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Univalence criteria for a new integral operator

Nicoleta Breaz [a], Virgil Pescar [b], Daniel Breaz [a,*]

[a] *Department of Mathematics, "1 Decembrie 1918" University of Alba Iulia, 510009, Alba Iulia, Romania*

[b] *Department of Mathematics, "Transilvania" University of Braşov, 500091 Braşov, Romania*

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ABSTRACT

In view of an integral operator $H_{\gamma_1,\gamma_2,\ldots,\gamma_{[\mathrm{Re}\,\eta]},\beta,\eta}$ for an analytic function f in the open unit disk $\mathcal{U}$, sufficient conditions for univalence of this integral operator are discussed.

1. Introduction

Let $\mathcal{A}$ be the class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the open unit disk $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$. Let S denote the subclass of $\mathcal{A}$ consisting of all univalent functions f in $\mathcal{U}$.

For $f \in \mathcal{A}$, the integral operator G_α is defined by

$$G_\alpha(z) = \int_0^z \left(\frac{f(u)}{u}\right)^{\frac{1}{\alpha}} \mathrm{d}u \tag{1.1}$$

for some complex numbers α ($\alpha \neq 0$).

In [1] Kim–Merkes have proved that the integral operator G_α is in the class $\mathcal{S}$ for $\frac{1}{|\alpha|} \leq \frac{1}{4}$ and $f \in \mathcal{S}$.

In [2] Pascu and Pescar defined the integral operator

$$H_{\alpha,\beta}(z) = \left[\beta \int_0^z u^{\beta-1} \left(\frac{f(u)}{u}\right)^{\alpha} \mathrm{d}u\right]^{\frac{1}{\beta}} \tag{1.2}$$

for α, β being complex numbers, $\beta \neq 0$ and $f \in \mathcal{A}$.

In [2–5] we have the sufficient conditions of univalence for the integral operator $H_{\alpha,\beta}$.

* Corresponding author.
E-mail addresses: nbreaz@uab.ro (N. Breaz), virgilpescar@unitbv.ro (V. Pescar), dbreaz@uab.ro, breazdaniel@yahoo.com (D. Breaz).

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Also, the integral operator J_γ for $f \in \mathcal{A}$ is given by

$$M_\gamma(z) = \left\{ \frac{1}{\gamma} \int_0^z u^{-1} (f(u))^{\frac{1}{\gamma}} \, \mathrm{d}u \right\}^\gamma \tag{1.3}$$

where γ is a complex number, $\gamma \neq 0$.

Miller and Mocanu [6] have shown that the integral operator M_γ is in the class S for $f \in \mathcal{S}^*$, $\gamma > 0$, $\mathcal{S}^*$ is the subclass of $\mathcal{S}$ consisting of all starlike functions f in $\mathcal{U}$.

We introduce the general integral operator

$$H_{\gamma_1,\gamma_2\dots,\gamma_{|[\mathrm{Re}\,\eta]|},\beta,\eta}(z) = \left\{ \eta\beta \int_0^z u^{\eta\beta-1} \left(\frac{f_1(u)}{u}\right)^{\frac{1}{\gamma_1}} \dots \left(\frac{f_{|[\mathrm{Re}\,\eta]|}(u)}{u}\right)^{\frac{1}{\gamma_{|[\mathrm{Re}\,\eta]|}}} \mathrm{d}u \right\}^{\frac{1}{\eta\beta}} \tag{1.4}$$

for $f_j \in \mathcal{A}$, γ_j, η, β complex numbers, where $\gamma_j \neq 0$, $\mathrm{Re}\,\eta \notin [0, 1)$, $j = \overline{1, |[\mathrm{Re}\,\eta]|}$, $\beta \neq 0$, $|[\mathrm{Re}\,\eta]|$ is the modulus of integer part of η.

If in (1.4) we take $\eta = 1$, $\gamma_1 = \frac{1}{\alpha}$, $f_1 = f$ we obtain the integral operator $H_{\alpha,\beta}$.

For $\eta\beta = 1$, $|[\mathrm{Re}\,\eta]| = 1$, $\gamma_1 = \alpha$, $f_1 = f$, for (1.4), we have the integral operator G_α.

For $\eta\beta = \frac{1}{\gamma}$, $\gamma_1 = \gamma$, $|[\mathrm{Re}\,\eta]| = 1$, $f_1 = f$, from (1.4) we obtain the integral operator M_γ.

2. Preliminary results

We need the following lemmas.

Lemma 2.1 ([7]). *Let α be a complex number,* $\mathrm{Re}\,\alpha > 0$ *and $f \in \mathcal{A}$. If*

$$\frac{1 - |z|^{2\mathrm{Re}\,\alpha}}{\mathrm{Re}\,\alpha} \left| \frac{zf''(z)}{f'(z)} \right| \leq 1 \tag{2.1}$$

for all $z \in \mathcal{U}$, then for any complex number β, $\mathrm{Re}\,\beta \geq \mathrm{Re}\,\alpha$ *the function*

$$F_\beta(z) = \left[\beta \int_0^z u^{\beta-1} f'(u) \mathrm{d}u \right]^{\frac{1}{\beta}} \tag{2.2}$$

is in the class S.

Lemma 2.2 (*Schwarz [8]*). *Let f be a function regular in the disk $\mathcal{U}_R = \{z \in \mathbb{C} : |z| < R\}$ with $|f(z)| < M$, M fixed. If $f(z)$ has one zero with multiplicity order bigger than m, for $z = 0$, then*

$$|f(z)| \leq \frac{M}{R^m} |z|^m, \quad z \in \mathcal{U}_R \tag{2.3}$$

the equality (in the inequality (2.3) for $z \neq 0$) can hold only if

$$f(z) = \mathrm{e}^{\mathrm{i}\theta} \frac{M}{R^m} z^m,$$

where θ is constant.

3. Main results

Theorem 3.1. *Let γ_j, β, η be complex numbers, $\beta \neq 0$,* $\mathrm{Re}\,\eta \notin [0, 1)$, $j = \overline{1, |[\mathrm{Re}\,\eta]|}$, $a = \sum_{j=1}^{|[\mathrm{Re}\,\eta]|} \mathrm{Re}\,\frac{1}{\gamma_j} > 0$ *and* $f_j \in \mathcal{A}$, $f_j(z) = z + b_{2j}z^2 + b_{3j}z^3 + \cdots$, $j = \overline{1, |[\mathrm{Re}\,\eta]|}$.

If

$$\left| \frac{zf_j'(z)}{f_j(z)} - 1 \right| \leq \frac{(2a+1)^{\frac{2a+1}{2a}}}{2\,|[\mathrm{Re}\,\eta]|} \left|\gamma_j\right|, \quad j = \overline{1, |[\mathrm{Re}\,\eta]|}, \tag{3.1}$$

for all $z \in \mathcal{U}$, and $\mathrm{Re}\,\eta\beta \geq a$, *then the function*

$$H_{\gamma_1,\gamma_2,\dots,\gamma_{|[\mathrm{Re}\,\eta]|},\beta,\eta}(z) = \left\{ \eta\beta \int_0^z u^{\eta\beta-1} \left(\frac{f_1(u)}{u}\right)^{\frac{1}{\gamma_1}} \dots \left(\frac{f_{|[\mathrm{Re}\,\eta]|}(u)}{u}\right)^{\frac{1}{\gamma_{|[\mathrm{Re}\,\eta]|}}} \mathrm{d}u \right\}^{\frac{1}{\eta\beta}} \tag{3.2}$$

is in the class $\mathcal{S}$.

Proof. We consider the function

$$g(z) = \int_0^z \left(\frac{f_1(u)}{u}\right)^{\frac{1}{\gamma_1}} \cdots \left(\frac{f_{|[\mathrm{Re}\,\eta]|}(u)}{u}\right)^{\frac{1}{\gamma_{|[\mathrm{Re}\,\eta]|}}} \mathrm{d}u. \tag{3.3}$$

The function g is regular in $\mathcal{U}$. We define the function $p(z) = \frac{zg''(z)}{g'(z)}$, $z \in \mathcal{U}$ and we obtain

$$p(z) = \frac{zg''(z)}{g'(z)} = \sum_{j=1}^{|[\mathrm{Re}\,\eta]|} \left[\frac{1}{\gamma_j}\left(\frac{zf_j'(z)}{f_j(z)} - 1\right)\right], \quad z \in \mathcal{U}. \tag{3.4}$$

From (3.1) and (3.4) we have

$$|p(z)| \le \frac{(2a+1)^{\frac{2a+1}{2a}}}{2} \tag{3.5}$$

for all $z \in \mathcal{U}$ and applying Lemma 2.2 we get

$$|p(z)| \le \frac{(2a+1)^{\frac{2a+1}{2a}}}{2}|z|, \quad z \in \mathcal{U}. \tag{3.6}$$

From (3.4) and (3.6) we have

$$\frac{1-|z|^{2a}}{a}\left|\frac{zg''(z)}{g'(z)}\right| \le \frac{(1-|z|^{2a})|z|}{a} \cdot \frac{(2a+1)^{\frac{2a+1}{2a}}}{2} \tag{3.7}$$

for all $z \in \mathcal{U}$.

Because

$$\max_{|z|\le 1} \frac{(1-|z|^{2a})|z|}{a} = \frac{2}{(2a+1)^{\frac{2a+1}{2a}}},$$

from (3.7) we have

$$\frac{1-|z|^{2a}}{a}\left|\frac{zg''(z)}{g'(z)}\right| \le 1 \tag{3.8}$$

for all $z \in \mathcal{U}$. So, by the Lemma 2.1, the integral operator $H_{\gamma_1,\gamma_2,\ldots,\gamma_{|[\mathrm{Re}\,\eta]|},\beta,\eta}$ belongs to the class $\mathcal{S}$. □

Corollary 3.2. *Let γ be a complex number, $a = \mathrm{Re}\,\frac{1}{\gamma} > 0$ and $f \in \mathcal{A}$, $f(z) = z + b_{21}z^2 + b_{31}z^3 + \cdots$.*

If

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \le \frac{(2a+1)^{\frac{2a+1}{2a}}}{2}|\gamma| \tag{3.9}$$

for all $z \in \mathcal{U}$, then the integral operator M_γ defined by (1.3) belongs to the class $\mathcal{S}$.

Proof. We take $\eta = 1$, $\beta = \frac{1}{\gamma}$, $\gamma_1 = \gamma$, $f_1 = f$ in Theorem 3.1. □

Corollary 3.3. *Let α, β be complex numbers, $\beta \ne 0$, $a = \mathrm{Re}\,\alpha > 0$, $f \in \mathcal{A}$, $f(z) = z + b_{21}z^2 + b_{31}z^3 + \cdots$.*

If

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \le \frac{(2a+1)^{\frac{2a+1}{2a}}}{2\,|\alpha|} \tag{3.10}$$

for all $z \in \mathcal{U}$, then for $\mathrm{Re}\,\beta \ge \mathrm{Re}\,\alpha$ the integral operator $H_{\alpha,\beta}$ is in the class $\mathcal{S}$.

Proof. For $\eta = 1$, $\gamma_1 = \frac{1}{\alpha}$, $f_1 = f$, from Theorem 3.1 we have Corollary 3.3. □

Corollary 3.4. *Let α be a complex number, $a = \mathrm{Re}\,\frac{1}{\alpha} \in (0, 1]$, $f \in \mathcal{A}$, $f(z) = z + b_{21}z^2 + b_{31}z^3 + \cdots$.*

If

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \le \frac{(2a+1)^{\frac{2a+1}{2a}}}{2}\,|\alpha| \tag{3.11}$$

for all $z \in \mathcal{U}$, then the integral operator G_α given by (1.1) is in the class $\mathcal{S}$.

Proof. We take $\eta\beta = 1$, $|[\mathrm{Re}\,\eta]| = 1$, $\gamma_1 = \alpha$, $f_1 = f$ in Theorem 3.1. □

Corollary 3.5. *Let α, η be complex numbers $a = |[\mathrm{Re}\,\eta]| \cdot \mathrm{Re}\,\frac{1}{\alpha}$, $\mathrm{Re}\,\eta \notin [0, 1)$, $a \in (0, 1]$ and $f_j \in \mathcal{A}$, $f_j(z) = z + b_{2j}z^2 + \cdots$, $j = \overline{1, |[\mathrm{Re}\,\eta]|}$.*

If

$$\left|\frac{zf_j'(z)}{f_j(z)} - 1\right| \leq \frac{(2a+1)^{\frac{2a+1}{2a}}}{2\,|[\mathrm{Re}\,\eta]|}|\alpha|, \quad j = \overline{1, |[\mathrm{Re}\,\eta]|} \tag{3.12}$$

for all $z \in \mathcal{U}$, then the function

$$L_\alpha(z) = \int_0^z \left(\frac{f_1(u)}{u}\right)^{\frac{1}{\alpha}} \cdots \left(\frac{f_{|[\mathrm{Re}\,\eta]|}(u)}{u}\right)^{\frac{1}{\alpha}} \mathrm{d}u \tag{3.13}$$

is in the class $\mathcal{S}$.

Proof. For $\eta\beta = 1$, $\gamma_1 = \gamma_2 = \cdots = \gamma_{|[\mathrm{Re}\,\eta]|} = \alpha$ from Theorem 3.1. we obtain the Corollary 3.5. □

Theorem 3.6. *Let $\gamma_j, \alpha, \beta, \eta$ be complex numbers, $\gamma_j \neq 0$, $\mathrm{Re}\,\eta \notin [0, 1)$, $\beta \neq 0$, $a = \mathrm{Re}\,\alpha > 0$, $j = \overline{1, |[\mathrm{Re}\,\eta]|}$ and $f_j \in \mathcal{S}$, $f_j(z) = z + \sum_{k=2}^{\infty} b_{kj}z^k$, $j = \overline{1, |[\mathrm{Re}\,\eta]|}$.*

If

$$\sum_{j=1}^{|[\mathrm{Re}\,\eta]|} \frac{1}{|\gamma_j|} \leq \frac{a}{2}, \quad \text{for } 0 < a < \frac{1}{2} \tag{3.14}$$

or

$$\sum_{j=1}^{|[\mathrm{Re}\,\eta]|} \frac{1}{|\gamma_j|} \leq \frac{1}{4}, \quad \text{for } a \geq \frac{1}{2} \tag{3.15}$$

then for $\mathrm{Re}\,\eta\beta \geq a$, the integral operator $H_{\gamma_1,\gamma_2,\ldots,\gamma_{|[\mathrm{Re}\,\eta]|},\beta,\eta}$ given by (1.4) is in the class $\mathcal{S}$.

Proof. We consider the function

$$g(z) = \int_0^z \left(\frac{f_1(u)}{u}\right)^{\frac{1}{\gamma_1}} \cdots \left(\frac{f_{|[\mathrm{Re}\,\eta]|}(u)}{u}\right)^{\frac{1}{\gamma_{|[\mathrm{Re}\,\eta]|}}} \mathrm{d}u. \tag{3.16}$$

The function g is regular in $\mathcal{U}$. We have

$$\frac{1-|z|^{2a}}{a}\left|\frac{zg''(z)}{g'(z)}\right| \leq \frac{1-|z|^{2a}}{a}\sum_{j=1}^{|[\mathrm{Re}\,\eta]|}\left[\frac{1}{|\gamma_j|}\left|\frac{zf_j'(z)}{f_j(z)} - 1\right|\right]. \tag{3.17}$$

Since $f_j \in \mathcal{S}$, $j = \overline{1, |[\mathrm{Re}\,\eta]|}$ we have

$$\left|\frac{zf_j'(z)}{f_j(z)}\right| \leq \frac{1+|z|}{1-|z|}, \quad z \in \mathcal{U}, j = \overline{1, |[\mathrm{Re}\,\eta]|}. \tag{3.18}$$

From (3.17) and (3.18) we obtain

$$\frac{1-|z|^{2a}}{a}\left|\frac{zg''(z)}{g'(z)}\right| \leq \frac{1-|z|^{2a}}{a}\frac{2}{1-|z|}\sum_{j=1}^{|[\mathrm{Re}\,\eta]|}\frac{1}{|\gamma_j|} \tag{3.19}$$

for all $z \in \mathcal{U}$.

For $0 < a < \frac{1}{2}$ we have

$$\max_{|z|\leq 1}\frac{1-|z|^{2a}}{1-|z|} = 1$$

and from (3.14), (3.19) we get

$$\frac{1-|z|^{2a}}{a}\left|\frac{zg''(z)}{g'(z)}\right| \leq 1, \quad z \in \mathcal{U}. \tag{3.20}$$

For $a \geq \frac{1}{2}$ we have

$$\max_{|z|\leq 1} \frac{1-|z|^{2a}}{1-|z|} = 2a$$

and from (3.15), (3.19) we obtain (3.20).

From (3.20) and Lemma 2.1 it results that the integral operator $H_{\gamma_1,\gamma_2,\ldots,\gamma_{|[\mathrm{Re}\,\eta]|},\beta,\eta}$ belongs to the class $\mathcal{S}$. □

Corollary 3.7. *Let α, γ be complex numbers, $a = \mathrm{Re}\,\alpha > 0$ and $f \in \mathcal{S}, f(z) = z + \sum_{k=2}^{\infty} b_{k1} z^k$.*

If

$$\frac{1}{|\gamma|} \leq \frac{a}{2}, \quad \text{for } 0 < a < \frac{1}{2} \tag{3.21}$$

or

$$\frac{1}{|\gamma|} \leq \frac{1}{4}, \quad \text{for } a \geq \frac{1}{2} \tag{3.22}$$

then for $\mathrm{Re}\,\frac{1}{\gamma} \geq a$, the integral operator M_γ given by (1.3) is in the class $\mathcal{S}$.

Proof. We take in Theorem 3.6, $\eta\beta = \frac{1}{\gamma}$, $|[\mathrm{Re}\,\eta]| = 1$, $\gamma_1 = \gamma$, $f_1 = f$. □

Corollary 3.8. *Let α, β, γ be complex numbers, $a = \mathrm{Re}\,\gamma > 0$ and $f \in \mathcal{S}, f(z) = z + \sum_{k=2}^{\infty} b_{k1} z^k$.*

If

$$|\alpha| \leq \frac{a}{2}, \quad \text{for } 0 < a < \frac{1}{2} \tag{3.23}$$

or

$$|\alpha| \leq \frac{1}{4}, \quad \text{for } a \geq \frac{1}{2} \tag{3.24}$$

then for $\mathrm{Re}\,\beta \geq a$, the integral operator $H_{\alpha,\beta}$ given by (1.2) belongs to the class $\mathcal{S}$.

Proof. For $\eta = 1, f_1 = f, \gamma_1 = \frac{1}{\alpha}$, from Theorem 3.6 we obtain Corollary 3.8. □

Corollary 3.9. *Let α, γ be complex numbers, $a = \mathrm{Re}\,\gamma \in (0, 1]$ and $f \in \mathcal{S}, f(z) = z + \sum_{k=2}^{\infty} b_{k1} z^k$.*

If

$$\frac{1}{|\alpha|} \leq \frac{a}{2}, \quad \text{for } 0 < a < \frac{1}{2} \tag{3.25}$$

or

$$\frac{1}{|\alpha|} \leq \frac{1}{4}, \quad \text{for } a \geq \frac{1}{2} \tag{3.26}$$

then the integral operator G_α given by (1.1) is in the class $\mathcal{S}$.

Proof. For $\eta\beta = 1$, $|[\mathrm{Re}\,\eta]| = 1$, $\gamma_1 = \alpha$, $f_1 = f$, from Theorem 3.6 we obtain Corollary 3.9. □

Corollary 3.10. *Let α, η, γ be complex numbers, $\mathrm{Re}\,\eta \notin [0, 1)$, $a = \mathrm{Re}\,\gamma \in (0, 1]$, $f_j \in \mathcal{S}, f_j(z) = z + \sum_{k=2}^{\infty} b_{kj} z^k$, $j = \overline{1, |[\mathrm{Re}\,\eta]|}$.*

If

$$\frac{|[\mathrm{Re}\,\eta]|}{|\alpha|} \leq \frac{a}{2}, \quad \text{for } 0 < a < \frac{1}{2} \tag{3.27}$$

or

$$\frac{|[\mathrm{Re}\,\eta]|}{|\alpha|} \leq \frac{1}{4}, \quad \text{for } a \geq \frac{1}{2} \tag{3.28}$$

then the integral operator L_α given by (3.13) is in the class $\mathcal{S}$.

Proof. For $\eta\beta = 1$, $\mathrm{Re}\,\eta \notin [0, 1)$, $\gamma_1 = \gamma_2 = \cdots = \gamma_{|[\mathrm{Re}\,\eta]|} = \alpha$, from Theorem 3.6. we obtain the Corollary 3.10. □

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