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Local Decomposition of Planar Harmonic Mappings

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Abstract. A necessary and sufficient condition is given for a planar harmonic mapping f to be locally decomposable as a univalent harmonic mapping F of an analytic function φ.

Keywords. Harmonic functions, harmonic mappings, analytic functions, univalent functions, complex dilatation.

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A *planar harmonic mapping*, or simply a *harmonic mapping*, f of a region $\mathcal{D}$ is a complex-valued function of the form $f = u + iv$, where u and v are real-valued harmonic functions of $\mathcal{D}$. It also takes the form $f = h + \overline{g}$, where h and g are analytic functions in $\mathcal{D}$ that are unique up to additive constants and single-valued if $\mathcal{D}$ is simply connected and possibly multiple-valued otherwise. The functions h and g are called the *analytic* and *co-analytic* parts of f respectively. Evidently, analytic and anti-analytic functions are harmonic mappings.

Every harmonic mapping of an analytic function is a harmonic mapping, but an analytic function of a harmonic mapping is not necessarily harmonic. If a harmonic mapping f is (locally) injective, then it is termed (locally) univalent. We use the term *conformal map* for a univalent analytic function.

The *Jacobian* and *second complex dilatation* of f are the functions $J = |h'|^2 - |g'|^2$ and $\omega = g'/h'$ respectively. Note that J and ω are single-valued functions on $\mathcal{D}$ whether h and g are single-valued or not. A result of Lewy [5] states that f is locally univalent at a point in $\mathcal{D}$ only if its Jacobian J is not zero there. Note that ω is either a non-constant meromorphic function or a (possibly infinite) constant in $\mathcal{D}$.

In [6] A. Lyzzaik asserts that f is an open map in $\mathcal{D}$ if and only if $|\omega| \neq 1$. In this case, the connectedness of $\mathcal{D}$ yields $|\omega| < 1$ and f is sense-preserving in $\mathcal{D}$, or $|\omega| > 1$ and f is sense-reversing in $\mathcal{D}$. Since $\overline{f}$ and f admit opposite orientations,

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we assume without loss of generality that $|\omega| < 1$ or, equivalently, f is sense-preserving. Under this assumption J will be positive in $\mathcal{D}$ with the possible exception of a set of isolated points; namely the common zero derivatives z_0 of h and g where the order of h' is at most that of g'.

Definition 1. Let $f = h + \overline{g}$ be a sense-preserving harmonic mapping of a region $\mathcal{D}$. A point $z_0 \in \mathcal{D}$ is called a *critical point* of f of *order* k if it is a zero of h' and g' of respective orders k and ℓ such that $k \leq \ell$.

Motivated by the fact that a harmonic mapping of an analytic function is a harmonic mapping and that a composition of sense-preserving functions is a sense-preserving function, P. Duren and W. Hengartner [3] gave a necessary and sufficient condition for a sense-preserving harmonic mapping f of a region $\mathcal{D}$ to have a decomposition $f = F \circ \varphi$ for some analytic function φ of $\mathcal{D}$ and some sense-preserving univalent harmonic mapping F of $\varphi(\mathcal{D})$; see P. Duren [4, pp. 149–151]. We state the above result in the following theorem.

Theorem A (P. Duren and W. Hengartner)**.** *Let f be a sense-preserving harmonic mapping of a region $\mathcal{D}$ with dilatation ω. A necessary and sufficient condition for f to have a decomposition $f = F \circ \varphi$ for some analytic function φ of $\mathcal{D}$ and some sense-preserving univalent harmonic mapping F of $\varphi(\mathcal{D})$ is that $\omega(z_1) = \omega(z_2)$ wherever $f(z_1) = f(z_2)$.*

Under this condition the decomposition of f is unique up to conformal mappings; that is, if f has another decomposition $f = \tilde{F} \circ \tilde{\varphi}$, then $F = \tilde{F} \circ \psi$ and $\varphi = \psi^{-1} \circ \tilde{\varphi}$ for some conformal mapping ψ of $\varphi(\mathcal{D})$.

The proof of the necessity part of Theorem A is immediate and comprises the proof of the implication $(a) \Rightarrow (b)$ of the next Theorem 1. The proof of sufficiency, however, is more involved and uses the Existence and Uniqueness Theorem of the Beltrami equation. Our proofs of the local decomposition stated in Theorem 1 do not use the Beltrami equation.

In view of Theorem A, P. Duren and W. Hengartner stated: "it seems likely that a closer study of the given harmonic function (in the theorem) will lead to a necessary and sufficient condition for the existence of a local decomposition". The purpose of this article is to assert the truth of the foregoing statement. In particular, we prove the following result.

Theorem 1. *Let f be a sense-preserving harmonic mapping of a region $\mathcal{D}$ with dilatation ω, and let $z_0 \in \mathcal{D}$. Then the following three statements are equivalent.*

(a) *f has a decomposition $f = F \circ \varphi$ for some analytic function φ of some open neighborhood U of z_0 and some sense-preserving univalent harmonic mapping F of $\varphi(U)$.*
(b) *There exists some open neighborhood W of z_0 in which $\omega(z_1) = \omega(z_2)$ wherever $f(z_1) = f(z_2)$.*

(c) *There exists some open neighborhood V of z_0 in which a single-valued analytic root $\tilde{\varphi}$ of $\omega - \omega_0$, $\omega(z_0) = \omega_0$, satisfies the property that $\tilde{\varphi}(z_1) = \tilde{\varphi}(z_2)$ if and only if $f(z_1) = f(z_2)$.*

If these statements hold, then f has the decomposition $f = \tilde{F} \circ \tilde{\varphi}$ in V which is unique up to conformal mappings; that is, given the decomposition in (a), there exists a conformal map ψ from $\varphi(U \cap V)$ onto $\tilde{\varphi}(U \cap V)$ such that $F = \tilde{F} \circ \psi$ and $\varphi = \psi^{-1} \circ \tilde{\varphi}$ in $\varphi(U \cap V)$ and $U \cap V$ respectively.

We call $f = \tilde{F} \circ \tilde{\varphi}$ the *canonical decomposition* of f at z_0. By a *single-valued root* $\tilde{\varphi}$ of $\omega - \omega_0$ we mean a function which satisfies $\omega - \omega_0 = (\tilde{\varphi})^r$ for some positive integer r.

We illustrate Theorem 1 by the following two examples. In Example 1, the theorem is used to establish the canonical decomposition, if it exists, of a harmonic mapping at a given critical point.

Example 1. Let f be the harmonic mapping

$$f(z) = \sin^2(2z) - \sin^2 \overline{z}$$

whose dilatation is $\omega(z) = 1/4 \sec(2z)$. It can be verified that near the origin $|\omega| < 1$ and f is sense-preserving. Moreover, f and ω have a critical point at the origin of the same order 1.

Assuming that f has the canonical decomposition $f = \tilde{F} \circ \tilde{\varphi}$ at the origin, then $\eta = \tilde{\varphi}(z) = \omega(z) - 1/4$ is an analytic function at the origin with a critical point there of order 1. By investigating the analytic and co-analytic parts of f in relation to $\tilde{\varphi}$, we find that both are functions of $\tilde{\varphi}$ in a manner leading to the desired harmonic mapping $\tilde{F}$; namely,

$$\tilde{F}(\eta) = 1 - \frac{1}{(4\eta + 1)^2} + \frac{2\,\overline{\eta}}{2\,\overline{\eta} + 1},$$

which is locally univalent at the origin since its Jacobian there is non-zero.

Note that f also has the same canonical decomposition $f = \tilde{F} \circ \tilde{\varphi}$ at every point $k\pi$, where k is an integer, since f and $\tilde{\varphi}$ are both periodic functions with period π.

Example 2 uses Theorem 1 to decide whether a harmonic mapping has the desired decomposition or not near a given critical point.

Example 2. Let f be the harmonic mapping

$$f(z) = z^2 - \frac{2}{3} z^3 + z^4 + \frac{1}{2} \overline{z}^2 - \frac{1}{3} \overline{z}^3 + \overline{z}^4 - \frac{2}{5} \overline{z}^5 + \frac{2}{3} \overline{z}^6$$

whose dilatation is $\omega(z) = z^2 + 1/2$. It can be verified that near the origin $|\omega| < 1$ and f is sense-preserving. Moreover, f and ω have a critical point at the origin of the same order 1.

If f has a decomposition of the required type, then, by Theorem 1, its canonical decomposition $f = \tilde{F} \circ \tilde{\varphi}$ near the origin has $\tilde{\varphi}(z) = z^2$ since $\tilde{\varphi}$ and f have the same valency near the origin. But this would imply that both the analytic and co-analytic parts of f are, like $\tilde{\varphi}$, even functions near the origin which yields a contradiction. Hence, f does not decompose as desired near the origin.

The proof of Theorem 1 requires two lemmas. We first state a requisite notion for the first lemma.

Definition 2. Let f be a complex-valued function of a region Ω, and let $z_0 \in \Omega$. The notation $f_{z_0} \sim z^n$ means that there exist an open neighborhood $U \subset \Omega$ of z_0 and sense-preserving homeomorphisms $h_1 \colon U \to (|\zeta| < 1)$ and $h_2 \colon \mathbb{C} \to \mathbb{C}$ such that $h_1(z_0) = h_2 \circ f(z_0) = 0$ and

$$h_2 \circ f \circ h_1^{-1}(\zeta) = \zeta^n, \qquad |\zeta| < 1.$$

The first lemma describes the local behavior of sense-preserving harmonic mappings near critical points. It shows that this behavior is topologically similar to that of an analytic function near a critical point.

This behavior extends to pseudo-analytic and anti-pseudo-analytic functions of the second kind; see [6] and [2]. Thus we are content to state Lemma 1 without proof as follows.

Lemma 1. *Let f be a sense-preserving harmonic mapping of a region $\mathcal{D}$, and let $z_0 \in \mathcal{D}$ be a critical point of f of order k. Then $f_{z_0} \sim z^{k+1}$.*

By virtue of this lemma, we can find $\epsilon > 0$ and a Jordan domain $V \subset \mathcal{D}$ containing z_0 such that for any given $0 \le \vartheta < 2\pi$ there exist exactly $k+1$ Jordan arcs $\alpha_{\vartheta,1}, \alpha_{\vartheta,2}, \ldots, \alpha_{\vartheta,k+1}$ which satisfy the following properties:

($\mathbf{P}_{\vartheta,1}$) each arc $\alpha_{\vartheta,j}$ starts from z_0 and ends in ∂V;
($\mathbf{P}_{\vartheta,1}$) each arc $\alpha_{\vartheta,j}$ maps under f homeomorphically onto the line segment $d_\vartheta \colon f(z_0) + te^{i\vartheta}$, $0 \le t \le \epsilon$;
($\mathbf{P}_{\vartheta,1}$) the arcs $\alpha_{\vartheta,j}$ are mutually disjoint except for z_0;
($\mathbf{P}_{\vartheta,1}$) the arcs $\alpha_{\vartheta,j}$ read in the order $\alpha_{\vartheta,1}, \alpha_{\vartheta,2}, \ldots, \alpha_{\vartheta,k+1}$ as V is swept positively about z_0;
($\mathbf{P}_{\vartheta,1}$) the bounded simply connected sub-regions $V_{\vartheta,j}$, $j = 1, 2, \ldots, k+1$, of V bounded by $\alpha_{\vartheta,j}$, $\alpha_{\vartheta,j+1}$, and ∂V, with $\alpha_{\vartheta,k+2} = \alpha_{\vartheta,1}$, and lying on the left-hand side of $\alpha_{\vartheta,j}$ maps under f homeomorphically onto the set $\{w : |w - f(z_0)| \le \epsilon\} \setminus d_\vartheta$.

For convenience, let $\mathbf{P}_\vartheta$ denote the properties $\mathbf{P}_{\vartheta,1}, \mathbf{P}_{\vartheta,2}, \ldots, \mathbf{P}_{\vartheta,5}$.

The second lemma describes, more closely than Lemma 1, the local behavior of a sense-preserving harmonic mapping near a critical point. It asserts the interesting fact that the behavior is not only topologically similar to that of an analytic function near a critical point but analytically as well.

Lemma 2. *Let f be a sense-preserving harmonic mapping of a region $\mathcal{D}$, and let $z_0 \in \mathcal{D}$ be a critical point of f of order k. Then we can find $\epsilon > 0$ and a Jordan domain $V \subset \mathcal{D}$ containing z_0 such that for any given $0 \leq \vartheta < 2\pi$ there exist exactly $k+1$ Jordan analytic arcs $\alpha_{\vartheta,1}, \alpha_{\vartheta,2}, \ldots, \alpha_{\vartheta,k+1}$ which satisfy the properties $\mathbf{P}_\vartheta$ and the additional property that the interior angle of every region $V_{\vartheta,j}$ at z_0 is exactly $2\pi/(k+1)$.*

Remark 1. Lewy's Theorem [5] asserts that if the Jacobian of a harmonic mapping f is zero at a point z_0, then f is not locally univalent at z_0. In this case two things may happen. Either f "folds" at z_0 or it has a critical point at z_0; see Definition 1.

The reader may well be aware that Kneser's proof of the Rado-Kneser-Choquet Theorem [4, p.29] shows that in both cases a straight line in some direction through $f(z_0)$ exists whose preimage set under f contains $2n$ Jordan arcs emanating from z_0 at equal angles. Lemma 2 concentrates on the case when z_0 is a "non-folding" critical point in which case it is shown that every straight line through $f(z_0)$ has a preimage set with the Kneser property.

Proof of Lemma 2. Assume without loss of generality that $f(z_0) = 0$. For a given $0 \leq \vartheta < 2\pi$ Lemma 1 yields an ϵ, a Jordan domain $V \subset \mathcal{D}$ containing z_0, and Jordan arcs $\alpha_{\vartheta,1}, \alpha_{\vartheta,2}, \ldots, \alpha_{\vartheta,k+1}$ which satisfy the properties $\mathbf{P}_\vartheta$. The same conclusion also follows for the harmonic mapping $e^{-i\vartheta} f$ that also has z_0 as a critical point of order $k+1$, however with d_0 replacing d_ϑ.

Thus it suffices to prove the lemma under the following assumptions:

(i) $f = h + \overline{g}$ is a harmonic mapping of $\mathcal{D}$;
(ii) $z_0 \in \mathcal{D}$ is a critical point of f of order k;
(iii) $f(z_0) = 0$;
(iv) there exist an $\epsilon > 0$, a Jordan domain $V \subset \mathcal{D}$ containing z_0, and Jordan arcs $\alpha_{0,1}, \alpha_{0,2}, \ldots, \alpha_{0,k+1}$ which satisfy the properties $\mathbf{P}_0$.

Let γ be the positively-directed boundary arc of V. For each $1 \leq j \leq k+1$, let ζ_j be the endpoint of $\alpha_{0,j}$. Clearly, each $f(\zeta_j)$ equals ϵ. There exist exactly $k+1$ points $\eta_1, \eta_2, \ldots, \eta_{k+1}$ in γ that map under f to $-\epsilon$. These points and the points ζ_j alternate in γ in the order $\zeta_1, \eta_1, \zeta_2, \eta_2, \ldots, \zeta_{k+1}, \eta_{k+1}$ as γ is traversed positively once.

We write

$$f(z) = F(z) + 2\Re g(z) \qquad (z \in \mathcal{D}), \tag{1}$$

where $F = h - g$. Since $f(z_0) = 0$, we assume without loss of generality that $h(z_0) = g(z_0) = 0$, and, consequently, $F(z_0) = 0$. We write

$$h(z) = \sum_{n=k+1} a_n z^n \ \text{ and } \ g(z) = \sum_{n=\ell+1} b_n z^n,$$

where $k \leq \ell$ and $a_{k+1}, b_{\ell+1} \neq 0$, with $|b_{k+1}| < |a_{k+1}|$ if $k = \ell$. This implies that F has a critical point of order k at z_0. In view of this, we choose ϵ sufficiently small so that the winding number, $n(F \circ \gamma, 0)$, of $F \circ \gamma$ about the origin is exactly $k+1$, and observe that the proof will be completed once we show that F maps each $\alpha_{0,j}$, $1 \leq j \leq k+1$, homeomorphically into the positive real axis.

For $1 \leq j \leq k+1$, let δ_j be the subarc of γ with initial and terminal points ζ_j and η_j respectively. Also, let δ'_j be the subarc of γ with initial and terminal points η_j and ζ_{j+1} respectively, with $\zeta_{k+2} = \zeta_1$. Clearly, f maps each δ_j and δ'_j homeomorphically onto the upper- and lower-half of the circle $|w| = \epsilon$ respectively. Since, by (1), $f(z)$ and $F(z)$ lie on the same horizontal line for every $z \in \mathcal{D}$, the function F maps each δ_j and δ'_j into the upper- and lower-half planes respectively. Hence $F(z), z \in \gamma$, is real if and only if z is either ζ_j or η_j for some $1 \leq j \leq k+1$.

We further conclude, since $n(F \circ \gamma, 0) = k+1$, that $F \circ \gamma$ meets (and crosses) each of the positive and negative real axes exactly $k+1$ times; namely at the points $F(\zeta_j)$ and $F(\eta_j)$. Note that, by (1), each $n(F \circ \delta_j, 0)$ and $n(F \circ \delta'_j, 0)$ takes one of the values $0, \pm 1/2$. Since

$$n(F \circ \gamma, 0) = \sum_{j=1}^{k+1} [n(F \circ \delta_j, 0) + n(F \circ \delta'_j, 0)] = k+1,$$

$n(F \circ \delta_j, 0) = n(F \circ \delta'_j, 0) = 1/2$ for every $1 \leq j \leq k+1$. This yields each $F(\zeta_j) > 0$ and each $F(\eta_j) < 0$.

By (1) once again, we conclude that each $F \circ \alpha_{\vartheta,j}$ is a subarc of the real axis with endpoints the origin and $F(\zeta_j) > 0$. Note that as z_0 is the only critical point of f in $\overline{V}$, it is likewise for F. Therefore, F maps each arc $\alpha_{\vartheta,j}$ homeomorphically onto the line segment $[0, F(\zeta_j)]$. This completes the proof of Lemma 2. ∎

Note that, as mentioned earlier, only the necessity part of Theorem A shall be used in the proof of Theorem 1 which we are now ready to establish.

Proof of Theorem 1. First, we establish the proof when f is locally univalent at z_0. In this case, (a) holds since f decomposes as $f = F \circ \varphi$ for any conformal map φ of some open neighborhood U of z_0 and univalent harmonic mapping $F = f \circ \varphi^{-1}$ of $\varphi(U)$. Also, since f is univalent in U, (b) holds at once. Now either $\omega'(z_0) \neq 0$ or ω has a zero derivative of order $r \geq 1$ at z_0. In the first case ω is locally conformal at z_0 and $\tilde{\varphi} = \omega - \omega_0$. In the second case $\tilde{\varphi}$ is any single-valued analytic $(r+1)$th root of $\omega - \omega_0$. In either case $\tilde{\varphi}$ is a conformal map of some open neighborhood V of z_0 and (c) holds in V.

Thus (a), (b) and (c) hold simultaneously if f is locally univalent at z_0. Furthermore, f has the canonical decomposition $f = \tilde{F} \circ \tilde{\varphi}$ at z_0, where $\tilde{F}$ is the univalent harmonic mapping $f \circ \tilde{\varphi}^{-1}$ of $\tilde{\varphi}(V)$. Clearly, the conformal map $\psi = \tilde{\varphi} \circ \varphi^{-1}$ from $\varphi(U \cap V)$ onto $\tilde{\varphi}(U \cap V)$ satisfies $\varphi = \psi^{-1} \circ \tilde{\varphi}$ and $F = \tilde{F} \circ \psi$ in $U \cap V$

and $\varphi(U \cap V)$ respectively. This proves the theorem when f is locally univalent at z_0.

Next, we proceed to prove the theorem when f has a critical point at z_0 of some order $k \geq 1$. First, we assume without loss of generality that $f(z_0) = 0$.

$(a) \Rightarrow (b)$. Suppose that (a) holds. Then we have $f(z_1) = f(z_2)$ if and only if $\varphi(z_1) = \varphi(z_2)$ for any $z_1, z_2 \in U$. Also,

$$f_z = (F_\zeta \circ \varphi)\varphi' \qquad \text{and} \qquad f_{\overline{z}} = (F_{\overline{\zeta}} \circ \varphi)\overline{\varphi'},$$

where $\zeta = \varphi(z)$, $z \in U$. It follows that

$$\omega = \frac{\overline{F_{\overline{\zeta}} \circ \varphi}}{F_\zeta \circ \varphi}$$

and $\omega(z_1) = \omega(z_2)$ wherever $f(z_1) = f(z_2)$ in U. This proves $(a) \Rightarrow (b)$.

$(b) \Rightarrow (c)$. Suppose that (b) holds. Since z_0 is a critical point of f of order k, Lemma 2 yields an $\epsilon > 0$, a Jordan domain V containing z_0 with $\overline{V} \subset W$, and $k+1$ Jordan analytic arcs $\alpha_{\vartheta,1}, \alpha_{\vartheta,2}, \ldots, \alpha_{\vartheta,k+1}$ which satisfy the properties $\mathbf{P}_\vartheta$ for every $0 \leq \vartheta < 2\pi$; for the rest of the proof see Figure 1.

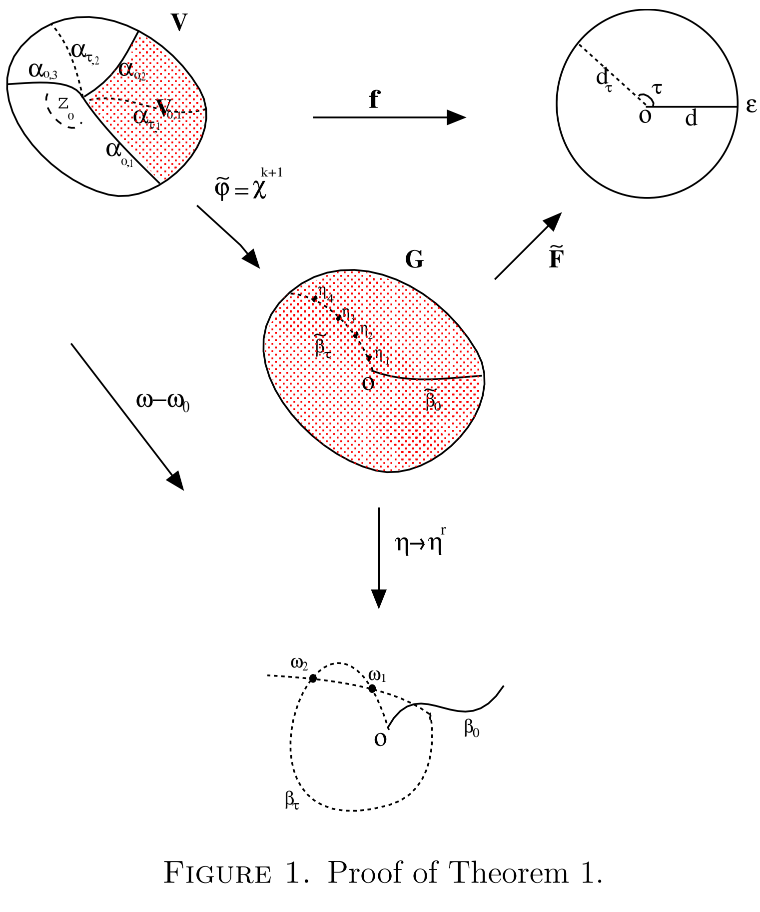

FIGURE 1. Proof of Theorem 1.

Let $\beta_{0,j} = (\omega - \omega_0)(\alpha_{0,j})$ for every $1 \le j \le k+1$. By property $\mathbf{P}_{0,2}$, for every $0 \le t \le \epsilon$ there exists a unique point $\varsigma_{t,j} \in \alpha_{0,j}$ such that $f(\varsigma_{t,j}) = t$ and $\alpha_{0,j} = \{\varsigma_{t,j} : 0 \le t \le \epsilon\}$. Then property (b) yields

$$\omega(\varsigma_{t,1}) = \omega(\varsigma_{t,2}) = \ldots = \omega(\varsigma_{t,k+1}), \qquad 0 \le t \le \epsilon,$$

and the arcs $\beta_{0,1}, \beta_{0,2}, \ldots, \beta_{0,k+1}$ coincide with an arc denoted by β_0. This implies at once that $\omega'(z_0) = 0$, and we can write

$$\omega - \omega_0 = \chi^{(k+1)r}, \tag{2}$$

for some positive integer r and some conformal map χ of some open neighborhood Ω of z_0 such that $\chi(z_0) = 0$. We choose ϵ sufficiently small so that $\overline{V} \subset \Omega$ and β_0 is a Jordan arc.

Define the analytic function $\tilde{\varphi}$ by

$$\tilde{\varphi}(z) = [\chi(z)]^{k+1} \qquad (z \in V),$$

and let $G = \tilde{\varphi}(V)$. Observe that $\tilde{\varphi}$ assumes every value in G at most $k+1$ times, counting multiplicity. Fix the value $\rho e^{i\tau}$, where $0 < \rho < \epsilon$ and $0 \le \tau < 2\pi$. By the construction of V, there exist $k+1$ distinct non-zero values $z_1, z_2, \ldots, z_{k+1}$ in V such that

$$f(z_1) = f(z_2) = \ldots = f(z_{k+1}) = \rho e^{i\tau}. \tag{3}$$

Note that, by properties $\mathbf{P}_{0,2}$ and $\mathbf{P}_{0,5}$ and Lemma 1, one and only one value z_j belongs to each region $V_{0,i} \cup (\alpha_{0,i})^\circ$, $1 \le i \le k+1$; $(\alpha_{0,i})^\circ$ is the interior of $\alpha_{0,i}$, For convenience, we label the values z_j so that $z_j \in V_{0,j} \cup (\alpha_{0,j})^\circ$ for every $1 \le j \le k+1$. By invoking (b), we conclude that

$$\omega(z_1) = \omega(z_2) = \ldots = \omega(z_{k+1}). \tag{4}$$

An application of Lemma 2 yields analytic Jordan arcs $\alpha_{\tau,1}, \alpha_{\tau,2}, \ldots, \alpha_{\tau,k+1}$ which satisfy the properties $\mathbf{P}_\tau$ and labeled such that $z_j \in (\alpha_{\tau,j})^\circ \subset V_{0,j} \cup (\alpha_{0,j})^\circ$. It follows from (4) that the arcs $(\omega - \omega_0)(\alpha_{\tau,1}), (\omega - \omega_0)(\alpha_{\tau,2}), \ldots, (\omega - \omega_0)(\alpha_{\tau,k+1})$ coincide with one analytic arc β_τ. Note that β_τ, unlike $d_\tau \neq 0$, cannot necessarily be made Jordan for all values τ.

We show that

$$\tilde{\varphi}(z_1) = \tilde{\varphi}(z_2) = \ldots = \tilde{\varphi}(z_{k+1}). \tag{5}$$

To do so, we consider two cases:

(I) $\tau = 0$. In this case each arc $\tilde{\varphi}(\alpha_{0,j})$ is, by (2), some rth root of the Jordan arc β_0 and, consequently, $\tilde{\varphi}(\alpha_{0,j})$ is a Jordan arc. This implies, by Lemma 2, that the arcs $\tilde{\varphi}(\alpha_{0,j})$ coincide with exactly one of the rth roots, $\tilde{\beta}_0$, of β_0; in particular $\tilde{\varphi}(z_j) \in \tilde{\beta}_0$ for all $1 \le j \le k+1$. Since β_0 is a Jordan arc, if $\tilde{\varphi}(z_i) \neq \tilde{\varphi}(z_j)$ for some $1 \le i \neq j \le k+1$, then

$$(\omega - \omega_0)(z_i) = [\tilde{\varphi}(z_i)]^r \neq [\tilde{\varphi}(z_j)]^r = (\omega - \omega_0)(z_j)$$

and $\omega(z_i) \neq \omega(z_j)$ which contradicts (4). This proves (5) for case (I).

(II) $0 < \tau < 2\pi$. In this case each arc $(\alpha_{\tau,j})^\circ$ lies exactly in one of the Jordan domains $V_{0,1}, V_{0,2}, \ldots, V_{0,k+1}$, say $V_{0,j}$. Since $\tilde{\varphi}\colon \alpha_{0,j} \to \tilde{\beta}_0$ is a homeomorphism, Lemma 2 shows at once that $\tilde{\varphi}$ is a conformal map from each $V_{0,j}$ onto $G \setminus \tilde{\beta}_0$. This implies that each $\tilde{\varphi}(\alpha_{\tau,j})$ is a Jordan arc and that, by Lemma 2, all these arcs coincide with exactly one of the rth roots, $\tilde{\beta}_\tau$, of β_τ; in particular, $\tilde{\varphi}(z_j) \in \tilde{\beta}_\tau$ for all $1 \le j \le k+1$. Recall that β_τ need not be a Jordan arc. If β_τ is a Jordan arc, then the same argument of case (I) applies and we conclude (5).

If not, then as an analytic arc, β_τ admits only a finite number of self-intersections; the origin is not a self-intersection. Let $\omega_1, \omega_2, \ldots, \omega_p$ be the points of self-intersection of β_τ, and let $\eta_1, \eta_2, \ldots, \eta_q$ be their rth roots in $\tilde{\beta}_\tau$ labeled according to their order of appearance in $\tilde{\beta}_\tau$ as it is traversed from the origin onward. Clearly, each η_j belongs to $(\tilde{\beta}_\tau)^\circ$. Since $\tilde{\varphi}\colon \alpha_{\tau,j} \to \tilde{\beta}_\tau$ is a homeomorphism for every $1 \le j \le k+1$, there exist exactly q points $z^1_{\tau,j}, z^2_{\tau,j}, \ldots, z^q_{\tau,j}$ in $\alpha_{\tau,j}$ which map under $\tilde{\varphi}$ to $\eta_1, \eta_2, \ldots, \eta_q$ respectively, and these points are labeled according to their order of appearance in $\alpha_{\tau,j}$ as it is traversed from the origin onward. Recall that the points z_j are on $\alpha_{\tau,j}$. We consider two exhaustive cases:

(A) There exists $1 \le \nu \le k+1$ such that $z_\nu \notin \{z^i_{\tau,j} : 1 \le i \le q\}$. Then $\omega(z_v) \ne \omega_i$ for all $1 \le i \le p$, and, by (4), $\omega(z_j) \ne \omega_i$ for all $1 \le i \le p$ and $1 \le j \le k+1$. Thus $z_j \ne z^i_{\tau,j}$ and $\tilde{\varphi}(z_j) \ne \eta_i$ for all i,j. This yields (5) or else $\tilde{\varphi}(z_i) \ne \tilde{\varphi}(z_j)$ for some $1 \le i \ne j \le k+1$ which by using the argument in (A) gives a contradiction.

(B) $z_j \in \{z^i_{\tau,j} : 1 \le i \le q\}$ for every $1 \le j \le k+1$. Suppose that $z_1 = z^\nu_{\tau,1}$ for some $1 \le \nu \le q$. We claim that $z_j = z^\nu_{\tau,j}$ for every $1 \le j \le k+1$. Let $w_i = f(z^i_{\tau,1})$ for every $1 \le i \le q$. Suppose that there exist i_0 and $j_0 \ne 1$ such that $f(z^{i_0}_{\tau,j_0}) \notin \{w_1, w_2, \ldots, w_q\}$. Then $z^{i_0}_{\tau,j_0}$ is one of $k+1$ values $z'_1, z'_2, \ldots, z'_{k+1}$, $z'_j \in \alpha_{\tau,j}$, in V which satisfy

$$f(z'_1) = f(z'_2) = \ldots = f(z'_{k+1}) \ne w_i, \qquad 1 \le i \le q.$$

We conclude at once that $z'_1 \ne z^i_{\tau,1}$ for all i and, by the proof of (a), $z'_j \ne z^i_{\tau,j}$ for all i,j. This yields a contradiction and $f(z^i_{\tau,j}) \in \{w_1, w_2, \ldots, w_q\}$ for all i,j. Recall that $f\colon \alpha_{\tau,j} \to d_\tau$ is a homeomorphism for every j. It follows that the points $w_1, w_2, \ldots, w_q$ are labeled according to their order of appearance in d_τ as it is traversed from the origin onward in line with the order of appearance of the values $z^i_{\tau,1}$ on $\alpha_{\tau,1}$. This implies that $f(z^i_{\tau,j}) = w_i$ for every i,j; in particular, the desired claim holds.

In view of the above, we assume that $z_j = z^\nu_{\tau,j}$ for some $1 \le \nu \le q$ and all $1 \le j \le k+1$. Denote by d^ν_τ the line segment $[w_{\nu-1}, w_\nu]$, with w_0 the origin, and by $\alpha^\nu_{\tau,j}$ the subarc of $\alpha_{\tau,j}$ with endpoints $z^{\nu-1}_{t,j}$ and $z^\nu_{t,j}$, with $z^0_{t,j}$ the origin. Clearly, $f\colon \alpha^\nu_{\tau,j} \to d^\nu_\tau$ is a homeomorphism for every j. Let w' be any interior point of d^ν_τ, and let $z'_{\tau,j}$ be the interior point of $\alpha_{\tau,j}$ satisfying $f(z'_{\tau,j}) = w'$ for every j. Obviously, $z'_{\tau,j} \to z_j$ as $w' \to w_\nu$ for all j, and, as shown in (A),

$$\tilde{\varphi}(z'_{\tau,1}) = \tilde{\varphi}(z'_{\tau,2}) = \ldots = \tilde{\varphi}(z'_{\tau,k+1}).$$

By letting $w' \to w_\nu$, the continuity of $\tilde{\varphi}$ yields (5).

This concludes the proof that (3) and (5) hold simultaneously in V.

Another conclusion is that the function $\tilde{\varphi}\colon V \to G$ assumes every value in G exactly $(k+1)$ times, counting multiplicity. For if $\eta \in G$ is a fixed non-zero value, then there exist $k+1$ non-zero values $z_1, z_2, \ldots, z_{k+1}$ in V which satisfy (3) with $\tilde{\varphi}(z_j) = \eta$ for some j. Consequently, (5) holds and $\tilde{\varphi}(z_j) = \eta$ for all j. Since $\tilde{\varphi}\colon V \to G$ is at most $(k+1)$-valent, the conclusion follows.

We show now that (c) holds. If $f(z) = f(z')$ for distinct points $z, z' \in V$, then z and z' are two of $k+1$ points $z_1, z_2, \ldots, z_{k+1}$ of V which satisfy (3). Hence (5) holds and $\tilde{\varphi}(z) = \tilde{\varphi}(z')$. Conversely, if $\tilde{\varphi}(z) = \tilde{\varphi}(z')$ and $f(z) \neq f(z')$ for some distinct points $z, z' \in V$, then there exist two mutually disjoint sets of distinct points $\{z_1, z_2, \ldots, z_{k+1}\}$ and $\{z'_1, z'_2, \ldots, z'_{k+1}\}$ of V such that

$$f(z_j) = f(z) \quad \text{and} \quad f(z'_j) = f(z'), \qquad 1 \leq j \leq k+1,$$

and consequently,

$$\tilde{\varphi}(z_j) = \tilde{\varphi}(z) = \tilde{\varphi}(z') = \tilde{\varphi}(z'_j), \qquad 1 \leq j \leq k+1,$$

which contradicts the fact that $\tilde{\varphi}\colon V \to G$ is $(k+1)$-valent. Hence, $f(z) = f(z')$ and $(b) \Rightarrow (c)$.

$(c) \Rightarrow (a)$ Suppose that (c) holds. We choose V so that $\tilde{\varphi}'$ is zero only at the origin. Let $G = \tilde{\varphi}(V)$, and let $\tilde{\eta} \in V$ be any point which satisfies $\tilde{\varphi}(\tilde{\eta}) = \eta$ for every $\eta \in G$. Define the function $\tilde{F}\colon G \to \mathbb{C}$ by $\tilde{F}(\eta) = f(\tilde{\eta})$. Note that $f(z_1) = f(z_2)$ if and only if $\tilde{\varphi}(z_1) = \tilde{\varphi}(z_2)$ wherever $z_1, z_2 \in V$. If $\tilde{\eta}, \tilde{\eta}' \in V$ are values which satisfy $\tilde{\varphi}(\tilde{\eta}) = \tilde{\varphi}(\tilde{\eta}')$, then $f(\tilde{\eta}) = f(\tilde{\eta}')$ and $\tilde{F}$ is well defined. Also, if $\eta_1, \eta_2 \in G$ are distinct with corresponding values $\tilde{\eta}_1, \tilde{\eta}_2 \in V$, then $\tilde{\varphi}(\tilde{\eta}_1) \neq \tilde{\varphi}(\tilde{\eta}_2)$, or $f(\tilde{\eta}_1) \neq f(\tilde{\eta}_2)$, and $\tilde{F}(\eta_1) = f(\tilde{\eta}_1) \neq f(\tilde{\eta}_2) = \tilde{F}(\eta_2)$. Thus $\tilde{F}$ is injective.

Observe that every non-zero $\eta \in V$ is a center of some disc $\Delta \subset V$ such that $\tilde{\varphi}^{-1}\colon \Delta \to V \setminus \{0\}$ is a conformal map and $f\colon \tilde{\varphi}^{-1}(\Delta) \to \mathbb{C} \setminus \{0\}$ is a univalent harmonic map. We conclude that $\tilde{F}$ is a non-vanishing univalent harmonic mapping in $G \setminus \{0\}$. But $\tilde{F}$ is locally bounded at the origin and $\tilde{F}(0) = 0$. Therefore, $\tilde{F}$ is a univalent harmonic mapping in G and $f = \tilde{F} \circ \tilde{\varphi}$ in V. Thus $(c) \Rightarrow (a)$.

This completes the proofs of the equivalences $(a) \Leftrightarrow (b) \Leftrightarrow (c)$, and also the proof that under the statements (a), (b), and (c) the function f has the canonical decomposition $f = \tilde{F} \circ \tilde{\varphi}$.

It remains to show the uniqueness of the canonical decomposition $f = \tilde{F} \circ \tilde{\varphi}$ near z_0. Suppose that f is also decomposable as in (a). Observe that $\varphi(z_1) = \varphi(z_2)$ if and only if $\tilde{\varphi}(z_1) = \tilde{\varphi}(z_2)$ for any $z_1, z_2 \in U \cap V$. Then, by essentially the same argument as above, the composition $\tilde{\varphi} \circ \varphi^{-1}$ yields the desired univalent analytic function ψ from $\varphi(U \cap V)$ onto $\tilde{\varphi}(U \cap V)$.

This completes the proof of Theorem 1. ■

An immediate consequence of Theorem 1 is the following corollary.

Corollary 1. *Let f be a sense-preserving harmonic mapping of a region $\mathcal{D}$ with dilatation ω, and let $z_0 \in \mathcal{D}$ be a critical point of f and $\omega - \omega_0$, $\omega(z_0) = \omega_0$, of orders k and m respectively. A necessary condition for f to have a decomposition $f = F \circ \varphi$ for some analytic function φ in some open neighborhood U of z_0 and some sense-preserving univalent harmonic mapping F in $\varphi(U)$ is that $(k+1)$ divides $(m+1)$.*

Example 2 shows that the condition of Corollary 1 is a necessary but not a sufficient condition for the existence of the required local decomposition.

The following Example 3 uses Corollary 1 to conclude that members of a familiar class of harmonic mappings do not have the desired local decomposition at the origin.

Example 3. Let k and m be positive integers satisfying $k < m$, let $k+1$ not divide $m+1$ and let f be the harmonic polynomial

$$f(z) = z^{k+1} + \frac{k+1}{m+1}\overline{z}^{m+1}.$$

The dilatation of f is $\omega(z) = z^{m-k}$. It is immediate that f is sense-preserving in the unit disc, and that f and ω admit the origin as a critical point of orders k and $m-k-1$ respectively. Since $k+1$ neither divides $m+1$ nor $m-k$, by Corollary 1, f has no decomposition of the desired form near the origin.

On the other hand, suppose that $k < m$ and $(m+1) = (k+1)r$, where $r = 2, 3, \ldots$. Then the dilatation of f is $\omega(z) = z^{(k+1)(r-1)}$, f is sense-preserving in the unit disc, and f and ω admit the origin as a critical point of orders k and $m-k-1$ respectively. In this case, f has the canonical decomposition $f = \tilde{F} \circ \tilde{\varphi}$, where $\eta = \tilde{\varphi} = z^{k+1} = \omega^{1/(r-1)}(z)$ and $\tilde{F} = \eta + \overline{\eta}^r/r$ is a univalent harmonic mapping in the unit disc.

We end the paper by asking whether it is possible to conclude the sufficiency part of Theorem A from Theorem 1 without depending on the Existence and Uniqueness Theorem of the Beltrami equation.

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