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Close-to-Convexity Criteria for Planar Harmonic Mappings

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Abstract We give a criteria for planar harmonic mappings to be univalent close-to-convex which settles a conjecture of P. T. Mocanu.

Keywords Close-to convex function · Planar harmonic mapping

1 Introduction

Recently, Mocanu [5] posed the following conjecture:

Conjecture 1 *If h and g are analytic functions of the open unit disc $\mathbb{D}$, with $h'(0) \neq 0$, that satisfy*

$$g'(z) = zh'(z) \tag{1}$$

and

$$\Re\left\{1 + z\frac{h''(z)}{h'(z)}\right\} > -\frac{1}{2} \tag{2}$$

for all $z \in \mathbb{D}$, then the harmonic mapping $f = h + \overline{g}$ is univalent in $\mathbb{D}$.

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The purpose of this note is to prove the conjecture by establishing the following stronger result:

Theorem 1 *Let* $f = h + \overline{g}$ *be a harmonic mapping of* $\mathbb{D}$, *with* $h'(0) \neq 0$, *that satisfies*

$$g'(z) = zh'(z) \tag{3}$$

and inequality (2) *for all* $z \in \mathbb{D}$. *Then* f *is a univalent close-to-convex mapping.*

2 Preliminaries

This section is devoted for the needed notation and preliminaries.

A simply connected proper subdomain of $\mathbb{C}$ is called *close-to-convex* if its complement in $\mathbb{C}$ is the union of closed half lines with pairwise disjoint interiors. A univalent analytic or harmonic function $f : \mathbb{D} \to \mathbb{C}$ is called *close-to-convex* if $f(\mathbb{D})$ is close-to-convex.

Let G be a Jordan domain of the Riemann sphere $\mathbb{C}^\infty$, and let f be a complex-valued function of G.

(a) Let $z_0 \in G$. We write $f_{z_0} \sim z^r$, where $r = 1, 2, \ldots$, if there exist an open neighborhood U of z_0 and sense-preserving homeomorphisms $h_1 : U \to (|\zeta| < 1)$ and $h_2 : \mathbb{C}^\infty \to \mathbb{C}^\infty$ such that $h_1(z_0) = h_2 \circ f(z_0) = 0$ and $h_2 \circ f \circ h_1^{-1}(\zeta) = \zeta^r$ $(|\zeta| < 1)$. On the other hand, if $h_2 \circ f \circ h_1^{-1}(\zeta) = \overline{\zeta}^r$ $(|\zeta| < 1)$, then we write $f_{z_0} \sim \overline{\zeta}^r$.

(b) Let $z_0 \in \partial G$, and let f be also defined on an open arc of ∂G containing z_0. We write $f_{z_0} \sim z^r$ $(z \in G)$ if there exist an open neighborhood U of z_0 and sense-preserving homeomorphisms $h_1 : U \cap \overline{G} \to (|\zeta| < 1, \Im\zeta \geq 0)$ and $h_2 : \mathbb{C}^\infty \to \mathbb{C}^\infty$ such that $h_1(z_0) = h_2 \circ f(z_0) = 0$ and $h_2 \circ f \circ h_1^{-1}(\zeta) = \zeta^r$ $(|\zeta| < 1, \Im\zeta \geq 0)$. On the other hand, if $h_2 \circ f \circ h_1^{-1}(\zeta) = \overline{\zeta}^r$ $(|\zeta| < 1, \Im\zeta \geq 0)$, then we write $f_{z_0} \sim \overline{\zeta}^r$ $(z \in G)$.

(c) Again, let $z_0 \in G$. We write $f_{z_0} \sim z^r, \overline{z}^s$, where $r, s = 1, 2, \ldots$, if there exists a cross-cut of G through z_0 that divides G into two Jordan domains G_1 and G_2 such that $f_{z_0} \sim z^r(G_1)$ and $f_{z_0} \sim \overline{z}^s(G_2)$. Note that G_1 and G_2 are unique since f must be sense-preserving in G_1 and sense-reversing in G_2. This definition states that the covering properties of f near z_0 resemble those of the mapping: $z \to z^r$, if $\Im z \geq 0$, and $z \to \overline{z}^s$, if $\Im z \leq 0$, near the origin.

Observe that the above definitions (a), (b), and (c) depend only on an open neighborhood of z_0 relative to G; these definitions may be found in Lyzzaik [4].

In this paper, we shall only use the notation $f_{z_0} \sim z, \overline{z}$, $f_{z_0} \sim z, \overline{z}^3$ and $f_{z_0} \sim z^3, \overline{z}$.

The paper is organized as follows. In Sect. 2, we prove Theorem 1. In Sect. 3 we give examples that illustrates Theorem 1, its strength and limitations.

3 Proof of Theorem 1

The ingredients of the proof of Theorem 1 are two well known results. The first is the following characterization for analytic close-to-convex functions due to Kaplan [3].

Lemma 1 *A necessary and sufficient condition for an analytic function* $h : \mathbb{D} \to \mathbb{C}$ *to be close-to-convex is that* h' *is nonvanishing and*

$$\int_{\theta_1}^{\theta_2} \Re\left\{1 + re^{i\theta}\frac{h''(re^{i\theta})}{h'(re^{i\theta})}\right\} > -\pi \tag{4}$$

whenever $\theta_1 < \theta_2$ *and* $0 < r < 1$.

The second ingredient is a sufficiency condition for close-to-convexity of harmonic mappings due to Clunie and Sheil-Small [1].

Lemma 2 *If a harmonic mapping* $f = h + \overline{g}$ *satisfies* $|g'(0)| < |h'(0)|$ *and the property that every analytic function* $h + \lambda g$, *where* $|\lambda| = 1$, *is close-to-convex, then* f *is also close-to-convex.*

Proof of Theorem 1 By hypothesis and (3) we have $h'(0) \neq 0$ and $g'(0) = 0$, respectively. Thus, by virtue of Lemma 2, it suffices to show that every analytic function $F_\lambda = h - \lambda g$, where $|\lambda| = 1$, is close-to-convex. We do this by using Lemma 1. Because of (3)

$$F'_\lambda(z) = h'(z) - \lambda g'(z) = (1 - \lambda z)h'(z)$$

and F'_λ is nonvanishing in $\mathbb{D}$. By logarithmic differentiation we can write

$$\Re\left\{1 + z\frac{F''_\lambda(z)}{F'_\lambda(z)}\right\} = \Re\left\{\frac{\lambda z}{\lambda z - 1}\right\} + \Re\left\{1 + z\frac{h''(z)}{h'(z)}\right\}. \tag{5}$$

Fix θ_1, θ_2, and r so that $\theta_1 < \theta_2$ and $0 < r < 1$. We claim that

$$\int_{\theta_1}^{\theta_2} \Re\left\{1 + re^{i\theta}\frac{F''_\lambda(re^{i\theta})}{F'_\lambda(re^{i\theta})}\right\} d\theta > -\pi. \tag{6}$$

Since F'_λ is nonvanishing, we may assume without loss of generality that $\theta_1 < \theta_2 < \theta_1 + 2\pi$.

By (2) we have

$$\int_{\theta_1}^{\theta_2} \Re\left\{1 + re^{i\theta}\frac{h''(re^{i\theta})}{h'(re^{i\theta})}\right\} d\theta > \frac{\theta_1 - \theta_2}{2}. \tag{7}$$

For $z \in \mathbb{D}$ one may verify that

$$\Re\left\{\frac{z}{z-1}\right\} = \frac{1}{2}\left\{\frac{z}{z-1} + \frac{\overline{z}}{\overline{z-1}}\right\} = \frac{1}{2}\frac{|1-z|^2 - (1-|z|^2)}{|1-z|^2}$$
$$= \frac{1}{2} - \frac{1}{2}\frac{1-|z|^2}{|1-z|^2}.$$

Replacing z by λz, with $z = re^{i\theta}$, and letting $\zeta = \overline{\lambda} r$ yields

$$\Re\left\{\frac{\lambda z}{\lambda z - 1}\right\} = \frac{1}{2} - \frac{1}{2}\frac{1-|z|^2}{|1-\lambda z|^2} = \frac{1}{2} - \frac{1}{2}\frac{1-|\zeta|^2}{|e^{i\theta}-\zeta|^2}$$
$$= \frac{1}{2} - \frac{1}{2}P_\zeta(\theta),$$

where

$$P_\zeta(\theta) = \frac{1-|\zeta|^2}{|e^{i\theta}-\zeta|^2}$$

is the Poisson Kernel. It follows that

$$\int_{\theta_1}^{\theta_2} \Re\left\{\frac{\lambda re^{i\theta}}{\lambda re^{i\theta}-1}\right\} d\theta = \frac{\theta_2-\theta_1}{2} - \frac{1}{2}\int_{\theta_1}^{\theta_2} P_\zeta(\theta) d\theta$$
$$> \frac{\theta_2-\theta_1}{2} - \frac{1}{2}\int_{0}^{2\pi} P_\zeta(\theta) d\theta = \frac{\theta_2-\theta_1}{2} - \pi. \tag{8}$$

By invoking (5), (8), and (7) consecutively we obtain

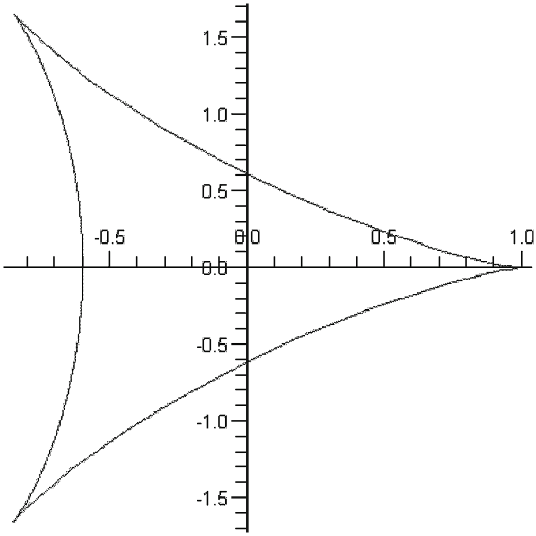
$$\int_{\theta_1}^{\theta_2} \Re\left\{1 + re^{i\theta}\frac{F_\lambda''(re^{i\theta})}{F_\lambda'(re^{i\theta})}\right\} d\theta = \int_{\theta_1}^{\theta_2} \Re\left\{\frac{\lambda re^{i\theta}}{\lambda re^{i\theta}-1}\right\} d\theta$$
$$+ \int_{\theta_1}^{\theta_2} \Re\left\{1 + re^{i\theta}\frac{h''(re^{i\theta})}{h'(re^{i\theta})}\right\} d\theta > \frac{\theta_2-\theta_1}{2} - \pi + \frac{\theta_1-\theta_2}{2} = -\pi.$$

This proves inequality (5). Therefore, every F_λ is a close-to-convex function and the proof of Theorem 1 is complete. □

4 Examples

Here we present examples, motivated by Mocanu [5], asserting that inequality (2) is best possible for the truth of Theorem 1.

Fig. 1 Image domain of f for $a = 0.3$ and $n = 1$



Fix $a,\ 0 \le a < 0.5$, and let $f = h + \overline{g}$ be the harmonic polynomial mapping whose

$$h(z) = z - az^2 \ \text{ and } \ g(z) = \frac{z^2}{2} - \frac{2az^3}{3}.$$

Note that h' is nonvanishing and $g'(z) = zh'(z)$ in $\mathbb{D}$; hence f is a locally univalent function of $\mathbb{D}$. It is easy to verify that h satisfies (2) for all a, $0 \le a \le 0.3$; in particular, for $a = 0.3$ we have

$$\inf_{z\in\mathbb{D}} \Re\left\{1 + z\frac{h''(z)}{h'(z)}\right\} = -\frac{1}{2}.$$

Thus, by Theorem 1, f is a close-to-convex function whenever $0 \le a \le 0.3$; see Fig. 1 for the image domain of f when $a = 0.3$.

Suppose next that $0.3 < a < 0.5$. We intend to show that f is not a close-to-convex function, indeed not univalent. Differentiation of both sides of the equation $f(e^{it}) = h(e^{it}) + \overline{g(e^{it})}$ yields

$$\frac{d}{dt} f(e^{it}) = -2e^{-it/2}\Im\phi(t), \tag{9}$$

where

$$\phi(t) = (1 - 2ae^{it})e^{i3t/2}.$$

We find the number of times that $\Im\phi(t)$ changes sign over $(-\pi - \epsilon, \pi)$ for sufficiently small positive ϵ. Consider the continuous branch of

$$\arg\phi(t) = \frac{3t}{2} + \arg(1 - 2ae^{it}). \tag{10}$$

where $\arg\phi(0) = 0$; thus the associated continuous branch of $\arg(1 - 2ae^{it})$ satisfies $\arg(1 - 2a) = 0$. Differentiation with respect to t gives

$$\begin{aligned}\frac{d}{dt}\arg\phi(t) &= \frac{3}{2} + \frac{d}{dt}\Im\log(1 - 2ae^{it})\\ &= \frac{3}{2} + \Re\left\{\frac{2ae^{it}}{2ae^{it} - 1}\right\}\\ &= 2 - \frac{1 - 4a^2}{2|2ae^{it} - 1|^2}\\ &= \frac{20a^2 - 16a\cos t + 3}{2|2ae^{it} - 1|^2}.\end{aligned} \tag{11}$$

It is immediate that $\arg\phi(t)$ increases steadily for all values t satisfying

$$\cos t \le \frac{20a^2 + 3}{16a}$$

and decreases steadily for the remaining values. But the inequality

$$\frac{20a^2 + 3}{16a} < 1$$

holds if and only if $0.3 < a < 0.5$.

Let

$$t_0 = \arccos\left(\frac{20a^2 + 3}{16a}\right).$$

Obviously, $0 < t_0 < \pi/2$, and $d\arg\phi(t)/dt$ is positive on the closed interval $[t_0, \pi]$ and negative on the closed interval $[0, t_0]$; hence $\phi(t)$ is a strictly increasing on the first interval and strictly decreasing on the second.

Observe that $\arg(1 - 2ae^{it})$ decreases by less than $\pi/2$ on $[0, t_0]$; thus likewise does $\arg\phi(t)$. But the latter is strictly increasing on $[t_0, \pi]$ and, by (10), attains a net variation of exactly $3\pi/2$ on $[0, \pi]$. Hence, as t varies from 0 to π, the graph of $\phi(t)$ starts from $\phi(0) = 1 - 2a > 0$ to move downward to $\phi(t_0)$ in the open fourth quadrant, then backs off to cross the positive real axis and proceeds thereafter to wind around the origin steadily in the positive direction by an angle of size $3\pi/2$ before it terminates at $\phi(\pi) = (1 + 2a)e^{i3\pi/2}$. Thus $\Im\phi(t)$ changes sign exactly twice in the open interval (t_0, π), with the terminal point $\phi(\pi)$ lying on the negative imaginary axis (Figs. 2, 3).

On the other hand $\phi(-t) = \overline{\phi(t)}$. This implies at once that $\Im\phi(t)$ changes sign exactly twice in the open interval $(-\pi, -t_0)$. Moreover, the graph of $\phi(t)$ on $[-t_0, t_0]$

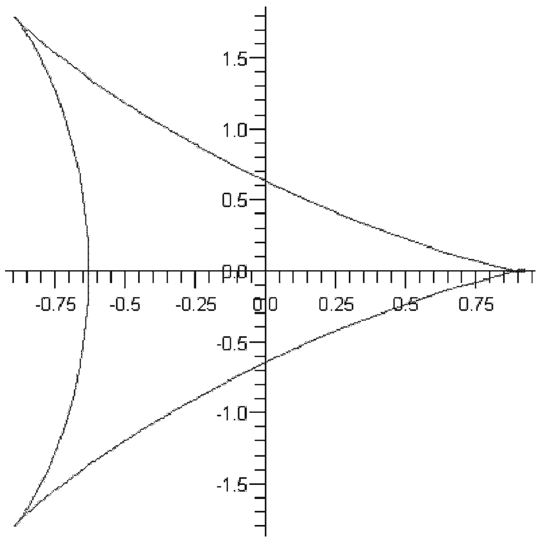


Fig. 2 The curve $f(e^{it}), t \in [-\pi, \pi]$, for $a = 0.4$ and $n = 1$

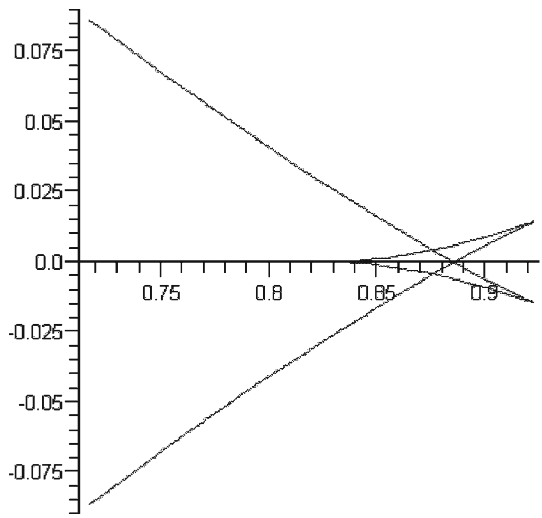


Fig. 3 The curve $f(e^{it}), t \in [-\pi/3, \pi/3]$, for $a = 0.4$ and $n = 1$

is symmetric about the real axis and crosses it at $\phi(0)$. Thus $\Im\phi(t)$ changes sign exactly five times over $(-\pi - \epsilon, \pi)$ for sufficiently small positive ϵ.

In view of (8), we conclude that the argument of the tangent vector of $f(\partial\mathbb{D})$, endowed with the direction inherited via f from the positive direction of $\partial\mathbb{D}$, is strictly decreasing except at exactly five distinct points, say $f(z_k)$, $1 \le k \le 5$, where at each point $f(\partial\mathbb{D})$ has a cusp of angle size zero.

Write $z_k = e^{it_k}$. Once again, by (8), we infer that the net variation of the arg $df(e^{it})$ on $[-\pi, \pi]\backslash\{t_k : 1 \le k \le 5\}$ is exactly $-\pi$. Since h' is nonvanishing on $\partial\mathbb{D}$, $f_{z_k} \sim z, \overline{z}^3$ or $f_{z_k} \sim z^3, \overline{z}$ and the interior angle of $f(\mathbb{D})$ at $f(z_k)$ is either 0 or π respectively;

[4]. But f is sense-preserving and locally univalent on $\overline{\mathbb{D}}$; hence $f_z \sim z$ $(z \in \overline{\mathbb{D}})$ for every $z \neq z_k$.

We argue now by contradiction to establish the non-univalence of f. Suppose that f is univalent in $\mathbb{D}$. Then the interior angle of $f(\mathbb{D})$ at every $f(z_k)$ is zero and consequently the curvature there is π. Thus, by the Gauss-Bonnet theorem [2, pp.264–283], $2\pi = -\pi + 5\pi = 4\pi$ and we have a contradiction.

Therefore, f is not a close-to-convex function of $\mathbb{D}$ whenever $0.3 < a < 0.5$.

We close the paper by expressing interest in finding sufficient conditions on h that would lead to the univalence of harmonic mappings of the unit disc $f = h + \overline{g}$ satisfying $g' = zh'$.

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