{J_{n+1}} \arg \sqrt{a(e^{it})} \leq \pi.$$

Let P be the $(n+1)-$sided polygon whose vertices are the points q_i; Obviously, P is a convex polygon in $\overline{\Omega}$. Let $\alpha(q_i)$, $1 \leq i \leq n+1$, be the size of the opening angle at q_i as seen from the inside of $f(U)$. Note that for $2 \leq i \leq n$, $\alpha(q_i)$ is also the size of the opening angle at q_i as seen from the inside of P. By Theorem B(c,d), since J_i is not a singleton,

$$\Delta_{J_i} \arg \sqrt{a(e^{it})} = \alpha(q_i), \qquad 2 \leq i \leq n.$$

Since $\Delta_J \arg \sqrt{a(e^{it})} = \sum_1^{n+1} \Delta_{J_i} \arg \sqrt{a(e^{it})}$, we obtain

$$\sum_2^n \alpha(q_i) \leq \Delta_J \arg \sqrt{a(e^{it})} \leq \sum_2^n \alpha(q_i) + 2\pi.$$

But the sum of the vertex angles of the polygon P is $(n-1)\pi$. Being convex, the vertex angles of P at q_1 and q_{n+1} has size between 0 and π. Hence,

$$(n-3)\pi < \sum_2^n \alpha(q_i) < (n-1)\pi$$

which with the previous double inequality yields (9).

(b) $n = \infty$; this case may happen as with Example 1 below. By Lemma 2, the jumps of f^* do not accumulate in J. Then for every positive integer k, there exists a compact subinterval J_k of J that contains at least k jumps of f^*. Thus, by (a), $\Delta_{J_k} \arg \sqrt{a(e^{it})} > (k-3)\pi$. But $\Delta_J \arg \sqrt{a(e^{it})} \geq \Delta_{J_k} \arg \sqrt{a(e^{it})}$ and k is arbitrary. This proves (b) and completes the proof of Lemma 2. ∎

Proof of Theorem 2. We assume with no loss of generality that $\theta_0 = 0$. Evidently, by Lemma 1, conditions (ai) and (aii) in the statement of Theorem 2 are equivalent.

(a) If f^* is constant on a left (right) interval J of 1, then a continues analytically across J such that $|a| \equiv 1$ on J. Moreover, by Theorem B(b), $\Delta_J \arg \sqrt{a(z)} < 2\pi$. This proves (a).

(b) If $\Delta_J \arg \sqrt{a(z)} < \infty$ for some left (right) interval J of 1, then, by the same argument above, all but finitely many ζ_n satisfy $\operatorname{Im} \zeta_n > 0$ and (5) holds. By invoking Lemma 3, f^* has a finite number of jumps in J which, if they exist, could be avoided by choosing J sufficiently small. In this case, since Ω is convex, f^* is constant on J and (b) follows. This completes the proof of Theorem 2. ∎

The following is an immediate consequence of Theorem 2.

Corollary 2. *Let f be a GRM from U onto a bounded convex domain Ω whose dilatation a is a Blaschke product of form* (2) *such that there exists a left (right) interval J of θ_0 across which a continues analytically with $|a| \equiv 1$ on J. Then a necessary and sufficient condition for f^* to be non-constant on any left (right) interval of θ_0 is that one of the following two conditions holds:*

(i) $\Delta_J \arg \sqrt{a(z)} = \infty$;
(ii) *there exists a subsequence $\{\xi_n\}$ of $\{\zeta_n\}$ converging to $e^{i\theta_0}$ such that* (8) *holds.*

The following corollary follows at once from Theorem 2 b and shows that the 'non-constancy' result of Theorem 1 is sharp.

Corollary 3. *Let f be a GRM from U onto a bounded convex domain Ω whose dilatation a is a Blaschke product of form* (2)*, where $m = 0$, $\operatorname{Im} \zeta_n > 0$ for all n, and*

$$\sum_{1}^{\infty} \frac{1 - |\zeta_n|}{|1 - \zeta_n|} < \infty. \tag{10}$$

Then f^* is constant in some left interval of 1.

Our next result examines the stability of a function f given in Theorem 2 upon the addition of finitely many zeros to its dilatation a. Indeed, we show that f is stable if a satisfies (4) and unstable otherwise.

Corollary 4. *Let a be a Blaschke product whose zeros $\{\zeta_n\}_{n=1}^{\infty}$ accumulate at $e^{i\theta_0}$, and let A be the Blaschke product whose zeros are those of a and an additional value ζ_0. Let f and F be the GRMs of the unit disk onto a bounded convex domain Ω whose dilatations are a and A respectively.*

(a) *If f^* is non-constant on any left or right interval of θ_0 then so is F^*.*
(b) *If f^* is constant on some left (right) interval J of θ_0, then for a suitable choice of ζ_0 the function F admits a jump in J.*

Proof. We may assume that $\theta_0 = 0$.

(a) If f^* is non-constant on any left (right) interval of 1, then, by Theorem 2, the zeros of a satisfy (4), and obviously so do the zeros of A. By Theorem 1, part (a) follows.

(b) Suppose that f^* is constant on some right interval $J = \{e^{it} : 0 < t < \delta\}$, where $0 < \delta < \pi$, of 1. Then f admits a harmonic extension beyond J and a is analytic on J and satisfies $|a(J)| \equiv 1$. Since Ω is convex, then, by Theorem B(e), a jump of F^* necessarily occurs on J if $\Delta_J \arg \sqrt{A(z)} > \pi$. Let $\Delta_J \arg \sqrt{a(z)} = \epsilon > 0$. Now for $\zeta \in U$, where $\arg \zeta > 0$, converging tangentially to 1,

$$\Delta_J \arg \sqrt{\frac{z-\zeta}{1-\bar{\zeta}z}} = \Delta_J \arg(z-\zeta) - \frac{1}{2}\Delta_J \arg z$$

converges to π. In particular, for an appropriate choice ζ_0 of ζ we have

$$\Delta_J \arg \sqrt{\frac{z-\zeta_0}{1-\bar{\zeta_0}z}} > \pi - \epsilon.$$

Then

$$\Delta_J \arg \sqrt{A(z)} > \epsilon + \Delta_J \arg \sqrt{\frac{z-\zeta_s}{1-\bar{\zeta_s}z}} > \pi.$$

This proves (b) for the case assumed. An analogous proof holds if f^* is constant on some left interval J of 1. This proves (b) and the proof of the corollary is complete. ∎

We exhibit now an example of a GRM f from the open unit disc onto itself whose dilatation a is a Blaschke product having zeros that satisfy (9) such that the jumps of the boundary function f^* are generally not entirely dependent on the zeros of the dilatation. This is done by showing that upon merely applying a rigid rotation to a, continuity of f^* at a point may turn into a jump at the same point.

Example 1. Consider the following:

(a) the infinite partition $\pi > t_0 > t_1 > \ldots > \pi/4$, where $\lim_{k\to\infty} t_k = \pi/4$;
(b) the open circular arcs $J_k = \{e^{it} : \pi/2^{k+1} < t < \pi/2^k\}$, $k = 0, 1, \ldots$;

(c) the function f^* defined by $f^*(J_k) = e^{it_k}$, $k = 0, 1, \ldots$, and the symmetry property $f^*(e^{-it}) = \overline{f^*(e^{it})}$;
(d) the function f is Poisson integral of f^*.

Then, by the Radó-Kneser-Choquet Theorem [1, Thm. 1.1], f is a univalent harmonic mapping of U into U that satisfies $f(0) = w_0 \in \mathbb{R}$ and $f_z(0) > 0$. In view of (c) and (d), the symmetry property $f(\overline{z}) = \overline{f(z)}$ holds for f. Consequently, the dilatation a of f is likewise symmetric and its zeros are symmetric about the real axis. Moreover, by Laugesen [10, Thm. 5], the dilatation a is an infinite Blaschke product having the property that for every θ, the radial limit $\lim_{r\to 1^-} a(re^{i\theta})$ exists and has modulus 1. Hence there exist only finitely many zeros of a on every radius of the unit disc. Since f^* is constant on every arc J_k, a continues analytically across J_k with $|a| \equiv 1$ there. Hence, the zeros of a accumulate in the set $\{e^{\pm i\pi/2^k} : k = 0, 1, \ldots\}$. But f^* is constant on intervals left and right of every point $\nu_k = e^{\pm i\pi/2^k}$, $k \neq 0$, which, by Corollary 1, neither side of the diameter of U ending at ν_k contains a sequence of zeros of a converging to ν_k. Hence, every ν_k is not an accumulation point of the zeros of a and, consequently, the zeros of a accumulate only at 1.

Because $\lim_{r\to 1^-} a(r)$ exists and has modulus 1 and a has the symmetry property, we conclude that either $a(z) = a_1(z)$ or $a(z) = a_{-1}(z)$, where

$$a_\eta(z) = \eta \prod_1^\infty \frac{(z-\zeta_n)(z-\overline{\zeta_n})}{(1-\overline{\zeta_n}z)(1-\zeta_n z)}, \qquad \eta = \pm 1.$$

Thus $a_\eta(1) = \eta$. But since $df^*(1) = \sqrt{2}i$, Theorem B(a) yields

$$\operatorname{Im}\Big(\sqrt{a_\eta(1)}\, df^*(1)\Big) = \pm \operatorname{Im}\Big(\eta^{1/2}\sqrt{2}i\Big) = \pm\sqrt{2}\operatorname{Re}\eta^{1/2} = 0.$$

Hence $\eta = -1$. We have thus exhibited a GRM from U onto U with dilatation a_{-1}.

Next, let F be the GRM from the open unit disc onto itself whose dilatation is a_1 and satisfies $F(0) = 0$ and $F_z(0) > 0$. It may be easily verified that $\overline{F(\overline{z})}$ is also a GRM from the open unit disc onto itself whose dilatation is $\overline{a_1(\overline{z})} = a_1(z)$ and which is normalized at the origin exactly like F. Then, by [4, Thm. 1], F satisfies the symmetry property $F(\overline{z}) = \overline{F(z)}$. Because infinitely many zeros of a lie on either side of the real axis, by Corollary 1, F^* is non-constant on any left or right interval of 1. Moreover, F^* is continuous at 1 or else it has a jump of pure imaginary size iC, where $C > 0$. Then, again by Theorem B(a), we obtain

$$\operatorname{Im}\Big(\sqrt{a(1)}dF^*(1)\Big) = \pm\operatorname{Im}(iC) = \pm C = 0.$$

Thus F^* is continuous at 1.

We sum-up the results of the paper regarding a GRM f from the open unit disc onto a bounded convex region whose dilatation a is an inner function as follows.

The dilatation could be replaced by an appropriate Blaschke product whose zeros ζ_n, $n = 1, 2, \ldots$. If $\lim_{r\to 1^-} a(re^{i\theta_0})$ exists and has modulus not equal to 1 or fails to exist, then f^* turns out to be continuous at $e^{i\theta_0}$, and if

$$\sum_n \frac{1-|\zeta_n|}{|e^{i\theta_0}-\zeta_n|} = \infty,$$

then f^* turns out to be non-constant on any right or left interval of $e^{i\theta_0}$. However, if

$$\sum_n \frac{1-|\zeta_n|}{|e^{i\theta_0}-\zeta_n|} < \infty,$$

then the constancy of f^* on either side of $e^{i\theta_0}$ is completely determined based on the location of the zeros ζ_n relative to the radius $[0, e^{i\theta_0}]$. This is unlikely with the continuity of f^* at $e^{i\theta_0}$ which cannot be attained by a general distribution condition on the zeros of a: Corollary 1 offers a case where a specific choice of the zeros of a yield continuity of f^* at some point and Example 1 shows that criteria for the continuity of f^* cannot in general be given by merely considering the zeros of a.

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