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Problems and Conjectures in Planar Harmonic Mappings

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Abstract. Planar harmonic mappings underly the theory of minimal surfaces in three space. The seminal paper [13] introduced a complex analytic approach for their studies. Ever since this approach has become an extensive field of research. These problems and conjectures were proposed by many colleagues throughout the past quarter of a century.

Keywords. harmonic mapping, analytic dilatation, convex domains, elliptic differential equation, nonparametric minimal surfaces.

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1. Introduction

A *harmonic mapping* f of a complex region G is a complex-valued function that satisfies Laplace's equation

$$\Delta f \equiv f_{xx} + f_{yy} = 0.$$

This function can be written as

$$f(z) = u(x, y) + iv(x, y), \quad z = x + iy,$$

where u and v are real-valued harmonic functions, and

$$f(z) = h(z) + \overline{g(z)}, \tag{1.1}$$

where h and g are analytic functions which are single-valued if G is simply-connected and possibly multiple-valued if G is otherwise. In the former case, the *second complex dilatation* is the meromorphic function $a = g'/h'$ or $a \equiv \infty$. It is known that $|a| < 1$ in G if and only if f is open and sense-preserving, and $|a| > 1$ in G if and only if f is open and sense-reversing.

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In the sequel, let $\mathbb{C}$, $\mathbb{D}$, and $\mathbb{T}$ denote the complex plane, the open unit disk, and the unit circle respectively.

Let $f:\ \mathbb{D} \to \mathbb{C}$ be a harmonic mapping. The aforementioned representation (1.1) is unique if it is assumed that $g(0) = 0$. Such functions admit the power series representation

$$f(z) = \sum_{n=-\infty}^{-1} c_n(f)\overline{z}^n + \sum_{0}^{\infty} c_n(f) z^n.$$

The mapping f is said to belong to the class S_H if it is univalent, sense-preserving on $\mathbb{D}$ and normalized by $c_0(f) = c_1(f) - 1 = 0$. If in addition, $c_{-1}(f) = 0$, then f belongs to the class S_H^0.

2. Harmonic mappings on Simply connected domains

Two approaches for the study of harmonic mappings on $\mathbb{D}$ are available. Let $f^*(e^{i\theta})$ be a Lebesgue integrable function on $\mathbb{T}$. Then the Poisson integral

$$f(z) = P[f^*] = \frac{1}{2\pi}\int_{-\pi}^{\pi} P(r, \varphi - \theta) f^*(e^{i\varphi})\, d\varphi, \quad z = re^{i\theta} \in \mathbb{D}, \tag{2.1}$$

where $P(r,t)$ is the *Poisson kernel* of $\mathbb{D}$, is a harmonic mapping of $\mathbb{D}$ whose unrestricted limit at every continuity point $e^{i\theta_0}$ of f^* is $f^*(e^{i\theta_0})$.

Open sense-preserving harmonic mappings of $\mathbb{D}$ also arise as solutions of linear elliptic partial differential equations of the form

$$\overline{f_{\overline{z}}}(z) = a(z) f_z(z), \quad z \in \mathbb{D}, \tag{2.2}$$

where a is an analytic function from $\mathbb{D}$ into itself; note that a is indeed the dilatation of f.

In the first approach, if f^* is a homeomorphism between $\mathbb{T} = \partial\mathbb{D}$ and the boundary of a bounded simply connected domain Ω and if $f(\mathbb{D}) \subset \Omega$, then the *Rado-Kneser-Choquet Theorem* [17, pp. 29-30] asserts that f is a univalent sense-preserving harmonic mapping of $\mathbb{D}$ onto Ω.

In the second approach, if $|a| < k < 1$ in $\mathbb{D}$, then it is classical that the existence of the *Riemann Mapping (RM)* of equation (2.2) holds; namely, for a given bounded simply connected domain Ω containing a point w_0 having a locally connected boundary, there exists a univalent solution f of (2.2) that satisfies $f(0) = w_0$ and $f_z(0) > 0$ and maps $\mathbb{D}$ onto Ω. If in addition Ω is a Jordan domain, then f extends to a homeomorphism from $\overline{\mathbb{D}}$ onto $\overline{\Omega}$. Some satisfactory generalizations of the above theorems do exist when f^* need not be continuous, one-to-one, or satisfies $\|a(z)\|_\infty = 1$.

Theorem A. [22] Let Ω be a bounded simply connected domain whose boundary $\partial\Omega$ is locally connected. Suppose that $a(\mathbb{D}) \subset \mathbb{D}$ and w_0 is a fixed point of Ω. Then there exists a univalent solution f of (1.1) having the following properties:

(a) $f(0) = w_0, f_z(0) > 0$ and $f(\mathbb{D}) \subset \Omega$.
(b) There is a countable set $E \subset \mathbb{T}$ such that the unrestricted limits $f^*(e^{it}) = \lim_{z \to e^{it}} f(z)$ exist on $\mathbb{T} \setminus E$ and they are on $\partial\Omega$.
(c) The functions
$$f^*_-(e^{it}) = \operatorname{ess}\lim_{s \uparrow t} f^*(e^{is}) \quad \text{and} \quad f^*_+(e^{it}) = \operatorname{ess}\lim_{s \downarrow t} f^*(e^{is})$$
exist on $\mathbb{T}$, belong to $\partial\Omega$ and are equal on $\mathbb{T} \setminus E$.
(d) The cluster set of f at $e^{it} \in E$ is the straight line segment joining $f^*_-(e^{it})$ to $f^*_+(e^{it})$.

The mapping f is termed a *Generalized Riemann Mapping (GRM) from* $\mathbb{D}$ *onto* Ω.

A natural generalization of the classical class S of normalized univalent functions on $\mathbb{D}$ is the class S_H of sense-preserving univalent harmonic mappings on $\mathbb{D}$ normalized by $h(0) = g(0) = h'(0) - 1 = 0$. However this class is not compact and the subclass S_H^0 where $g'(0) = 0$ is compact and often more appropriate for our purposes.

3. Open Problems

Problem 3.1. (W. Hengartner) Characterize unbounded domains $f(\mathbb{D})$ where f is a univalent harmonic mapping on $\mathbb{D}$ and the dilatation $a(z)$ is a finite Blaschke product.

Reference: [4].

Problem 3.2. **a)** (J. Clunie and T. Sheil-Small) For $f \in S_H^0$, find the best bound for $|c_2(f)|$. It is conjectured to be $5/2$.

b) (R. S. Laugesen) One may ask the more general question: If for $f \in S_H^0$ the inequality $|c_2(f)| \leq 2 + ||a||_\infty/2$ holds, then are the extremal functions f the rotations of the Koebe-type harmonic function $k(z)$ defined by $h(z) - g(z) = z/(1-z)^2$ and $a(z) = ||a||_\infty z$.

c) (J. Clunie and T. Sheil-Small) For $f \in S_H^0$, it is conjectured that
$$\big||c_n| - |c_{-n}|\big| \leq n.$$

d) (J. Clunie and T. Sheil-Small) For $f \in S_H^0$, it is conjectured that the disk $\{w : \ |w| < 1/6\} \subset f(\mathbb{D})$.

e) (J. Clunie and T. Sheil-Small) For $f \in S_H^0$, it is proposed that the lower bound for the inner mapping radius of $f(\mathbb{D})$ is $2/3$.

f) (J. Clunie and T. Sheil-Small) For $f \in S_H$, it is proposed that the upper bound for the inner radius is of $f(\mathbb{D})$ is $\pi/2$.

g) (J. Clunie and T. Sheil-Small) For $f \in S_H$, it is proposed that the sharp coefficient bounds for f are $|c_n(f)| \le (2n^2+1)/3$.

Remark: Partial results for subclasses exist as well as estimates for e) and f).

References: [13], [16].

Problem 3.3. **a)** (T. Sheil-Small) For $f \in S_H$, the proposed radius of convexity is $3-\sqrt{8}$.

b) (P. L. Duren) Find the radius of starlikeness for starlike mappings in S_H.

c) (P. L. Duren) Find the radius of convexity for harmonic isomorphisms of $\mathbb{D}$.

d) (T. Sheil-Small) For $f = h + \overline{g} \in S_H$, the proposed radius of univalence for h is $1/\sqrt{3}$.

Remarks:

1) The radius of convexity for close-to-convex mappings in S_H is $3-\sqrt{8}$.

2) The radius of convexity for convex mappings in S_H is $\sqrt{2}-1$.

3) The suggested extremal function for d) is the Koebe-type mapping for which $h(z) = (z + z^3/3)/(1-z)^2$.

References: [13], [38], [39].

Problem 3.4. (W. Hengartner) Let Ω be a simply connected Jordan domain convex in the horizontal direction. We further assume that $\partial\Omega$ is the union of two connected arcs, one that is convex and another that is concave with respect to Ω. Determine the maximum valency of the harmonic extension of a homeomorphism from $\mathbb{T}$ onto $\partial\Omega$.

Problem 3.5. Let $f : \mathbb{D} \xrightarrow{onto} \mathbb{D}$ denote a univalent harmonic mapping whose complex dilatation function is $a(z)$.

a) (A.Weitsman) Is there an f whose dilatation a is an infinite Blaschke product?

b) (R. S. Laugesen) Find conditions on the boundary values of f such that its dilatation a is an infinite Blaschke product.

c) (R. S. Laugesen) Find conditions on the boundary values of a harmonic mapping f such that its dilatation a is a singular inner function.

Remarks:

1) If the image domain in a) is some bounded convex domain then the answer is yes.

2) For b) to hold, the accumulation points of the set of zeros of $a(z)$ should be $\mathbb{T}$. On the other hand, Laugesen gave an example of an inner function $a(z)$ for which b) holds.

3) Let E be a closed countable subset of $\mathbb{D}$. Suppose that f jumps at each point of E and is constant on each component of $\mathbb{D} \setminus E$, then a is a Blaschke product.

References: [10] , [32].

Problem 3.6. (T. Sheil-Small) Does there exist a circle mapping $f(e^{it}) = e^{i\phi(t)}$ where $\phi(t)$ is a non decreasing function such that $\phi(2\pi) - \phi(0) = 4\pi$ whose harmonic extension f is beyond an arbitrary large valency.

Remark: There exist circle mappings of the designated type whose harmonic extensions f are $6-$ and $8-$valent. It seems that this result is true for any finite valency.

Reference: [7].

Problem 3.7. (A. Wilmshurst) It is conjectured that if $f = p + \overline{q}$ is a harmonic polynomial where the degree of p is n and the degree of q is m, where $1 \leq m < n - 1$, then f has at most $m(m-1) + 3n - 2$ zeros.

Remark: True for m=1.

References: [40], [28].

Problem 3.8. (P. L. Duren) Let f be a RM or GRM associated with a dilatation function a.

a) Let $||a(z)||_\infty \leq k < 1$ and Ω be a Jordan domain. Is the RM unique ?

b) Let $||a(z)||_\infty < 1$. The existence proof of the GRM is nonconstructive and difficult. Find a constructive proof.

c) Let $||a(z)||_\infty < 1$ and Ω a Jordan domain. Is the GRM unique?

Remark: If Ω is a symmetric domain then a) is true, and if it is a strictly starlike domain then c) holds true even when $||a||_\infty \leq 1$.

References: [2], [8], [6], [22], [19].

Problem 3.9. (P. L. Duren) Let f be a univalent harmonic self-mapping of $\mathbb{D}$ whose dilatation a is a square of an analytic function. It is conjectured that the minimum of $|c_1(f)| + |c_{-1}(f)|$ is attained for the mapping that maps $\mathbb{D}$ onto a circumscribed square.

Problem 3.10. (W. Hengartner)) Let $f_1, f_2, ..., f_n$ be n complex-valued pluri-harmonic functions defined on the unit ball $\mathbb{B}$ of C^n.

a) Is it true that $f = (f_1, f_2, ..., f_n)$ is locally univalent if and only if its Jacobian does not vanish ?

b) If $f = (f_1, f_2, ..., f_n)$ is a homeomorphism from $\partial\mathbb{B}$ onto $\partial\mathbb{B}$. Then is it true that f is a homeomorphism from $\mathbb{B}$ onto $\mathbb{B}$?

Remarks:

1) For n = 1 the answer to a) and b) is yes.

2) If $f_1, f_2, ..., f_n$ are a complex-valued harmonic functions on $\mathbb{B}$ then the answer to a) and b) is no.

3) If $f_1, f_2, ..., f_n$ are analytic on $\mathbb{B}$ then the answer to a) is yes.

References: [29], [31], [33], [37].

Problem 3.11. (W. Hengartner) Let $f = p\overline{q}$, where p is an analytic polynomial of degree n and q is an the analytic polynomial of degree m, and let $q(z) \neq$ *const* $p(z)$. Find a sharp upper bound for the valency of f.

References: [5], [1]

Problem 3.12. (D. Bshouty) Let Σ_H be the class of normalized univalent harmonic mappings $f(z)$ defined on the exterior of the unit disk, Δ, and normalized at infinity by $f(z) = z + O(1)$. Find Grunsky type inequalities for the class Σ_H.

Remark: There exists an area theorem for the class.

Reference: [35].

Problem 3.13. (A. Weitsman) Let $\big(X_1(z), X_2(z), X_3(z)\big)$, $z \in \mathbb{D}$, be a Wierstrass representation of a minimal surface which may have self-intersections. Geometers call this a conformal minimal immersion of $\mathbb{D}$. Then $f(z) = X_1(z) + iX_2(z)$ is a harmonic mapping where $a(z)$ is the square of a meromorphic function in $\mathbb{D}$. It is an open problem weather a conformal minimal immersion of $\mathbb{D}$ can be proper. This would mean that

$$X_1^2(z) + X_2^2(z) + X_3^2(z) \to \infty \text{ as } |z| \to 1.$$

It is not hard to construct a harmonic mapping $f(z)$ which is proper, i.e. $|f(z)| \to \infty$ as $|z| \to 1$ but to do this and make $a(z)$ the square of a meromorphic function is another matter.

Problem 3.14. (A. Lyzzaik) If

$$\Re\left\{1+z\frac{h''}{h'}\right\}>-\frac{1}{2}$$

and $g' = zh'$, then $f = h + \lambda\overline{g}$, $|\lambda| = 1$ is close to convex. What is the maximum valency of f if $g' = z^2h'$?

Reference: [9].

Problem 3.15. (J. C. C. Nitsche, D. Kalaj) Let $f(z)$ be a univalent mapping of $\mathbb{D}$ onto itself satisfying $f(0) = 0$. Find the sharp bound for

$$\inf_{z\in\mathbb{D}}(|f_z|^2+|f_{\overline{z}}|^2).$$

Remarks:

1) The sharp bound at the origin is known.

2) The bound $1/\pi^2$ is known.

3) The conjecture is $2/\pi^2$.

4)The result was generalized for mappings onto convex domains.

References: [21], [20], [26], [27].

Problem 3.16. (A. Lyzzaik) Set $\mathbb{T}_\rho = \{z:\ |z| = \rho\}$ and $\mathbb{A}_\rho = \{z:\ \rho < |z| < 1\}$ for $0 < \rho < 1$. Let Ω be a bounded convex domain, $w_0 \in \Omega$ and f is a univalent harmonic mapping from $\mathbb{A}_\rho$ onto $\Omega \setminus w_0$ that extends across $\mathbb{T}_\rho$ continuously. It is known that

$$f(z) = h(z) - h(\rho^2/\overline{z}) + w_0 + c\log(|z|/\rho),$$

where h is an analytic function in $\mathbb{A}_{\rho^2}$. It was also shown that h is close-to-convex function of $\mathbb{A}_{\rho^2}$ in the sense that $h = H \circ \phi$, where H is a close-to-convex function of $\mathbb{D}$ and $\phi : \mathbb{A}_{\rho^2} \to \mathbb{A}_0$ is a homeomorphism. The conjecture now is: h is a convex function in the same sense.

References: [3], [34].

Problem 3.17. (A. Lyzzaik) Let Λ be a multiply connected domain bounded by the Jordan curves $\alpha_1, \alpha_2, ..., \alpha_n$ with α_1 forming the outer boundary, and let Ω be a Jordan domain. Suppose that f^* is a sense-preserving weak homeomorphism between α_1 and $\partial\Omega$. Find sufficient conditions (on Ω and /or f) that yield a harmonic extension f of f^* which maps Λ homeomorphically onto Ω minus $n-1$ points.

Remark:

1) If Ω is convex then such an extension always exists.

2) If Λ is an annulus then such an extension exists.

References: [18], [3], [34].

Problem 3.18. (S. Ponnusamy) For $M \geq 1$ we set $\mathbb{D}_M = \{z : |z| < M\}$. For a normalized harmonic mapping $f : \mathbb{D} \to \mathbb{D}_M$ such that $f(0) = J_f(0) - 1 = 0$, it is conjectured that

a) $|c_n| + |c_{-n}| \leq M - 1/M, \ \ n > 2.$

b) $|f_z| + |f_{\bar{z}}| \leq \frac{4}{\pi}(M - 1/M)/(1 - |z|^2).$

References: [12], [14].

Problem 3.19. (A. Lyzzaik, D. Bshouty and A. Weitsman) Let $a(z)$ be a Blaschke product and f the GRM from $\mathbb{D}$ onto $\mathbb{D}$ associated with a. Let $f^*(e^{it})$ denote the radial boundary values of f. If $\frac{df^*}{dt}(e^{i\theta})$ exists, is it true that a has finitely many zeros in any Stolz angle at $e^{i\theta}$?

Remark: If $|\frac{df^*}{dt}(e^{i\theta})| \leq c$, where c is a specific constant, then the result is true.

Reference: [11].

Problem 3.20. (T. Iwaniec, L.V. Kovalev and J. Onninen) (Generalized Nitsche bound) Let $\mathbb{A} = \mathbb{A}(r, R) = \{z : r < |z| < R\}$ and $\mathbb{A}^* = \mathbb{A}(r_*, R_*)$ be a pair of circular annuli. Suppose that $f \colon \mathbb{A} \to \mathbb{A}^*$ is a harmonic mapping not homotopic to a constant within the class of continuous mappings from $\mathbb{A}$ to $\mathbb{A}^*$. Then

$$\frac{R_*}{r_*} \geq \frac{1}{2}\left(\sqrt{\frac{R}{r}} + \sqrt{\frac{r}{R}}\right). \tag{3.1}$$

If f is in addition injective, then

$$\frac{R_*}{r_*} \geq \frac{1}{2}\left(\frac{R}{r} + \frac{r}{R}\right). \tag{3.2}$$

Remark: The mapping $f(z) = z + 1/\bar{z}$ and the domain $\mathbb{A} = \mathbb{A}(1/R, R)$ turn (3.1) into an equality. The mapping f also shows the sharpness of (3.2) when restricted to the annulus $\mathbb{A}(1, R)$.

Reference: [23].

Problem 3.21. (T. Iwaniec, L.V. Kovalev and J. Onninen) The *affine Modulus* of a doubly connected domain $\Omega \subset \mathbb{C}$ is defined by

$$\mathrm{Mod}_{@}\Omega = \sup\{\mathrm{Mod}\ \phi(\Omega) : \ \ \phi : \mathbb{C} \to \mathbb{C} \text{ affine}\}.$$

Let Ω and Ω^* be doubly connected domains in $\mathbb{C}$ such that

$$\mathrm{Mod}_{@}\Omega^* > \mathrm{Mod}\Omega. \tag{3.3}$$

Then there exists a harmonic homeomorphism $f\colon \Omega \to \Omega^*$ unless $\mathbb{C} \setminus \Omega^*$ is bounded. In the latter case there is no such f. Does equality in (3.3) (with both sides finite) suffice for the existence of f?

Reference: [25].

Problem 3.22. (T. Iwaniec, L.V. Kovalev and J. Onninen) Let f be a harmonic K-quasiconformal homeomorphism of $A(1,R)$ onto $A(1,R_*)$. It is conjectured that

$$R_* \le \frac{k+1}{2}R - \frac{K-1}{2}\frac{1}{R}.$$

Equality is attained, uniquely modulo conformal automorphisms, for

$$f(z) = \frac{k+1}{2}z - \frac{K-1}{2}\frac{1}{\bar{z}}.$$

Remark: The corresponding lower bound is known.

Reference: [24].

Problem 3.23. (T. Iwaniec, L.V. Kovalev and J. Onninen) It is conjectured that if f is a harmonic homeomorphism from a doubly connected domain Ω onto Ω^* then

$$\mathrm{Mod}_{@}\Omega^* \ge \log\cosh \mathrm{Mod}(\Omega).$$

Remark: If true then it is sharp for Ω^* a circular ring.

Reference: [25].

Problem 3.24. (T. Iwaniec, L.V. Kovalev and J. Onninen) We write $\Omega_1 \tilde{\hookrightarrow} \Omega_2$ when Ω_1 is a domain contained in a doubly connected domain Ω_2 in such a way that Ω_1 separates the boundary components of Ω_2. The monotonicity of the modulus can be expressed by saying that $\Omega_1 \tilde{\hookrightarrow} \Omega_2$ implies Mod $\Omega_1 \le$ Mod Ω_2 and $\mathrm{Mod}_{@}\ \Omega_1 \le \mathrm{Mod}_{@}\ \Omega_2$.

a) (Domain Comparison Principle) Let Ω and Ω^* be doubly connected domains such that Mod $\Omega < \infty$ and there exists a harmonic homeomorphism $f : \Omega \xrightarrow{\text{onto}} \Omega^*$. If $\Omega_0 \tilde{\hookrightarrow} \Omega$, does there exist a harmonic homeomorphism $f_0 : \Omega_0 \xrightarrow{\text{onto}} \Omega^*$.

b) (Target Comparison Principle) Let Ω and Ω^* be doubly connected domains such that there exists a harmonic homeomorphism $f : \Omega \xrightarrow{\text{onto}} \Omega^*$. If Mod $\Omega_0^* < \infty$ and $\Omega^* \tilde{\hookrightarrow} \Omega_0^*$, does there exist a harmonic homeomorphism $h_0 : \Omega \xrightarrow{\text{onto}} \Omega_0^*$.

Reference: [25].

Problem 3.25. (N. T. Koh and L.V. Kovalev) Let the Lebesgue area of a set E be denoted by $|E|$.

a) Let $f : \mathbb{D} \xrightarrow{\text{into}} \mathbb{D}$ be a univalent harmonic mapping and let $\mathbb{D}_r = \{z : |z| < r\}$, then

$$|f(\mathbb{D}_r)| \leq |\mathbb{D}_r|.$$

b) Is the above true for any harmonic mappings of that type?

Remark: True for univalent harmonic mappings of $\mathbb{D}$ onto $\mathbb{D}$.

Reference: [KK].

Problem 3.26. (M. Dorff, M. Nowak and Woloszkiewicz)

a) Let

$$f_0(z) = \frac{z - z^2/2}{(1-z)^2} + \overline{\frac{z^2/2}{(1-z)^2}}$$

and $f = h + \overline{g}$ be in K_H^0, the subclass of convex mappings in S_H^0, with $h(z)+g(z) = z/(1-z)$ and dilatation $a(z) = (z+\alpha)/(1+\alpha z)$ with $\alpha \in (-1, 1)$. Then $f_0 * f \in S_H^0$ and is convex in the direction of the real axis. Determine other values of $\alpha \in \mathbb{D}$ for which the previous result holds.

b) Consider other right half-plane mappings f_n formed by shearing $h_n(z) - g_n(z) = z/(1-z)$ with dilatations $a_n(z) = e^{i\theta} z^n$. Determine the values of n for which $f_n * f$ are univalent.

Reference: [15]

Problem 3.27. (M. Dorff, M. Nowak and Woloszkiewicz)

a) Determine what and how many fundamentally different (i.e., not rotations or not scalings) images can be constructed when taking the convex combination of two harmonic n-gon maps. Extend this to convex combinations of minimal graphs.

b) Determine which combinations are possible and what images can be constructed when taking the convex combination of a harmonic m-gon and n-gon, where $m < n$. Extend this to convex combinations of minimal graphs.

Problem 3.28. (M.Vuorinen) Find the best constant $c(K)$ such that for every quasiconformal harmonic map f from $\mathbb{D}$ onto $\mathbb{D}$ and $f(0) = 0$ we have

$$|f(x)| \geq \frac{|x|}{c(K)}.$$

Remark: Estimates for $c(K)$ exist with the property that $c(K) \to 1$ when $K \to 1$.

References: [27], [36].

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