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Exterior Univalent Harmonic Mappings With Finite Blaschke Dilatations

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Abstract. In this article we characterize the univalent harmonic mappings from the exterior of the unit disk, Δ, onto a simply connected domain Ω containing infinity and which are solutions of the system of elliptic partial differential equations $\overline{f_{\bar z}(z)} = a(z) f_z(z)$ where the second dilatation function $a(z)$ is a finite Blaschke product. At the end of this article, we apply our results to nonparametric minimal surfaces having the property that the image of its Gauss map is the upper half-sphere covered once or twice.

1 Introduction

In this article we study univalent harmonic mappings defined on the exterior of the unit disk, $\Delta = \{z : |z| > 1\}$ which keep infinity fixed. Without loss of generality, we may assume that f preserves the orientation. By [2], such mappings can be identified as non-constant solutions of the elliptic partial differential equation

$$\overline{f_{\bar z}(z)} = a(z) f_z(z) \quad \text{on } \Delta \tag{1.1}$$

where the second dilatation function $a(z) = \sum_{k=0}^{\infty} \alpha_k \frac{1}{z^k}$ of f belongs to $H(\Delta)$ and $|a(z)| < 1$ on Δ. It then follows [3], that f admits the representation

$$f(z) = A\Big(z - \frac{1}{\bar z}\Big) + B\Big(\bar z - \frac{1}{z}\Big) + C \log |z|^2 + F(z), \quad A \neq 0, \tag{1.2}$$

where F is a bounded harmonic function in Δ. Furthermore, substituting (1.2) in (1.1) and comparing the constant and the $\frac{1}{z}$ coefficients in (1.2), we get the following relations between the constants A, B and C

$$B = \overline{\alpha_0 A} \quad \text{and} \quad C = \frac{\alpha_1 \overline{\alpha_0} A + \overline{\alpha_1 A}}{1 - |\alpha_0|^2}. \tag{1.3}$$

In other words, B and C depend only on A, $a(\infty)$ and $a'(\infty)$ and are completely independent of the boundary values $f^*(e^{it}) = F^*(e^{it})$.

Suppose that Ω is a simply connected domain of $\bar{\mathbb{C}}$ containing infinity whose boundary is locally connected. Let $a \in H(\Delta)$ and $|a(z)| \le k < 1$ on Δ. Given $\arg A$, then there is a univalent solution of (1.1) which is of the form (1.2) and which maps Δ onto Ω. The

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uniqueness of such mappings is still in question and is only known for limited special cases of Ω. If $a(z)$ admits an analytic extension across an open interval J of the unit circle and satisfies $|a(e^{it})| = 1$, $e^{it} \in J$ then the existence theorem does not hold any more. This can be explained by the fact that in this case $f(e^{it})$ and $a(e^{it})$ are closely related on J. We investigate the consequences of this relationship in the case where $a(z)$ is a finite Blaschke product.

We first give in Section 2, a short summary of results obtained by the authors on the behaviour of the boundary correspondence $f^*(e^{it})$ in the case where the dilatation function a admits a continuous extension across a subinterval I of the unit circle.

In Section 3, we restrict ourselves to the case where a is a finite Blaschke product. We show that the number of complete resting points of f^* is $N - 2$ where N is the degree of the Blaschke product. It follows then that the image $\Omega = f(\Delta)$ contains at most $N - 2$ points of convexity. If $N = 1$, then by Theorem 2.1, Ω is a one point punctured plane and there exists no univalent solution of (1.1) which maps Δ onto the unbounded component of the complement of any compact continuum. If $N = 2$, then $\mathbb{C} \setminus \Omega$ is a convex set, *i.e.*, $\partial\Omega$ is either a point, a linear line segment or a convex Jordan curve. Further details about this case can be found in Section 5.2.

In Section 4, we characterize univalent solutions of (1.1) which map Δ onto Ω by means of their boundary correspondence. It provides simultaneously a constructive existence proof for special cases where a is a finite Blaschke product.

Finally, we consider in Section 5 the special cases $N = 1, 2$ and 4 and in Section 6, we apply our results to minimal surfaces.

2 Boundary Behaviour of Harmonic Maps on an Interval where $|a| = 1$, Summary

In this section we give a short summary of results proved by the authors in earlier papers. We adapt the statements for harmonic mappings defined in $\Delta = \bar{\mathbb{C}} \setminus \bar{U}$.

Let

$$f(z) = A\Big(z - \frac{1}{\bar{z}}\Big) + B\Big(\bar{z} - \frac{1}{z}\Big) + C\log|z|^2 + F(z), \quad A \neq 0$$

be a univalent harmonic and orientation preserving mapping defined on Δ, where B and C satisfy (1. 3). Our first result characterizes such harmonic mappings for which $f(\Delta)$ is a one point punctured plane, *i.e.*, $f(\Delta) = \bar{\mathbb{C}} \setminus \{q\}$ for some point $q \in \mathbb{C}$.

Theorem 2.1 **[3]** *Let f be a univalent harmonic and orientation-preserving mapping defined on Δ and normalized by (1. 2). Then $f(\Delta) = \bar{\mathbb{C}} \setminus \{q\}$ for some point $\mathbb{C}$ if, and only if f is of the form*

$$f(z) = A\Big(z - \frac{1}{\bar{z}}\Big) + B\Big(\bar{z} - \frac{1}{z}\Big) + C\log|z|^2 + D$$

where D is a constant. These mappings are solutions of the elliptic differential equation (1.1) where $a(z)$ is of the form

$$a(z) = \frac{\bar{A}}{A}\frac{(1 - p_1 z)(1 - p_2 z)}{(z - \overline{p_1})(z - \overline{p_2})}, \quad |p_1| < 1 \text{ and } |p_2| \leq 1. \tag{2.1}$$

Conversely, any univalent harmonic mapping which is of the form (1.2) and which is a solution of (1.1) with respect to a dilatation function $a(z)$ of the form (2.1) has the property that $f(\Delta) = \bar{\mathbb{C}} \setminus \{q\}$ for some point $q \in \mathbb{C}$.

The next result is an existence theorem.

Theorem 2.2 **[3]** *Let Ω be a simply connected proper subdomain of $\bar{\mathbb{C}}$ containing infinity whose boundary $\partial\Omega$ is locally connected. Let $a(z) \in H(\Delta)$, $|a(z)| < 1$ on Δ. For a fixed* $\arg(A)$*, unless $a(z)$ is of the form (2.1), there is a univalent solution of (1.1) satisfying the following properties:*

(1) f is of the form (1.2)
(2) $f(\Delta) \subset \Omega$.
(3) The unrestricted limits $f^*(e^{it}) = \lim_{z \to e^{it}} f(e^{it})$ exist on $\partial U \setminus E$ and belong to $\partial\Omega$ where E is a countable set.
(4) For $e^{it} \in E$, the cluster set of f at e^{it} is a straight-line segment joining two points of $\partial\Omega$.
(5) If Ω is the complement of a compact convex set, or if $|a(z)] \leq k < 1$ then $f(\Delta) = \Omega$.
(6) If $\mathbb{C} \setminus \Omega$ is strictly starlike with respect to a finite point p, and if $|a(z)| \leq k < 1$ for all $z \in \Delta$, then f is uniquely determined.

Suppose that $a(z)$ admits an analytic extension across an interval I of the unit circle and that its absolute value there is one. Then the boundary values of f depend strongly on the values of a.

Theorem 2.3 **[BH, Corollary 2.5]** *Let Ω be a simply connected proper subdomain of $\bar{\mathbb{C}}$ containing infinity and suppose that the boundary $\partial\Omega$ is locally connected. Let $a(z)$ be an analytic function on Δ, $|a| < 1$ on Δ and suppose that $a(z)$ has an analytic extension across a subinterval $I = \{e^{it} : \alpha < t < \beta\}$ of the unit circle $\partial\Delta$ such that $|a(z)| \equiv 1$ on I. Let $f(z)$ be a univalent solution of (1.1) which maps Δ onto Ω and which is of the form (1.2). Finally, let e^{it} be an interior point of I. Then the following properties hold:*

(1) The boundary correspondence $f^*(e^{it})$ satisfies

$$f^*(e^{it}) - \overline{a(e^{it}) f^*(e^{it})} + \overline{\int^t f^*(e^{it})\, da(e^{it})} = \text{const.} \tag{2.2}$$

(2) If f^* jumps at e^{it}, (which must and can happen only when $f^*(I)$ contains a linear segment) then we have

$$\arg[f^*(e^{i(t+0)}) - f^*(e^{i(t-0)})] = -\frac{1}{2}\arg\big(a(e^{it})\big) \mod \pi. \tag{2.3}$$

(3) If f^* is continuous at e^{it}, then we have

$$\lim_{h \to 0} \operatorname{Im}\left\{\sqrt{a(e^{it})}\left[\frac{f^*(e^{i(t+h)}) - f^*(e^{i(t-h)})}{h}\right]\right\} = 0. \tag{2.4}$$

(4) If f^* is not constant on a subinterval of I, then the right limit

$$\lim_{h\downarrow 0}\arg[f^*(e^{i(t+h)})-f^*(e^{i(t-0)})]=-\frac{1}{2}\arg a(e^{it})\mod \pi. \tag{2.5}$$

The relation (2.2) can be expressed in the differential form

$$df^*(e^{it})-\overline{a(e^{it})\,df^*(e^{it})}=0,\quad \text{on}\quad I \tag{2.6}$$

or equivalently by

$$\operatorname{Im}\big(\sqrt{a(e^{it})}\,df^*(e^{it})\big)=0. \tag{2.7}$$

Hence, unless $df^*(e^{it})=0$, we have

$$\arg\ df^*(e^{it})=-\frac{1}{2}\arg a(e^{it})\mod \pi,\quad \text{on}\quad I. \tag{2.8}$$

Definition 2.4 (1) We call $\beta(\tau)$ a regulated function on the interval $[a,b]$ if the one-sided limits $\beta(\tau+0)$ and $\beta(\tau-0)$ exist for all $t\in[a,b]$.

(2) Let Ω be a simply connected domain of $\bar{\mathbb{C}}$ containing infinity and suppose that the boundary $\partial\Omega$ is locally connected (every prime end is a singleton). Let ϕ be a conformal mapping from Δ onto Ω keeping infinity fixed. We call Ω a regulated domain if for each prime end $q=w(\tau)=\phi(e^{i\tau})$ of $\partial\Omega$ the direction angle of the forward (half-)tangent at $w(\tau)$,

$$\beta(q)=\lim_{s\downarrow\tau}\arg[w(s)-w(\tau)]=\lim_{s\downarrow\tau}\arg[w(s)-q], \tag{2.9}$$

exists and defines a regulated function. For more details see [4].

In what follows, we need the following definition.

Definition 2.5 Let Ω be a simply connected regulated domain of $\bar{\mathbb{C}}$ containing infinity and let f be a univalent harmonic orientation-preserving mapping from Δ onto Ω. Let q be a prime end of $\partial\Omega$.

(1) If q does not belong to a jump of f^*, we define $\gamma(q)$ and $\delta(q)$ by $(f^*)^{-1}(q)=J(q)=\{e^{it},\gamma(q)\le t\le\delta(q)\}$.
(2) If q is an interior point of a jump, *i.e.*, $q=\lambda f^*(e^{i(t+0)})+(1-\lambda)f^*(e^{i(t-0)})$, $0<\lambda<1$, then define $\gamma(q)=\delta(q)=t$.
(3) If q is the end point $f^*(e^{i(t-0)})$ of a jump, then define $\gamma(q)$ as in (1) and put $\delta(q)=t$.
(4) If q is the end point $f^*(e^{i(t+0)})$ of a jump, then put $\gamma(q)=t$ and define $\delta(q)$ as in (1).

Observe that the cluster sets $C(f^*,e^{i\gamma(q)})$ and $C(f^*,e^{i\delta(q)})$ contain q but they may also contain other points if a jump appears. Furthermore, if $J(q)=(f^*)^{-1}(q)$ is a continuum then $|a|\equiv 1$ on $J(q)$. Finally, relation 2.5 implies that

$$\beta(q)=\lim_{h\downarrow 0}\arg[f^*(e^{i(\delta(q)+h)})-f^*(e^{i(\delta(q)-0)})]=-\frac{1}{2}\arg a(e^{i\delta(q)})\mod \pi \tag{2.10}$$

exists at each prime end $q \in f^*(I)$.

Let Ω be a regulated domain containing infinity and let f be a univalent harmonic and orientation-preserving mapping from Δ onto Ω. Suppose that $a(z)$, as defined by (1.1), admits an analytic continuation across an open interval $I \subset \partial\Delta$ such that $|a(z)| = 1$ there. Let q be a prime end in $f^*(I)$. Denote by $\alpha(q)$ the opening angle at q as seen from the inside of Ω. Set $A(t) = \arg a(e^{it})$, $e^{it} \in I$ as a continuous function and define $\triangle A(q) = \frac{1}{2}\left[A\big(\delta(q)\big) - A\big(\gamma(q)\big)\right]$. Then we have the following relation between $\alpha(q)$ and $\triangle A(q)$.

Theorem 2.6 [**1**] *Let Ω be a simply connected regulated domain of $\bar{\mathbb{C}}$ containing infinity and let f be a univalent harmonic orientation-preserving mapping from Δ onto Ω normalized as in (1.2). Suppose that $a(z)$, as defined by (1.1), admits an analytic continuation across an interval $I \subset \partial\Delta$ such that $|a(z)| \equiv 1$ there. Let q be a prime end in $f^*(I)$ such that $J(q) = (f^*)^{-1}(q) \subset I^0$. With the above notation, we have the following properties:*

(1) If $0 \le \alpha(q) < \pi$, then $\alpha(q) = -\triangle A(q)$.
(2) If $\pi \le \alpha(q) \le 2\pi$, then either $\alpha(q) = -\triangle A(q)$ or $\alpha(q) = -\triangle A(q) + \pi$.

Theorem 2.6 states that the total change of $-\frac{1}{2}\arg a(e^{it})$ over the interval $J(q) = f^{-1}(q)$ is either equal to the opening angle $\alpha(q)$ as seen from the inside of the domain or, if $\pi \le \alpha(q) \le 2\pi$, it can also be $\alpha(q) - \pi$. We shall use the following notation.

Definition 2.7 A prime end $q_0 \in \partial\Omega$ is said to be a *complete resting point* of f^* if $-\triangle A(q_0) = \alpha(q_0)$.

Remark 2.8 (1) If the prime end q is an interior point of a linear segment of $f^*(I)$, then either q is an interior point of a jump of f in which case $\triangle A(q) = 0$ or the inverse image $f^{-1}(q)$ is not a singleton and we have $\triangle A(q) = -\pi$.

(2) Each prime end with an opening angle $\alpha(q)$ strictly less than π is a complete resting point of f^*. In particular if $\alpha(q) = 0$, then $(f^*)^{-1}(q)$ is a singleton yet q is still a complete resting point of f^*. On the other hand, if $\alpha(q) > \pi$, it may happen that $(f^*)^{-1}(q)$ is an interval of $\partial\Delta$ with nonempty interior but q is not a complete resting point.

Proof of Theorem 2.6 The cases $0 \le \alpha < 2\pi$ where given in [BH, Theorem 2.13]. It remains to consider the case where $\alpha = 2\pi$. Suppose that $\alpha(q_0) = 2\pi$, $q_0 \in f^*(I)$, is not a tip of a linear segment. We shall proceed as in the proof of Theorem 3.3 in [1]. Let $I_1 = \{e^{it}, t_1 \le t \le t_2\}$ be a closed subinterval of I containing $f^{-1}(q_0)$. We define

$$(2.11)\quad B(t) = -\pi \sum_{\alpha(q)=-\triangle A(q)} H_q(t) + \pi \sum_{\substack{\alpha(q)=2\pi \text{ and} \\ \delta(q)=\gamma(q)}} H_q(t) - \frac{1}{2}A(t), \quad B\big(\delta(q_0)\big) = \beta(q_0),$$

where $H_q(t)$ is the Heavyside function $H_q(t) = 0$ if $t < \delta(q)$ and $H_q(t) = 1$ if $\delta(q) \le t$. The first sum is taken over the set of all complete resting points and the second sum is taken over all prime ends $q \in f^*(I_1)$ satisfying $\alpha(q) = 2\pi$ and $\delta(q) = \gamma(q)$. Let us remark that for each prime end $q \in f^*(I_1)$ we have $B\big(\delta(q)\big) = \beta(q)$ and $B\big(\gamma(q)\big) = \beta_L(q)$ where $\beta_L(q) - \pi$ is the direction angle of the backward half-tangent of $\partial\Omega$. We want to show that the first sum in (2.11) contains only finitely many terms and that there are no terms in the second sum. We begin by showing that the second sum contains only finitely many terms. Indeed, if not there is a sequence of points $q_j \in f^*(I_1)$ which converges from one side to a point $q_1 \in f^*(I_1)$ for which $\beta(q_j) - \beta_L(q_j)$ converges to $-\pi$ as $j \to \infty$. Therefore, the sequence $\beta(q_j)$ does not converge which contradicts the hypothesis that Ω is a regulated domain.

Next, since $B(t_2) - B(t_1)$ is finite, we conclude that n_R, the number of complete resting points of $f^*(I_1)$, is finite. Let now $\alpha(q_0) = 2\pi$, $q_0 \in f^*(I)$. If q_0 is not a tip of a linear segment, then there must be infinitely many complete resting points in each neighborhood of q_0 which contradicts the last conclusion. If q_0 is a tip of a linear segment, then the case $\delta(q_0) = \gamma(q_0)$ is excluded since q_0 cannot be an interior point of a jump. Therefore $\alpha(q_0) = 2\pi$ is excluded and (2.11) reduces to

$$B(t_2) - B(t_1) = -\pi n_R - \frac{1}{2}[A(t_2) - A(t_1)]. \tag{2.12}$$

The same arguments which prove Theorem 2.13 in [1] show that

$$\alpha(q) = -\triangle A(q) = 2\pi \quad \text{or} \quad \alpha(q) = -\triangle A(q) + \pi$$

and the proof of Theorem 2.6 is complete.

3 A Geometric Characterization of the Image Domain

Let

$$a(z) = e^{i\gamma} \prod_{k=1}^{N} \frac{1 - p_k z}{z - \overline{p_k}} = \sum_{k=0}^{\infty} \alpha_k z^{-k}, \quad |p_k| < 1,\ 1 \le k \le N, \tag{3.1}$$

be a finite Blaschke product of degree N defined on Δ the exterior of the unit disk U and let

$$f(z) = A\Big(z - \frac{1}{\bar{z}}\Big) + B\Big(\bar{z} - \frac{1}{z}\Big) + C \log |z|^2 + F(z)$$

be a univalent solution of the partial differential equation (1.1), $\overline{f_{\bar{z}}(z)} = a(z) f_z(z)$, where F is a bounded harmonic function in Δ and where

$$B = \overline{\alpha_0 A} \quad \text{and} \quad C = \frac{\alpha_1 \overline{\alpha_0} A + \overline{\alpha_1 A}}{1 - |\alpha_0|^2}.$$

We are interested in characterizing the image domains Ω of such mappings. As we have already seen, the boundary correspondence has to satisfy the relation (2.7),

$$\operatorname{Im}\Big(\sqrt{a(e^{it})}\, df^*(e^{it})\Big) = 0,$$

f^*—almost everywhere on the unit circle ∂U. Note that every harmonic function f of the form (1.2) whose coefficients B and C satisfy (1.3) has the property

$$\overline{f_{\bar{z}}(z)} - a(z) f_z(z) = O\Big(\frac{1}{z^2}\Big) \tag{3.2}$$

near infinity.

Definition 3.1 Let Ω be a simply connected regulated domain of $\bar{\mathbb{C}}$ containing infinity.

(1) We say that a prime end $q \in \partial\Omega$ is a point of convexity (with respect to Ω) if there exists a neighborhood V of $f^{-1}(q)$ and a line segment L containing q as an interior point such that $L \setminus \{q\}$ lies in the exterior of $f(\Delta \cap V)$.

(2) We say that a prime end $q \in \partial\Omega$ is a point of convexity (with respect to Ω) if there exists a neighborhood V of $f^{-1}(q)$ and a line segment L containing q as an interior point such that $L \setminus \{q\}$ lies in the interior of $f(\Delta \cap V)$.

If the complement $\mathbb{C} \setminus \Omega$ of Ω is a convex set, then, Ω has no points of convexity. This is in contrast to bounded Jordan domains in $\mathbb{C}$ which have at least three points of convexity. If a is a finite Blaschke product, then we conclude from (2.12) that there are only finitely many prime ends which are points of convexity with opening angle $\alpha(q) < \pi$. It follows then that each point of convexity is a complete resting point of f^*.

Our next result is a direct consequence of Theorem 2.6 and combines the number of complete resting points with the degree of the Blaschke product.

Theorem 3.2 *Let Ω be a regulated domain containing infinity and let $a(z)$ be a Blaschke product of degree $N \geq 2$ on Δ. Let*

$$f(z) = A\Big(z - \frac{1}{\bar{z}}\Big) + B\Big(\bar{z} - \frac{1}{z}\Big) + C\log|z|^2 + F(z), \quad A \neq 0,$$

be a univalent solution of (1.1) with respect to this Blaschke product which maps Δ onto Ω. Then Ω is either the punctured plane or it is a domain whose boundary has at most $N-2$ points of convexity. Furthermore, the number of complete resting points of f^* is equal to $N-2$.

Proof Recall the relation (2.12),

$$B(t_2) - B(t_1) = -\pi n_R - \frac{1}{2}[A(t_2) - A(t_1)].$$

Choose a prime end $q_0 \in \partial\Omega$ and put $t_1 = \delta(q_0)$ and $t_2 = \delta(q_0) + 2\pi$. Then we have $B\big(\delta(q_0) + 2\pi\big) - B\big(\delta(q_0)\big) = 2\pi$ and $\frac{1}{2}\big[A\big(\delta(q_0) + 2\pi\big) - A\big(\delta(q_0)\big)\big] = -N\pi$. Theorem 3.2 then follows immediately.

Remarks 3.3 (1) If $N = 1$, then by Theorem 2.1, there is no univalent solution of (1.1) which maps Δ onto the unbounded component of the complement of a compact continuum.

(2) If $N = 2$, then $f(\Delta)$ is either a one point punctured plane or it is the complement of a compact convex set.

4 The Inverse Problem

Recall that a univalent and orientation-preserving harmonic mapping defined on the exterior of the unit disk, Δ, which keeps infinity fixed, is necessarily of the form (1.2). In particular, if f is a solution of (1.1), where $a(z)$ is a Blaschke product of degree $N \geq 2$, then the boundary function f^* satisfies the relation (2.6) or equivalently (2.7) or (2.8). Furthermore, there are $N-2$ prime ends which are complete resting points of f^* and hence, the image domain Ω has at most $N-2$ points of convexity. It is a natural question to ask if these boundary conditions are sufficient that a solution of the Dirichlet problem is a univalent harmonic solution of (1.1) which maps Δ onto Ω. Unfortunately the answer is negative. For instance, there is no univalent harmonic mapping $f(z) = Az + O(1)$, $A > 0$, which is a univalent solution of $\overline{f_{\bar z}(z)} = \frac{1}{z^2} f_z(z)$ and maps Δ onto Δ. Moreover, we shall see in Section 5, that there is no univalent harmonic mapping $f(z) = Az + O(1)$, which is a univalent solution of $\overline{f_{\bar z}(z)} = \frac{1}{z^4} f_z(z)$ and maps Δ onto

$$\Omega = \{w : |w-1| > 2\} \cap \{w : |w+1| > 1\}$$

even if (2.6) is satisfied on the boundary. In other words, additional conditions must be satisfied.

Suppose now that f is of the form (1.2). Then $F(\frac{1}{\zeta})$ is a bounded harmonic function of ζ in the unit disk U. Using the Poisson formula with respect to the boundary function F^* of F, we have

$$\begin{aligned} F(\frac{1}{\zeta}) &= \frac{1}{2\pi} \int_0^{2\pi} \operatorname{Re}\Big(\frac{e^{it}+\zeta}{e^{it}-\zeta}\Big) F^*(e^{-it})\, dt \\ &= -\frac{1}{2\pi} \int_0^{-2\pi} \operatorname{Re}\Big(\frac{e^{-it}+\zeta}{e^{-it}-\zeta}\Big) F^*(e^{it})\, dt \end{aligned}$$

for ζ in U. Since $F^*(e^{it}) = f^*(e^{it})$, we get for z in Δ,

$$\begin{aligned} F(z) &= \frac{1}{2\pi} \int_0^{2\pi} \operatorname{Re}\Big(\frac{e^{-it}+\frac{1}{z}}{e^{-it}-\frac{1}{z}}\Big) f^*(e^{it})\, dt \\ &= -\frac{1}{2\pi} \int_0^{2\pi} \operatorname{Re}\Big(\frac{e^{it}+z}{e^{it}-z}\Big) f^*(e^{it})\, dt \\ &= -\frac{1}{4\pi} \int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z} f^*(e^{it})\, dt - \frac{1}{4\pi} \int_0^{2\pi} \frac{e^{-it}+\bar z}{e^{it}-\bar z} f^*(e^{it})\, dt. \end{aligned}$$

Therefore, we get

$$F_z(z) = -\frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it}}{(e^{it}-z)^2} f^*(e^{it})\, dt \tag{4.1}$$

$$\overline{F_{\bar z}(z)} = -\frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it}}{(e^{it}-z)^2} \overline{f^*(e^{it})}\, dt. \tag{4.2}$$

Next, using (4.1) and integration by parts, yields

$$\begin{aligned} \frac{1}{2\pi i}\int_0^{2\pi}\frac{df^*(e^{it})}{e^{it}-z} &= -\frac{1}{2\pi i}\int_0^{2\pi}\frac{d}{dt}\Big(\frac{1}{e^{it}-z}\Big)f^*(e^{it})\,dt \\ &= -\frac{1}{2\pi i}\int_0^{2\pi}\frac{-ie^{it}}{(e^{it}-z)^2}f^*(e^{it})\,dt \\ &= \frac{1}{2\pi}\int_0^{2\pi}\frac{e^{it}}{(e^{it}-z)^2}f^*(e^{it})\,dt \\ &= -F_z(z) \\ &= \frac{B}{z^2}+\frac{C}{z}+A-f_z(z), \end{aligned} \tag{4.3}$$

Similarly, we have

$$\begin{aligned} \frac{1}{2\pi i}\int_0^{2\pi}\frac{\overline{df^*(e^{it})}}{e^{it}-z} &= \frac{1}{2\pi}\int_0^{2\pi}\frac{e^{it}}{(e^{it}-z)^2}\overline{f^*(e^{it})}\,dt \\ &= -\overline{F_{\bar z}(z)} \\ &= \frac{\bar A}{z^2}+\frac{\bar C}{z}+\bar B-\overline{f_{\bar z}(z)}. \end{aligned} \tag{4.4}$$

In what follows we need the following property of boundary correspondence.

Definition 4.1 Let Ω be a regulated Jordan domain of $\bar{\mathbb{C}}$ which contains infinity and let ϕ be a conformal univalent mapping from the Δ onto Ω keeping infinity fixed. We say that a (orientation-preserving) mapping $f^*(e^{it})$ from $\partial\Delta$ into $\partial\Omega$ is a quasihomeomorphism, if its image contains at least three non-collinear points of $\partial\Omega$ and if $\phi^{-1}\circ f^*(e^{it})$ is the pointwise limit of a sequence of (orientation-preserving) homeomorphisms from $\partial\Delta$ onto $\partial\Omega$. If, in addition, the linear segments from $f^*(e^{i(t-0)})$ to $f^*(e^{i(t+0)})$ are parts of $\partial\Omega$, then we call $f^*(e^{it})$ a quasihomeomorphism from $\partial\Delta$ *onto* $\partial\Omega$.

Theorem 4.2 *Let*

$$a(z)=e^{i\gamma}\frac{1}{z^{n_0}}\prod_{k=1}^{m}\Big[\frac{1-p_kz}{z-\overline{p_k}}\Big]^{n_k}=\sum_{k=0}^{\infty}\alpha_kz^{-k}, \tag{4.5}$$

$n_0\geq 0$, $n_k>0$ and $0<|p_k|<1$, if $1\leq k\leq m$, $p_k\neq p_j$ if $i\neq j$ and $\sum_{k=0}^m n_k=N$, be a finite Blaschke product of degree N and let Ω be a regulated Jordan domain of $\bar{\mathbb{C}}$ which contains infinity and whose boundary has at most $N-2$ points of convexity. Let $f^*(e^{it})$ be a positively oriented quasihomeomorphism from the unit circle $\partial\Delta$ onto $\partial\Omega$ satisfying (2.7), i.e.,

$$\operatorname{Im}\Big(\sqrt{a(e^{it})}\,df^*(e^{it})\Big)=0$$

f^*-a.e. on $\partial\Delta$. Then the mapping

$$f(z)=A\Big(z-\frac{1}{\bar z}\Big)+B\Big(\bar z-\frac{1}{z}\Big)+C\log|z|^2-\frac{1}{2\pi}\int_0^{2\pi}\operatorname{Re}\Big(\frac{e^{it}+z}{e^{it}-z}\Big)f^*(e^{it})\,dt \tag{4.6}$$

is a univalent solution of (1.1),

$$\overline{f_{\bar{z}}(z)} = a(z) f_z(z)$$

which maps Δ *onto* Ω*, if and only if*

$$\frac{1}{2\pi i}\int_0^{2\pi} \frac{a(z)-a(e^{it})}{z-e^{it}}\, df^*(e^{it}) = \left(\bar{B} + \frac{\bar{C}}{z} + \frac{\bar{A}}{z^2}\right) - a(z)\left(A + \frac{C}{z} + \frac{B}{z^2}\right) \tag{4.7}$$

holds for all z in Δ.

Remark 4.3 The two conditions (2.7) and (4.7) imply the relation (1.3) between the constants A, B and C. Indeed, let $a(z) = \sum_{k=0}^{\infty} \alpha_k z^{-k}$. Since

$$\frac{a(z)-a(e^{it})}{z-e^{it}} = \frac{\alpha_0 - a(e^{it})}{z} + O\left(\frac{1}{z^2}\right)$$

near infinity, $\int_0^{2\pi} df^*(e^{it}) = 0$ and $\int_0^{2\pi} a(e^{it})\, df^*(e^{it}) = \int_0^{2\pi} \overline{df^*(e^{it})} = 0$, the assertion follows by comparing the constant and the $\frac{1}{z}$ coefficients in (4.7).

Proof Note that (2.7) is equivalent to (2.6),

$$\overline{df^*(e^{it})} = a(e^{it})\, df^*(e^{it})$$

f^*-a.e. on $\partial\Delta$. Therefore, we get from (4.3) and (4.4)

$$\begin{aligned}
\overline{f_{\bar{z}}(z)} - a(z) f_z(z) &= -\frac{1}{2\pi i}\int_0^{2\pi} \frac{\overline{df^*(e^{it})} - a(z)\, df^*(e^{it})}{(e^{it}-z)} \\
&\quad + \left(\bar{B} + \frac{\bar{C}}{z} + \frac{\bar{A}}{z^2}\right) - a(z)\left(A + \frac{C}{z} + \frac{B}{z^2}\right) \\
&= -\frac{1}{2\pi i}\int_0^{2\pi} \frac{a(e^{it}) - a(z)}{(e^{it}-z)}\, df^*(e^{it}) \\
&\quad + \left(\bar{B} + \frac{\bar{C}}{z} + \frac{\bar{A}}{z^2}\right) - a(z)\left(A + \frac{C}{z} + \frac{B}{z^2}\right).
\end{aligned} \tag{4.8}$$

Hence, f is a solution of (1.1) if and only if (4.8) holds. Next, observe that (4.7) implies that $f(z) = (Az + B\bar{z})\big(1 + o(1)\big)$ in a neighborhood of infinity, where $|B| = |\alpha_0||A| < |A|$. Therefore, for large r, the image $f(|z| = r)$ is a closed Jordan curve. Applying the argument principle, which holds for solutions of (1.1) in the annulus $1 < |z| < r$, we conclude that f is univalent on Δ and that $f(\Delta) = \Omega$. ∎

In what follows, we show that condition (4.7) in Theorem 4.2 can be replaced by (1.3) and a system of $[\frac{N}{2}]$ ($[x]$ denotes the integer part of a positive number x) equations from the following relations.

(1) For $0 < |p_k| < 1$ and $1 \leq j \leq n_k$,

$$\frac{1}{2\pi i}\int_0^{2\pi} \frac{e^{ijt}\,df^*(e^{it})}{(e^{it}-\overline{p_k})^j} = (j\overline{p_k}A + C) + \frac{B}{\overline{p_k}}\delta_{j,1}. \tag{4.9}$$

(2) If $n_0 = 2$, then

$$\int_0^{2\pi} e^{-it}\,df^*(e^{it}) = 2\pi i(A - \overline{\alpha_2 A}) \tag{4.10}$$

and if $n_0 \geq 3$, then

$$\begin{aligned} \int_0^{2\pi} e^{-it}\,df^*(e^{it}) &= 2\pi i A, \\ \int_0^{2\pi} e^{-ijt}\,df^*(e^{it}) &= 0, \quad 2 \leq j \leq n_0 - 2, \\ \int_0^{2\pi} e^{-i(n_0-1)t}\,df^*(e^{it}) &= -2\pi i\overline{\alpha_{n_0}A}. \end{aligned} \tag{4.11}$$

Set $p(z) = \prod_{k=1}^m (1 - p_k z)^{n_k}$ and $q(z) = z^{n_0}\prod_{k=1}^m (z - \overline{p_k})^{n_k}$. Then, multiplying (4.7) by $e^{-i\gamma/2}z^2q(z)$, we deduce that

$$\begin{aligned} &\frac{e^{-i\gamma/2}z^2q(z)}{2\pi i}\int_0^{2\pi} \frac{a(e^{it}) - a(z)}{(e^{it}-z)}\,df^*(e^{it}) \\ &\quad = e^{-i\gamma/2}q(z)(\bar{B}z^2 + \bar{C}z + \bar{A}) - e^{i\gamma/2}p(z)(Az^2 + Cz + B). \end{aligned} \tag{4.12}$$

Since $a(z)$ is a Blaschke product of degree N we conclude that the left hand side of (4.12) is a polynomial $t(z)$ of degree at most $N + 1$. We claim that

$$t(z) \equiv z^{N+2}\,\overline{t\left(\frac{1}{\bar{z}}\right)}. \tag{4.13}$$

Indeed, in view of (2.6) we have

$$t(z) = \frac{e^{-i\gamma/2}q(z)}{2\pi i}\int_0^{2\pi} \frac{z^2\overline{df^*(e^{it})}}{(e^{it}-z)} - \frac{e^{i\gamma/2}p(z)}{2\pi i}\int_0^{2\pi} \frac{z^2\,df^*(e^{it})}{(e^{it}-z)}$$

so that

$$z^{N+2}\,\overline{t\left(\frac{1}{\bar{z}}\right)} = \frac{e^{i\gamma/2}p(z)}{2\pi i}\int_0^{2\pi} \frac{e^{it}z\,df^*(e^{it})}{(e^{it}-z)} - \frac{e^{-i\gamma/2}q(z)}{2\pi i}\int_0^{2\pi} \frac{e^{it}z\overline{df^*(e^{it})}}{(e^{it}-z)}.$$

Since

$$\int_0^{2\pi} \frac{z^2\,df^*(e^{it})}{(e^{it}-z)} - \int_0^{2\pi} \frac{e^{it}z\,df^*(e^{it})}{(e^{it}-z)} = -z\int_0^{2\pi} df^*(e^{it}) = 0,$$

the claim follows. Next, the right hand side of (4.12) is a polynomial $s(z)$ of degree at most $N+2$ and it is readily seen that relation (4.13) also holds for $s(z)$. Hence, (4.7) reduces to $2+[\frac{N}{2}]$ independent conditions. These conditions may be chosen by comparing the values of any arbitrary $2+[\frac{N}{2}]$ independent linear functionals on the linear space of polynomials. Taking into consideration the two relations in (1.3), we remain with $[\frac{N}{2}]$ equations.

In particular, consider the point evaluations and derivations of order $j, 1 \leq j \leq n_k - 1$ at the points $\frac{1}{p_k}$. Then, since $p(z)$ and its first $n_k - 1$ derivatives vanish at $\frac{1}{p_k}$, we conclude that the above mentioned point evaluation and derivation functionals are identical for

$$\frac{1}{2\pi i}\int_0^{2\pi} \frac{a(e^{it})\, df^*(e^{it})}{(e^{it}-z)} \quad \text{and} \quad \left(\frac{\bar{B}}{z^2}+\frac{\bar{C}}{z}+\bar{A}\right)$$

which imply (4.9). Suppose now that $n_0 \geq 2$. We now consider the point evaluations and derivations of order $j, 1 \leq j \leq n_0+1$ at the origin. Then $q(z)$ and its first n_0-1 derivatives vanish there and we have $B = C = 0$. Therefore, the above mentioned point evaluation and derivation functionals are identical for

$$-\frac{1}{2\pi i}\int_0^{2\pi} \frac{z^2 p(z)}{(e^{it}-z)}\, df^*(e^{it}) \quad \text{and} \quad \bar{A}e^{-i\gamma}q(z) - Ap(z)z^2.$$

We now divide both expressions by $z^2 p(z)$ and conclude that the expressions

$$\frac{1}{2\pi i}\int_0^{2\pi} \frac{df^*(e^{it})}{(e^{it}-z)} \quad \text{and} \quad A - \frac{\bar{A}}{z^2 a(z)}$$

have $n_0 - 1$ equal coefficients when developed in power series around the origin. Since $\frac{1}{a(z)} = \overline{a(\frac{1}{\bar{z}})}$, the power series expansion of $\frac{1}{a(z)}$ at the origin is $\sum_{k=n_0}^{\infty} \overline{\alpha_k} z^k$ so that

$$A - \frac{\bar{A}}{z^2 a(z)} = A - \overline{A\alpha_{n_0}} z^{n_0-2} + O(z^{n_0-1})$$

near the origin. If $n_0 = 2$, then (4.10) follows. If $n_0 \geq 3$, (4.11) follows. Summarizing, we have

Theorem 4.4 *The necessary and sufficient condition (4.7) in Theorem 4.2 can be replaced by any set of* $[\frac{N}{2}]$ *linear functionals which together with the condition (1.3) form a linearly independent set. In particular, we may choose them from the set of equations (4.9) and/or (4.10) or (4.11).*

Remark 4.5 The necessity of the equations (4.9) to (4.11) can also be obtained by the simple observation that $\frac{\overline{f_{\bar{z}}(z)}}{a(z)}$ has to be an analytic function on Δ.

5 Special Cases

In this section we shall meet again the same restrictions on $a(z)$ and $\arg(A)$ which we have met in Theorem 2.1 and 2.2 and we shall get more information about the possible values of the constant A. We suppose again that $a(z)$ is a finite Blaschke product. Applying an appropriate Möbius premapping, we may assume without loss of generality that one of the factors of $a(z)$ is $\frac{1}{z}$ and that we may put $\gamma = 0$ in (3.1). This fact implies that $B = 0$ (see (1.2)) and it will reduce considerably the calculations hereafter. Since $|a(e^{it})| \equiv 1$ the relation (2.7) is enforced on $\partial\Delta$.

5.1 The Case $a(z) = 1/z$

By Theorem 2.1, we know that the only univalent solutions of (1.1) are of the form

$$f(z) = Az + 2\bar{A}\log|z| - \frac{A}{\bar{z}}$$

which maps Δ onto $\bar{\mathbb{C}}$ minus a point. The fact that Ω has to be a one point punctured plane can be deduced directly from the relation (2.7) which is satisfied on $\partial\Delta$ if and only if $df^* \equiv 0$ *i.e.*, if and only if $\partial\Omega$ is a point.

5.2 The Case $a(z) = \frac{1}{z}\frac{1-pz}{z-\bar{p}}$, $|p| < 1$

In this case (1.3) becomes

$$B = 0 \quad \text{and} \quad C = -\overline{pA}. \tag{5.1}$$

It remains to choose one independent condition of (5.1). We choose in (4.9) $p_1 = p$ and $j = 1$ and we get

$$\frac{1}{2\pi i}\int_0^{2\pi} \frac{e^{it}\,df^*(e^{it})}{e^{it} - \bar{p}} = \bar{p}A + C = 2i\bar{p}\,\mathrm{Im}(A).$$

Therefore,

$$\int_0^{2\pi} \frac{df^*(e^{it})}{e^{it} - \bar{p}} = \frac{1}{\bar{p}}\int_0^{2\pi}\Big[\frac{e^{it}}{e^{it} - \bar{p}} - 1\Big]\,df^*(e^{it}) = -4\pi\,\mathrm{Im}(A). \tag{5.2}$$

We may rewrite (5.2) in the form

$$\int_0^{2\pi} \frac{\sqrt{a(e^{it})}\,df^*(e^{it})}{|1 - \bar{p}e^{it}|} = -4\pi\,\mathrm{Im}\{A\}. \tag{5.3}$$

By Theorem 3.2, $\partial\Omega$ is a closed convex curve with no resting points. Therefore, $\sqrt{a(e^{it})}\,df^*(e^{it}) = \pm|df^*(e^{it})|$ does not change sign and (5.3) reduces to

$$\int_0^{2\pi} \frac{|df^*(e^{it})|}{|1 - \bar{p}e^{it}|} = \pm 4\pi\,\mathrm{Im}\{A\}. \tag{5.4}$$

The relation (5.4) shows immediately that $f(\partial U)$ is a singleton if and only if $\mathrm{Im}(A) = 0$.

We may summarize the above discussion as follows.

Theorem 5.1 *Let* $a(z) = \frac{1}{z}\frac{1-pz}{z-\bar{p}}$, $|p| < 1$ *and assume that*

$$f(z) = A\Big(z - \frac{1}{\bar{z}}\Big) + B\Big(\bar{z} - \frac{1}{z}\Big) + C\log|z|^2 + O(1)$$

is a univalent solution of (1.1) defined on Δ.

(1) *If* $\operatorname{Im}(A) = 0$, *then* $f(\Delta) = \bar{\mathbb{C}} \setminus \{q\}$ *for some point* $q \in \mathbb{C}$.

(2) *If* $\operatorname{Im}(A) \neq 0$, *then* $\partial\Omega$ *is either a straight-line segment or a convex closed Jordan curve whose weighted length is given by*

$$\int_0^{2\pi} \frac{|df^*(e^{it})|}{|1 - \bar{p}e^{it}|} = \pm 4\pi \operatorname{Im}(A).$$

In particular, if $a(z) = \frac{1}{z^2}$, *then* $4\pi|\operatorname{Im}(A)|$ *is the euclidean length of* $\partial\Omega$.

The existence proof of Theorem 2.2 which was given in [3] is not elementary and uses non-constructive arguments such as Schauder's fixed point theorem. Applying our Theorem 3.2 and Theorem 4.2 we get a new elementary proof of Theorem 2.2 for the special case $a(z) = \frac{1}{z}\frac{1-pz}{z-\bar{p}}$.

Theorem 5.2 *Let* Ω *be a simply connected domain of* $\bar{\mathbb{C}}$ *containing infinity. Let* $a(z) = \frac{1}{z}\frac{1-pz}{z-\bar{p}}$, $|p| < 1$, *and let* $\alpha, 0 < |\alpha| < \pi$, *be given. Then there exists a univalent solution* f *of (1.1) of the form*

$$f(z) = |A|e^{i\alpha}z\big(1 + o(1)\big),$$

which maps Δ *onto* Ω *if and only if* $\partial\Omega$ *is either a straight-line segment or a convex closed Jordan curve. Furthermore,* f *is unique.*

Proof Put $\operatorname{Im}(A) = \pm\frac{1}{4\pi}\int_0^{2\pi}\frac{|df^*(e^{it})|}{|1-\bar{p}e^{it}|}$ and $\operatorname{Re}(A) = \operatorname{Im}(A)\operatorname{cotan}(\alpha)$, where the sign is positive if and only if $0 < \alpha < \pi$. Construct the boundary correspondence $f^*(e^{it}) = F^*(e^{it})$ using the relation (2.7). There are essentially two possibilities. We choose the one for which (5.2) holds.

As an immediate consequence, we get the following interesting result:

Corollary 5.3 *Let* $a(z) = \frac{1}{z}\frac{1-pz}{z-\bar{p}}$, $|p| < 1$, *and suppose that the complement of* Ω *is a convex continuum. Consider the family* $\mathfrak{F}(a,\Omega)$ *of all univalent solutions* $f(z) = Az\big(1+o(1)\big)$ *of (1.1) which map* Δ *onto* Ω. *Then the variability region of* A *are the two horizontal straight-lines* $\operatorname{Im}(A) = \text{constant}$, *where the two constants are determined by (5.4).*

The next example illustrates the above statements.

Example 5.4 Let $a(z) = \frac{1}{z^2}$ and let $\Omega = \Delta$. By the existence and uniqueness theorem, Theorem 5.1 (Theorem 2.2 respectively), we conclude that

$$\mathfrak{F}\Big(\frac{1}{z^2},\Delta\Big) = \Big\{ f_\alpha(z) = \frac{e^{i\alpha}}{|1-e^{2i\alpha}|}\Big(z - \frac{e^{2i\alpha}}{\bar{z}}\Big), 0 < |\alpha| < \pi \Big\}.$$

Since $\operatorname{Im}(A) = \frac{sin(\alpha)}{|1-e^{2i\alpha}|} = \pm\frac{1}{2}$, we see that (5.4) holds. Indeed, we have $4\pi|\operatorname{Im}(A)| = 2\pi$, which is the euclidean length of $\partial\Omega$. Furthermore, $\operatorname{Re}(A) = \frac{1}{2}\cot(\alpha)$ which implies that the region of variability of A consists of the two horizontal straight lines $\operatorname{Im}(A) = \pm\frac{1}{2}$. One also realizes that there is no univalent solution of $\overline{f_{\bar{z}}(z)} = \frac{1}{z^2}f_z(z)$ normalized at infinity by $f(z) = Az\big(1 + o(1)\big)$, A real, which maps Δ onto itself.

5.3 The Case $a(z) = \frac{1}{z^2}\,(\frac{1-zp}{z-\bar{p}})^2$, $|p| < 1$

In this case, we have $N = 4$. By Theorem 4.4, condition (4.7) may be replaced by any two linear functionals which together with (1.3) form an independent linear set. Since $a(z) = O(\frac{1}{z^2})$ near infinity, we have $B = C = 0$. Equation (4.10) and the substitution $j = 1$ and $p_1 = p$ in (4.9) yield

$$\frac{1}{2\pi i}\int_0^{2\pi} e^{-it}\, df^*(e^{it}) = A - \overline{p^2 A} \tag{5.5}$$

and

$$\frac{1}{2\pi i}\int_0^{2\pi} \frac{e^{it}\, df^*(e^{it})}{e^{it} - \bar{p}} = \bar{p}A.$$

Applying the second derivation functional at the origin to (4.12), we may replace any of these two relations by

$$\frac{1}{2\pi i}\int_0^{2\pi} e^{-2it}\, df^*(e^{it}) = 2(1 - |p|^2)\overline{pA}. \tag{5.6}$$

Example 5.5 The special case where $p = 0$, *i.e.* $a(z) = 1/z^4$, has a nice geometric interpretation. First note that Theorem 3.2 implies the existence of two prime ends which are complete resting points and which may be points of convexity. These points separate the boundary of the image into two arcs Γ_1 and Γ_2. For $i = 1, 2$ let ℓ_i denote the euclidean length of Γ_i. Now, (5.6) reduces to

$$\frac{1}{2\pi i}\int_0^{2\pi} \sqrt{a(e^{it})}\, df^*(e^{it}) = \frac{1}{2\pi i}\int_0^{2\pi} \varepsilon(t)|df^*(e^{it})| = 0$$

where $\varepsilon(t) = 1$ on one of the arcs, say Γ_1 and $\varepsilon(t) = -1$ on Γ_2. We conclude that the euclidean lengths of both arcs are equal, *i.e.* $\ell_1 = \ell_2$. Of course, $f^*(e^{it})$ is still enforced by

$$\frac{1}{2\pi i}\int_0^{2\pi} e^{it}\varepsilon(t)|df^*(e^{it})| = A. \tag{5.7}$$

In the particular case where $\Omega = \Delta$, (5.7) can be easily interpreted as follows. Since $f^*(e^{it})$ and $df^*(e^{it})$ are orthogonal on $\partial\Omega$ we have the relation

$$\arg\, df^*(e^{it}) - \arg f^*(e^{it}) = \pi/2.$$

From (2.7) we conclude that

$$\arg\, df^*(e^{it}) - 2t = 0 \quad \text{or} \quad -\pi$$

as long as $df^*(e^{it}) \neq 0$, *i.e.*, except at the tips of Γ_i which are now the complete resting points of $f^*(e^{it})$. We deduce that for some constants c_i we have

$$\arg f^*(e^{it}) = 2t + c_i$$

on Γ_i. Using the representation $f^*(e^{it}) = e^{i\arg f^*(e^{it})}$ we get

$$|df^*(e^{it})| = |ie^{i\arg f^*(e^{it})} d\arg f^*(e^{it})| = |d\arg f^*(e^{it})| = 2,$$

as long as $df^*(e^{it}) \neq 0$. We shall set now the resting points at $f^*(e^{is})$, $t_0 - \frac{\pi}{2} < s < t_0$, and at $f^*(e^{is})$, $t_0 + \frac{\pi}{2} < s < t_0 + \pi$, so that (5.7) reduces to

$$\pm\Big(\frac{1}{2\pi i}\int_{t_0}^{t_0+\pi/2} 2e^{it}\,dt - \frac{1}{2\pi i}\int_{t_0+\pi}^{t_0+3\pi/2} 2e^{it}\,dt\Big) = A.$$

Putting $\zeta = e^{it_0}$, we get

$$A = \frac{\pm 2(1-i)\zeta}{\pi}.$$

6 An Application to Minimal Surfaces

We now apply our results of Section 5 to minimal surfaces. Let Ω, be as before, a simply connected proper subdomain of $\bar{\mathbb{C}}$ containing infinity and let $S = (u, v, s)$, $s = s(u, v)$, be a nonparametric surface defined over Ω. Then S is a minimal surface if, and only if there exists a univalent complex-valued harmonic mapping $f = u + iv$ from a domain D of $\bar{\mathbb{C}}$ onto Ω such that

$$s_z^2 = -\overline{f_{\bar z}(z)}\, f_z = -a f_z^2 \tag{6.1}$$

holds on D where a is the second dilatation function of f. If Ω is the one point punctured plane, then D is either conformally equivalent to Δ or to $\mathbb{C}\setminus\{0\}$; if not then we may choose $D = \Delta$. Historically, the function $i\sqrt{a}$ is called the *Weierstrass parameter* of the minimal surface and the Gauss map of S is given by the normal vector

$$\vec{N} = \frac{(2\,\mathrm{Im}\,\sqrt{a}, 2\,\mathrm{Re}\,\sqrt{a}, 1-|a|)}{1+|a|}. \tag{6.2}$$

It is interesting to note that $\vec{N}$ depends only on the second dilatation function a and that $\sqrt{a}$ is an analytic function on D. Hence, the study of nonparametric minimal surfaces over Ω with a given Gauss map leads us to the problem of finding univalent harmonic maps from Δ ($\mathbb{C}\setminus\{0\}$ respectively) onto Ω which are solutions of (1.1) and where a is the square of an analytic function on D.

6.1 Minimal Surfaces With Univalent Gauss Maps Onto the Upper Half-Sphere

We first start by considering the case where the image of the Gauss map is the upper half-sphere covered once. By (6.1), $a(z)$ has to be a square of an analytic function on Δ and (6.2) implies that $a(z)$ is the square of a single Blaschke factor. Applying an appropriate Möbius premapping, we may choose the isothermic parameters x and y such that $a(z) = \frac{1}{z^2}$. Therefore, we have $B = C = 0$. Integrating (6.1) we get

$$s(z) = \pm 2\,\mathrm{Im}\{A\log z + O(1)\}$$

which defines a real valued function on Δ if and only if A is pure imaginary. In particular there is no nonparametric minimal surface over the one point punctured plane whose Gauss map has the property that its image is the upper half-sphere covered once. By Theorem 3.2, there is no complete resting point of f^* and hence Ω contains no prime end which is a point of convexity. Summarizing we get

Theorem 6.1 *Let Ω be a simply connected regulated proper subdomain of $\bar{\mathbb{C}}$ containing infinity. Then there exists a nonparametric minimal surface S over Ω whose Gauss map has the property that its image is the upper half-sphere covered exactly once if, and only if the complement $\mathbb{C} \setminus \Omega$ is a compact convex continuum. S is uniquely determined up to translation and reflection with respect to the (u, v)-plane. Furthermore, we have*

$$f(z) = i|A|z + O(1) \quad \textit{and} \quad s(z) = |A| \ln |z| + O(1)$$

near infinity.

Example 6.2 Let us return to Example 5.4. We are looking for a nonparametric minimal surface over Δ with a univalent Gauss map onto the upper half-sphere. The univalent harmonic and orientation-preserving mappings from Δ onto $\Omega = \Delta$ whose second dilatation function is $a(z) = \frac{1}{z^2}$ are $f(z) = \pm\frac{i}{2}(z + \frac{1}{\bar{z}})$. Both describe the same minimal surfaces S, which expressed in the isothermic parameters x and y, are

$$S = \left(-y + \frac{y}{x^2 + y^2}, x + \frac{x}{x^2 + y^2}, \frac{\pm 1}{2} \ln(x^2 + y^2) + \text{const.}\right).$$

6.2 Minimal Surfaces Whose Gauss Map Images Are the Upper Sphere Covered Twice

In this case we may choose the isothermic parameters in such away that $a(z) = (\frac{1-zp}{z(z-\bar{p})})^2$. The complement of the orthogonal projection Ω of S is either a convex set or $\partial\Omega$ has at most two prime ends which are points of convexity. As we have seen in Section 5.3, the conditions (5.5) and (5.6) are necessary and sufficient in order that the solution of the Dirichlet problem with the boundary values induced by the relation (2.7) is a univalent solution of (1.1). For the special case $a(z) = \frac{1}{z^4}$ we immediately get

Theorem 6.3 *Let Ω be a simply connected regulated proper subdomain of $\bar{\mathbb{C}}$ containing infinity.*

(1) If $\partial\Omega$ has at most one point of convexity, then there exist nonparametric minimal surfaces over Ω whose Gauss map has the property that its image is the upper half-sphere covered twice.
(2) If $\partial\Omega$ contains two prime ends q_1 and q_2 which are points of convexity, then there exists a nonparametric minimal surface S over Ω whose Gauss map has the property that its image is the upper half-sphere covered twice if and only if q_1 and q_2 divide $\partial\Omega$ in two parts of equal euclidean length.
(3) In all other cases there are no nonparametric minimal surfaces having the above property.

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