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ON A CONJECTURE OF D. STYER REGARDING UNIVALENT GEOMETRIC AND ANNULAR STARLIKE FUNCTIONS

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ABSTRACT. The aim of this paper is two-fold. First, to give a direct proof for the already established result of Styer which states that a univalent geometrically starlike function f is a univalent annular starlike function if f is bounded. Second, to show that the boundedness condition of f is necessary, thus disproving a conjecture of Styer.

1. INTRODUCTION

A region Ω of the complex plane $\mathbb{C}$ is called *starlike with respect to* $w_0 \in \Omega$ if for every point $w \in \Omega$ the closed line segment $[w_0, w] = \{(1-t)w_0 + tw : 0 \leq t \leq 1\}$ lies in Ω; in this case w_0 is called a *star center point* of Ω. It is well known that the set of star center points of a region Ω, if nonempty, is convex. A univalent function f of the open unit disc $\mathbb{D}$ is said to be *geometrically starlike with respect to* w_0 if $f(\mathbb{D})$ is starlike with respect to w_0. Denote by $S_g(w_0)$ the class of all such functions. It is well known that a univalent function f of $\mathbb{D}$ satisfying $f(0) = w_0$ is geometrically starlike with respect to w_0 if and only if $\Re\{zf'(z)/(f(z) - w_0)\} > 0$ in $\mathbb{D}$; the condition that $f(0) = w_0$ is necessary. Designate by $S_a(w_0)$ the class of univalent functions f of $\mathbb{D}$ for which $\Re\{zf'(z)/(f(z) - w_0)\} > 0$ when $\rho < |z| < 1$ for some $0 < \rho < 1$; this is the class of *annular starlike functions with respect to* w_0. The terminology "geometric starlike" and "annular starlike" is due to Hummel [4].

It is immediate that $S_a(w_0)$ is a subset of $S_g(w_0)$. Nonetheless the fact that the set-inclusion is proper is not immediate. Individual examples and subclasses of functions $f \in S_g(w_0) \setminus S_a(w_0)$ were given consecutively by Bender [1], Hengartner and Schober [3], and Goodman and Saff [2]. Also, Styer [5] recently demonstrated in a noncomputational way a broad class of functions $f \in S_g(w_0) \setminus S_a(w_0)$. Furthermore, he proved, using a result about geometrically starlike functions [6], the following result.

Theorem 1. *If $f \in S_g(w_0)$ is bounded, and if w_0 is an interior point of the set of star center points of $f(\mathbb{D})$, then $f \in S_a(w_0)$.*

Following this, Styer [5] made the following conjecture.

Conjecture 1. *Theorem 1 holds for unbounded functions $f \in S_g(w_0)$.*

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The purpose of this paper is two-fold. First we establish the following analytic characterization for bounded univalent functions f of $\mathbb{D}$ for which the set of star center points of $f(\mathbb{D})$ has a nonempty interior.

Theorem 2. *Let f be a bounded univalent function of $\mathbb{D}$, and let $f(z_0) = w_0$ for $z_0 \in \mathbb{D}$. A necessary and sufficient condition for w_0 to be an interior point of the set of star center points of $f(\mathbb{D})$ is that there exists a positive α such that*

$$(1) \qquad \left|\arg\left\{\Psi(z, z_0) z \frac{f'(z)}{f(z) - w_0}\right\}\right| < \pi/2 - \alpha, \qquad z \in \mathbb{D},$$

where $\Psi(z, z_0) = (z - z_0)(1 - \overline{z}_0 z)/z$, $\Psi(z, 0) \equiv 1$, and arg *is the principal argument function.*

The result yields at once Theorem 1 (without using Styer's result [6].) The second purpose of the paper is to disprove Conjecture 1.

2. Proof of Theorem 2

For the necessity proof, let Δ be a compact disc centered at w_0 and lying in the set of star center points of $f(\mathbb{D})$, and let K be a compact subset of $f(\mathbb{D})$ containing Δ in its interior. For $a \in \partial f(\mathbb{D})$, let b and c be the points of $\partial\Delta$ for which the line segments $[b, a]$ and $[c, a]$ are tangent to Δ and lie, except for a, on the left- and right-hand side of the line segment $[w_0, a]$ respectively. Denote by Δ_a the region bounded by $[b, a]$, $[c, a]$, and the major arc of $\partial\Delta$ ending in b and c. Note that $\overline{\Delta_a} \setminus \{a\}$ lies in $f(\mathbb{D})$ for all $a \in \partial\Delta$.

It is easily seen that the collection $\{\Delta_a : a \in \partial f(\mathbb{D})\}$ is an open covering of K, and that by compactness $K \subset \Delta_{a_1} \cup \Delta_{a_2} \cup \cdots \cup \Delta_{a_r}$ for some $a_1, a_2, \cdots, a_r$ in $\partial f(\mathbb{D})$. Observe that no Δ_{a_i} is contained in $\bigcup_{j \neq i} \Delta_{a_j}$ and that the open-closed segments $(w_0, a_i]$ are mutually disjoint. Re-label the points a_i so that each $0 \leq \arg(a_i - w_0) < 2\pi$ and $\arg(a_i - w_0)$ strictly increases with i. With $a = a_i$, let $b = b_i$ and $c = c_i$. Note that for $1 \leq i \leq r$, the points $c_i, c_{i+1}, b_i, b_{i+1}$, with $c_1 = c_{r+1}$ and $b_1 = b_{r+1}$, are distinct and they appear on $\partial\Delta$ as listed when it is positively traversed. Denote by d_i the point of intersection of the open segments (a_i, b_i) and (a_{i+1}, c_{i+1}), with $a_1 = a_{i+1}$. Let $G = \bigcup_{i=1}^{r} \Delta_{a_i}$. Then G is a simply connected $(2r)$-polygonal region with vertices a_i and d_i, $K \subset G$, and G admits Δ as a subset of its set of star center points. Furthermore, the size of each of the interior and exterior angles of ∂G at the vertices a_i and d_i is less than π. Round the corners of G at each a_i from inside of G and at each d_i from outside of G in a manner that preserves the properties of G; denote, for convenience, the resulting region by G. Let F be the Riemann mapping from $\mathbb{D}$ onto G with $F(z_0) = w_0$ and $\arg F'(z_0) = \arg f'(z_0)$.

With $a \in \partial f(\mathbb{D})$, let 2α be a positive lower bound of the size of the angles $\angle bac$. Evidently $0 < \alpha < \pi/2$. Henceforth, assume $a \in \partial G$, and let b, c and Δ_a be as defined above. Because $G \subset f(\mathbb{D})$, the size of $\angle bac$ is at least 2α for all $a \in \partial G$. Observe that the angle between the outward normal vector of ∂G at a and the vector from w_0 to a is at most $\pi/2 - \alpha$. Since ∂G is smooth, this yields $|\arg[zF'(z)/(F(z) - w_0)]| \leq \pi/2 - \alpha$ for $z \in \partial\mathbb{D}$. Furthermore, by [8, Theorem 3.2], there exists $0 < \rho < 1$ such that $\arg[zF'(z)/(F(z) - w_0)]$ is continuous for $\rho \leq |z| \leq 1$. Since $\arg \Psi(z, z_0) \equiv 0$ for $|z| = 1$, F is analytic in $\mathbb{D}$, and $\Psi(z, z_0)zF'(z)/(F(z) - w_0)$ is a continuous and nonvanishing function in the closed unit disc, inequality (1) holds for F by the maximum principle.

To complete the proof, exhaust Ω by an increasing sequence of compact subsets K_n each containing Δ in its interior, and let F_n be the function F associated with K_n. Because $F_n(\mathbb{D}) \to f(\mathbb{D})$ as $n \to \infty$, $F_n(z_0) = w_0$ and $\arg F_n'(z_0) = \arg f'(z_0)$, the Carathéodory kernel theorem yields the local uniform convergence $F_n \to f$ as $n \to \infty$. We conclude that inequality (1) holds for f since it holds for each F_n. This ends the proof of necessity.

For another proof of necessity, $f \in S_g(w_0)$ means that the function

$$g(\zeta) = f\left(\frac{\zeta + z_0}{1 + \overline{z}_0\zeta}\right) - w_0$$

is a univalent starlike function that satisfies $g(0) = 0$. That is, $\Re[\zeta g'(\zeta)/g(\zeta)] > 0$ in $\mathbb{D}$, or, by letting $z = (\zeta - z_0)/(1 - \overline{z}_0\zeta)$,

$$\Re\left\{\Psi(z, z_0) z \frac{f'(z)}{f(z) - w_0}\right\} > 0, \qquad z \in \mathbb{D},$$

or, since $\arg \Psi(z, z_0) \equiv 0$ for $|z| = 1$, for $\epsilon > 0$ there exists ρ, $0 < \rho < 1$, such that

$$|\arg z f'(z)/(f(z) - w_0)| < \pi/2 + \epsilon, \qquad \rho < |z| < 1.$$

The rest of the proof now proceeds as in the proof of Styer [5, Theorem 1] without using his result about geometrically starlike functions [6]. This ends the second necessity proof.

For the sufficiency proof, inequality (1) yields ρ, $0 < \rho < 1$, such that

$$|\arg \frac{z f'(z)}{f(z) - w_0}| < \pi/2 - \alpha/2, \qquad \rho < |z| < 1.$$

Note that $\arg\{z f'(z)/(f(z) - w)\}$ is a continuous function in (z, w) for values z, $\rho < |z| < 1$, and w sufficiently close to w_0. Fix σ, $\rho < \sigma < 1$. By a compactness argument, there exists a compact disc Δ centered at w_0 such that

$$|\arg \frac{z f'(z)}{f(z) - w}| < \pi/2 - \alpha/3, \qquad |z| = \sigma, \qquad w \in \Delta.$$

Thus, for $|z| = \sigma$ and $w \in \Delta$, $\Re\{z f'(z)/(f(z) - w)\} > 0$ and the closed-open line segment $[w, f(z))$ lies in the image set of f as restricted to the open disc $|z| < \rho$. It follows that for every z, $\rho < |z| < 1$, the closed line segment $[w, f(z)]$ lies in $f(\mathbb{D})$. It is immediate that this property extends for all $z \in \mathbb{D}$. This ends the sufficiency proof and completes the proof of Theorem 2. □

3. Disproof of Conjecture 1

In this section we disprove Conjecture 1 [5].

For a fixed α, $1 < \alpha < 2$, let

$$f(z) = \left(\frac{1 + z}{1 - z}\right)^\alpha, \qquad z \in \mathbb{D}.$$

This is a univalent (close-to-convex) function of $\mathbb{D}$ whose image set $f(\mathbb{D})$ is the wedge $\{w : |\arg w| < \alpha\pi/2\}$ for which the set of star center points is $E = \{w : |\arg w| \leq (2 - \alpha)\pi/2\}$. With $f(\overline{z}) = \overline{f(z)}$, the function preserves symmetry about the real axis. Denote by Γ_r the curve parameterized by $f(re^{i\theta})$, $0 \leq \theta \leq 2\pi$. Then Γ_r is a Jordan curve that is symmetric about the real axis with its upper-half (the part that lies in the closed upper-half plane) starting and terminating at the points $((1 + r)/(1 - r))^\alpha$ and $((1 - r)/(1 + r))^\alpha$ respectively.

Henceforth, we show that $f \notin S_a(w)$ for every w in some real open interval $(0, b)$, which obviously lies in the interior of E.

Direct computation yields

$$\begin{aligned} \Re\left\{1 + z\frac{f''(z)}{f'(z)}\right\} &= \Re\left\{\frac{1+2\alpha z+z^2}{1-z^2}\right\} \\ &= \frac{1-|z|^2}{|1-z^2|^2}\Re(1+2\alpha z+|z|^2). \end{aligned}$$

By letting $R(r,\theta) = \Re\{1 + zf''(z)/f'(z)\}$ for $z = re^{i\theta}$, we obtain

$$R(r,\theta) = \frac{1-r^2}{|1-r^2e^{2i\theta}|^2}(1 + 2\alpha r\cos\theta + r^2).$$

Note that $(1+r^2)/(2\alpha r) < 1$ if and only if r belongs to the open interval $I = (\alpha - \sqrt{\alpha^2-1}, 1)$. Fix $r \in I$, and let $\Theta_r = \cos^{-1}(-(1+r^2)/(2\alpha r))$, where $\pi/2 < \Theta_r < \pi$. The function $R(r,\theta)$ is positive if $-\Theta_r < \theta < \Theta_r$, negative if $\Theta_r < \theta < 2\pi + \Theta_r$, and zero if $\theta = \pm\Theta_r$. Let $z_r = re^{i\Theta_r}$. Since $R(r,\theta)$ is the rate of change of the argument of the tangent vector to Γ_r at $re^{i\theta}$, Γ_r has exactly two points of inflection, namely the points $f(z_r)$ and $f(\overline{z_r})$. Denote by L_r the tangent line to Γ_r at $f(z_r)$, and let b_r be the point of intersection, if it exists, of this line with the real axis. (In fact, because of symmetry, the tangent lines to Γ_r at $f(z_r)$ and $f(\overline{z_r})$ are symmetric about the real axis and meet the real axis at the same point b_r, if it exists.)

Note that a parametric equation of L_r is given by $f(z_r) + itz_rf'(z_r)$, $-\infty < t < \infty$, which meets the real axis if $\Im f(z_r) + t\Re(z_rf'(z_r)) = 0$, or if t assumes the value $t_r = -\Re(z_rf'(z_r))/\Im f(z_r)$. Thus b_r exists and

$$b_r = f(z_r) - iz_rf'(z_r)\frac{\Re(z_rf'(z_r))}{\Im f(z_r)}.$$

Being real,

$$\begin{aligned} b_r &= \Re f(z_r) + \frac{\Re(z_rf'(z_r))\Im(z_rf'(z_r))}{\Im f(z_r)} \\ &= \frac{\Re f(z_r)\Im f(z_r) + \Re(z_rf'(z_r))\Im(z_rf'(z_r))}{\Im f(z_r)} \\ &= \frac{\Im[f(z)^2 + (z_rf'(z_r)^2)]}{2\Im f(z_r)} \\ &= \frac{\Im[f(z)^2\{1 + (z_rf'(z_r)/f(z_r))^2\}]}{2\Im f(z_r)} \\ &= \frac{\Im[f(z)^2\{1 + (2\alpha z_r/(1-z_r^2))^2\}]}{2\Im f(z_r)}. \end{aligned} \tag{2}$$

Note that $\lim_{r\to 1^-} z_r = (-1 + i\sqrt{\alpha^2-1})/\alpha$ since $\lim_{r\to 1^-}\Theta_r = \cos^{-1}(-1/\alpha)$, and consequently,

$$1 + \left(\frac{2\alpha z_r}{(1-z_r^2)}\right)^2 \to 1 + \frac{\alpha^4}{1-\alpha^2}, \qquad \text{as} \qquad r \to 1^-. \tag{3}$$

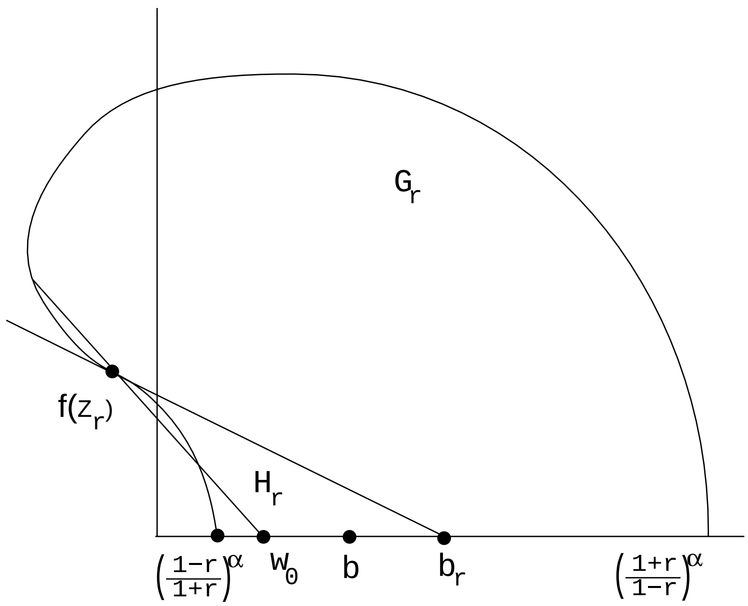

FIGURE 1. Disproof of Conjecture 1

Furthermore, by setting $\Theta = \cos^{-1}(1/\alpha)$,

$$\begin{aligned} \frac{\Im(f(z_r)^2)}{2\Im f(z_r)} &\to \frac{\Im[(1+e^{i\Theta})^{2\alpha}/(1-e^{i\Theta})^{2\alpha}]}{2\Im[(1+e^{i\Theta})^{\alpha}/(1-e^{i\Theta})^{\alpha}]} \qquad \text{as } r \to 1^- \\ &= [\cot(\Theta/2)]^{\alpha}\cos(\alpha\pi/2) \\ &= [(\alpha-1)/(\alpha+1)]^{\alpha/2}\cos(\alpha\pi/2). \end{aligned} \tag{4}$$

Thus if $b = \lim_{r\to 1^-} b_r$, then, by using equations (3) and (4) in equation (2), we conclude that

$$b = [1 + \alpha^4/(1-\alpha^2)][(\alpha-1)/(\alpha+1)]^{\alpha/2}\cos(\alpha\pi/2),$$

which is obviously positive.

Fix $w_0 \in (0, b)$; see Figure 1. We show that $f \notin S_a(w_0)$. It can be easily verified that $\lim_{r\to 1^-} f(r) = \infty$, $\lim_{r\to 1^-} f(-r) = 0$,

$$\lim_{r\to 1^-} f(z_r) = e^{i\alpha\pi/2}\left(\frac{1-\alpha}{1+\alpha}\right)^{\alpha/2}$$

lies in the open second quadrant of $\mathbb{C}$, and

$$\lim_{r\to 1^-} t_r = \frac{\alpha^2}{\sqrt{\alpha^2-1}}$$

is positive. In view of this, there exists ρ, $0 < \rho < 1$, such that for any r, $\rho < r < 1$, the points w_0, b_r and b lie in the interior region of Γ_r with $b_r > w_0$, $f(z_r)$ lies in the open second quadrant of $\mathbb{C}$, and t_r is positive. The latter fact implies that the tangent vector to Γ_r at $f(z_r)$ has the same direction as the vector from $f(z_r)$ to b_r. Endow this direction to L_r, and denote by δ_r and σ_r the subarcs of Γ_r parameterized by $f(re^{i\theta})$, $0 \le \theta \le \Theta_r$, and $f(re^{i\theta})$, $\Theta_r \le \theta \le \pi$, respectively. By invoking the mapping properties of f on the circle $|z| = r$, the arc δ_r and the line segments

$[f(z_r), b_r]$ and $[b_r, ((1+r)/(1-r))^\alpha]$ bound a convex Jordan subregion, G_r, of the upper-half of the interior region of Γ_r; note that G_r lies on the left-hand side of L_r. Also, the arc σ_r and the line segments $[f(z_r), b_r]$ and $[((1-r)/(1+r))^\alpha, b_r]$ bound a Jordan subregion, H_r, of the upper-half of the interior region of Γ_r; note also that H_r lies on the right-hand side of L_r. By virtue of the direction of the tangent vector to Γ_r at $f(z_r)$ and the fact that $((1-r)/(1+r))^\alpha < w_0 < b_r$, we conclude that the ray from w_0 through $f(z_r)$ crosses, consecutively, once each of the interiors of the arcs σ_r and δ_r in a manner that yields a cross-cut in G_r. Therefore, $f \notin S_a(w_0)$ for every $w_0 \in (0, b)$ and Conjecture 1 is false.

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