Solutions in Complex Analysis: Part 2

(Some more solutions to some problems taken from Complex Variables and Applications, R.V. Churchill, 2nd Edition)

Q.1. Show that

$$I = \int_0^\infty \frac{\sin^2 x}{x^2} dx = \frac{\pi}{2}$$

The standard way is to use Residue theory. An easier way does exist. We know that

$$\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}$$

Use integration by parts, namely

$$\int u dv = uv - \int v du$$

Let

$$u = \sin^2 x \Rightarrow du = 2\cos x \sin x dx = \sin 2x dx$$
$$dv = \frac{1}{x^2} dx \Rightarrow v = -\frac{1}{x}$$

from which we find that

$$\int_0^\infty \frac{\sin^2 x}{x^2} dx = -\frac{1}{x}\sin^2 x\Big|_0^\infty + \int_0^\infty \frac{\sin 2x dx}{x} = 0 + \int_0^\infty \frac{\sin 2x dx}{x}$$

The second integral is familiar

$$\int_0^\infty \frac{\sin(2x) dx}{x} = \int_0^\infty \frac{\sin(2x) d(2x)}{(2x)} = \frac{\pi}{2}$$

So then we have shown that

$$\int_0^\infty \frac{\sin^2 x}{x^2} dx = \frac{\pi}{2}$$

A second approach is to use the Theory of Residue.

There is a pole of order m=2 at z=0. Note the function is symmetric.

$$2\int_0^\infty \frac{\sin^2 x}{x^2}dx + \oint_{z\to 0} \frac{\sin^2 z}{z^2}dz + \oint_{z\to\infty} \frac{\sin^2 z}{z^2}dz = 2\pi i \operatorname{Res}(z=0)$$

Evidently, the contour at infinity is zero.

The contour around z=0 is a half-circle that is

$$\oint_{z=0} \frac{\sin^2 z}{z^2}dz = \left[\frac{1}{2}\oint_{z=0} \frac{(1-\cos 2z)}{z^2}dz\right]$$
$$= \operatorname{Im}\left[\frac{1}{2}\oint_{z=0} \frac{(1-e^{2iz})}{z^2}dz\right]$$

The Residue at z=0 is then

$$R(z=0) = -i\pi\left(\frac{1}{2}\right)\left(\operatorname{Im}\left[\frac{d}{dz}\left(z^2\frac{(1-e^{2iz})}{z^2}\right)\right]\Big|_{z=0}\right)$$
$$= -\pi$$

From which

$$2\int_0^\infty \frac{\sin^2 x}{x^2}dx - \pi = 0$$

So that

$$\int_0^\infty \frac{\sin^2 x}{x^2}dx = \frac{\pi}{2}$$

Q.2. Show that

$$I = \int_0^\infty \frac{\ln x}{x^2+1} = 0$$

Try first by a convenient substitution, that is let

$$y = \ln x \Rightarrow x = e^y \Rightarrow dx = e^y dy$$

which means that

$$\begin{aligned} I &= \int_{-\infty}^{\infty} \frac{y}{\left(e^{2y}+1\right)} e^y dy \\ &= \int_{-\infty}^{\infty} \frac{e^y}{e^y} \frac{y}{\left(e^y + e^{-y}\right)} dy \\ &= \frac{1}{2} \int_{-\infty}^{\infty} y\operatorname{sech}(y) dy \end{aligned}$$

Take particular note of the change in the lower limit of integration.

There are several ways of convincing yourself that the integral is equal to zero. For instance, the hyperbolic secant function is symmetric and so by inspection, the integral is equal to 0.

The hyperbolic secant can be expressed as a sum over even powers in y, so that the composite function has odd powers,

$$\operatorname{sech}(y) = \sum_{n=1}^{\infty} F_n (-1)^n y^{2n} \Rightarrow y\operatorname{sech}(y) = \sum_{n=1}^{\infty} F_n (-1)^n y^{2n+1}$$

The integral then once again becomes a sum over even powers

$$I = \frac{1}{2} \int_{-\infty}^{\infty} \sum_{n=1}^{\infty} F_n (-1)^n y^{2n+1} dy = \frac{1}{2} \sum_{n=1}^{\infty} \frac{F_n}{2n+2} (-1)^n y^{2n+2}$$

Which given the limits of integration once again yields 0.

In terms of residue theory, the proof is somewhat more complicated.

$$I = \int_0^\infty \frac{\ln x}{x^2 + 1} dx$$

Consider the transformation

$$y = \ln x \Rightarrow x = e^y$$

from which we find (note the lower limit of integration)

$$\begin{aligned} I &= \int_0^\infty \frac{ye^y}{e^{2y} + 1} dy \\ &= \int_{-\infty}^\infty \left(\frac{e^y}{e^y} \right) \frac{y}{\left(e^y + e^{-y}\right)} dy \\ &= \frac{1}{2} \int_{-\infty}^\infty \frac{y}{\cosh y} dy \end{aligned}$$

Transform from the hyperbolic to the trigonometric space

$$y = i\theta \Rightarrow \cosh y = \cos\theta$$

From which we have

$$I = -\frac{1}{2} \int_{-\infty}^\infty \frac{\theta}{\cos\theta} d\theta$$

The poles for this integral are countably infinite

$$\theta \pm \left(2n + 1\right)\frac{\pi}{2} = 0$$

By inspection the poles are matched and opposite in value

$$R\left(\theta+(2n+1)\frac{\pi}{2}=0\right)=\pm(2\pi i)\frac{\left[\frac{d}{d\theta}\theta\right]}{\left[\frac{d}{d\theta}\cos\theta\right]}\Big|_{\theta=-(2n+1)\frac{\pi}{2}}$$

$$=2\pi i(-1)^n$$

$$R\left(\theta-(2n+1)\frac{\pi}{2}=0\right)=\pm(2\pi i)\frac{\left[\frac{d}{d\theta}\theta\right]}{\left[\frac{d}{d\theta}\cos\theta\right]}\Big|_{\theta=(2n+1)\frac{\pi}{2}}$$

$$=2\pi i(-1)^{n+1}$$

From which we find that

$$\int_0^\infty \frac{\ln x}{x^2+1}dx=\sum_{n=0}^{\infty}2\pi i(-1)^n+2\pi i(-1)^{n+1}=0$$

Q.3. Show that

$$I=\int_0^\infty \frac{(\ln x)^2}{x^2+1}dx=\frac{\pi^3}{8}$$

Use a half circle in the upper plane with an exclusion at z=0.

$$\lim_{\substack{\rho\to 0\\ R\to\infty}}\int_{-R}^{-\rho}\frac{(\ln z)^2}{z^2+1}dz+\lim_{\substack{\rho\to 0\\ R\to\infty}}\int_{\rho}^{R}\frac{(\ln z)^2}{z^2+1}dz+\oint_{C\rho}\frac{(\ln z)^2}{z^2+1}dz+\oint_{CR}\frac{(\ln z)^2}{z^2+1}dz=2\pi i\text{Res}(z=i)$$

The Residue at z= i is

$$\text{Res}(z=i)=\left[(z-i)\frac{(\ln z)^2}{(z-i)(z+i)}\right]\Big|_{z=i}$$

$$=\frac{1}{2i}(\ln i)^2=\frac{1}{2i}\left(\ln 1+i\frac{\pi}{2}\right)^2=-\frac{\pi^2}{8i}$$

By inspection we see that the two contour integrals are equal to zero

The two remaining integrals

$$\lim_{\substack{\rho\to 0\\ R\to\infty}}\int_{-\rho}^{-R}\frac{(\ln z)^2}{z^2+1}dz+\lim_{\substack{\rho\to 0\\ R\to\infty}}\int_{\rho}^{R}\frac{(\ln z)^2}{z^2+1}dz=\lim_{\substack{\rho\to 0\\ R\to\infty}}\int_{-\rho}^{-R}\frac{(\ln r)^2}{r^2+1}dr+\lim_{\substack{\rho\to 0\\ R\to\infty}}\int_{\rho}^{R}\frac{(\ln r+i\pi)^2}{r^2+1}dr$$

$$=2\lim_{\substack{\rho\to 0\\ R\to\infty}}\int_{\rho}^{R}\frac{(\ln r)^2}{r^2+1}dr+2\pi i\lim_{\substack{\rho\to 0\\ R\to\infty}}\int_{\rho}^{R}\frac{(\ln r)dr}{r^2+1}-\pi^2\lim_{\substack{\rho\to 0\\ R\to\infty}}\int_{\rho}^{R}\frac{dr}{r^2+1}$$

Given that

$$\int_0^\infty\frac{dr}{r^2+1}=\frac{\pi}{2}$$

Equating the Real terms across the equal sign yields

$$2\lim_{\substack{\rho\to 0\\ R\to\infty}}\int_{\rho}^{R}\frac{(\ln r)^2}{r^2+1}dr-\pi^2\lim_{\substack{\rho\to 0\\ R\to\infty}}\int_{\rho}^{R}\frac{dr}{r^2+1}=2\lim_{\substack{\rho\to 0\\ R\to\infty}}\int_{\rho}^{R}\frac{(\ln r)^2}{r^2+1}dr-\pi^2\left(\frac{\pi}{2}\right)$$

$$=2\pi i\left(-\frac{\pi^2}{8i}\right)$$

$$=-\frac{\pi^3}{4}$$

From which we find

$$\int_0^\infty\frac{(\ln r)^2}{r^2+1}dr=\frac{1}{2}\left(\frac{\pi^3}{2}-\frac{\pi^3}{4}\right)=\frac{\pi^3}{8}$$

Equating the Imaginary terms we find as well that

$$\int_0^\infty\frac{(\ln x)dx}{x^2+1}=0$$

Q.4. Show that $$I=\int_0^\infty \frac{(\ln x)}{(x^2+1)^2}dx=-\frac{\pi}{4}$$

There are four contributions to this integral over the upper half circle and a pole at z=i,

$$\lim_{\substack{\rho\to 0\\ R\to\infty}}\int_{-R}^{-\rho}\frac{(\ln r+i\pi)}{(r^2+1)^2}dr+\lim_{\substack{\rho\to 0\\ R\to\infty}}\int_{\rho}^{R}\frac{(\ln r)}{(r^2+1)^2}dr+\oint_{z\to 0}\frac{(\ln z)}{(z^2+1)^2}dz+\oint_{z\to\infty}\frac{(\ln z)}{(z^2+1)^2}dz=2\pi i\text{Res}(z=i)$$

By inspection the contour integrals at z=0 and at infinity are both zero. We see then that

$$2\int_0^\infty\frac{(\ln r)}{(r^2+1)^2}dr+i\pi\int_0^\infty\frac{dr}{(r^2+1)^2}=2\pi i\text{Res}(z=i)$$

The Residue at z=i is

$$\text{Res}(z=i)=\left[\frac{d}{dz}(z-i)^2\left(\frac{\ln z}{(z-i)^2(z+i)^2}\right)\right]\Big|_{z=i}$$

$$=\left[\frac{1}{z}\frac{1}{(z+i)^2}-\frac{2\ln z}{(z+i)^3}\right]\Big|_{z=i}=\frac{i}{4}+\frac{\pi}{8}$$

Equating the Real terms across the equal sign

$$2\int_0^\infty\frac{(\ln r)}{(r^2+1)^2}dr=2\pi i\left(\frac{i}{4}\right)\Rightarrow\int_0^\infty\frac{(\ln r)}{(r^2+1)^2}dr=-\frac{\pi}{4}$$

Equating the Imaginary terms we find that

$$i\pi\int_0^\infty\frac{dr}{(r^2+1)^2}=2\pi i\left(\frac{\pi}{8}\right)\Rightarrow\int_0^\infty\frac{dr}{(r^2+1)^2}=\frac{\pi}{4}$$