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Convexity properties for some general integral operators on uniformly analytic functions classes

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ABSTRACT

In this paper the authors prove some properties for two general integral operators on the classes $\beta - \mathcal{UCV}(\alpha)$ and $\beta - \mathcal{S}_p(\alpha)$.

1. Introduction

Let $\mathcal{A}$ denotes the class of functions f of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ which are analytic in the open unit disk $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$. Denote by $\mathcal{S}$, the class of univalent functions.

A function $f \in \mathcal{S}$ is a convex function of order α, $0 \le \alpha < 1$ and denote this class by $\mathcal{K}(\alpha)$ if f satisfies the inequality

$$\mathrm{Re}\left\{\frac{zf''(z)}{f'(z)} + 1\right\} > \alpha, \quad z \in \mathcal{U}.$$

A function $f \in \mathcal{A}$ is said to be in the class of β-uniformly convex functions of order α, denoted by $\beta - \mathcal{UCV}(\alpha)$ if

$$\mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)} - \alpha\right\} \ge \beta\left|\frac{zf''(z)}{f'(z)} - 1\right| \tag{1}$$

for $-1 \le \alpha \le 1$, $\beta > 0$ and $z \in \mathcal{U}$.

A function $f \in \mathcal{A}$ is said to be in the class of uniformly β-starlike functions of order α, denoted by $\beta - \mathcal{S}_p(\alpha)$ if

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)} - \alpha\right\} \ge \beta\left|\frac{zf'(z)}{f(z)} - 1\right| \tag{2}$$

for $-1 \le \alpha \le 1$, $\beta > 0$ and $z \in \mathcal{U}$.

The classes $\beta - \mathcal{UCV}(\alpha)$ and $\beta - \mathcal{S}_p(\alpha)$ were studied in [1].

We consider the next general integral operators defined by

$$F_{\gamma_1,\gamma_2,\ldots,\gamma_n}(z) = \int_0^z \left(f_1'(t)\right)^{\gamma_1} \cdots \left(f_n'(t)\right)^{\gamma_n} \mathrm{d}t$$

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where $f_i \in \mathcal{A}$, $\gamma_i > 0$, $i = 1, 2, \ldots, n$ and

$$G_{\gamma_1,\gamma_2,\ldots,\gamma_n}(z) = \int_0^z \left(\frac{f_1(t)}{t}\right)^{\gamma_1} \cdots \left(\frac{f_n(t)}{t}\right)^{\gamma_n} \mathrm{d}t$$

with $f_j \in \mathcal{A}$, $\gamma_j > 0$, $j = 1, 2, \ldots, n$.

Remark. The operator $F_{\gamma_1,\gamma_2,\ldots,\gamma_n}$ was introduced by Breaz et al., in [2] and $G_{\gamma_1,\gamma_2,\ldots,\gamma_n}$ by Breaz and Breaz in paper [3].

2. Main results

Theorem 2.1. *If $f_i \in \beta_i - \mathcal{UCV}(\alpha_i)$, with $-1 \le \alpha_i < 1$, $\beta_i > 0$ for all $i \in \{1, \ldots, n\}$ and $\sum_{i=1}^n \gamma_i \le \frac{1}{2}$ then $F_{\gamma_1,\gamma_2,\ldots,\gamma_n} \in \mathcal{K}(\rho)$, where $\rho = 1 + \sum_{i=1}^n \gamma_i(\alpha_i - 1)$.*

Proof. We have

$$\frac{zF''_{\gamma_1,\gamma_2,\ldots,\gamma_n}(z)}{F'_{\gamma_1,\gamma_2,\ldots,\gamma_n}(z)} = \gamma_1 \frac{zf_1''(z)}{f_1'(z)} + \cdots + \gamma_n \frac{zf_n''(z)}{f_n'(z)}$$

$$\begin{aligned}\mathrm{Re}\frac{zF''_{\gamma_1,\gamma_2,\ldots,\gamma_n}(z)}{F'_{\gamma_1,\gamma_2,\ldots,\gamma_n}(z)} &= \gamma_1 \mathrm{Re}\frac{zf_1''(z)}{f_1'(z)} + \cdots + \gamma_n \mathrm{Re}\frac{zf_n''(z)}{f_n'(z)} \\ &= \mathrm{Re}\gamma_1 \left(1 + \frac{zf_1''(z)}{f_1'(z)} - \alpha_1\right) - \gamma_1 + \gamma_1\alpha_1 + \cdots + \mathrm{Re}\gamma_n \left(1 + \frac{zf_n''(z)}{f_n'(z)} - \alpha_n\right) - \gamma_n + \gamma_n\alpha_n.\end{aligned}$$

Since $f_i \in \beta_i - \mathcal{UCV}(\alpha_i)$, for all $i \in \{1, \ldots, n\}$ we apply (1) in above equality and we obtain that:

$$\begin{aligned}\mathrm{Re}\frac{zF''_{\gamma_1,\gamma_2,\ldots,\gamma_n}(z)}{F'_{\gamma_1,\gamma_2,\ldots,\gamma_n}(z)} &\ge \gamma_1 \cdot \beta_1 \left|\frac{zf_1''(z)}{f_1'(z)} - 1\right| + \cdots + \gamma_n \cdot \beta_n \left|\frac{zf_n''(z)}{f_n'(z)} - 1\right| - \sum_{i=1}^n \gamma_i + \sum_{i=1}^n \gamma_i\alpha_i \\ &\ge \sum_{i=1}^n \gamma_i (\alpha_i - 1).\end{aligned}$$

From the last relation we have:

$$\mathrm{Re}\left(\frac{zF''_{\gamma_1,\gamma_2,\ldots,\gamma_n}(z)}{F'_{\gamma_1,\gamma_2,\ldots,\gamma_n}(z)} + 1\right) \ge 1 + \sum_{i=1}^n \gamma_i (\alpha_i - 1),$$

which implies that $F_{\gamma_1,\gamma_2,\ldots,\gamma_n}$ is in the class $\mathcal{K}(\rho)$, where $\rho = 1 + \sum_{i=1}^n \gamma_i (\alpha_i - 1)$. □

Since $-1 \le \alpha_i < 1$ for all $i \in \{1, \ldots, n\}$ and $\sum_{i=1}^n \gamma_i \le \frac{1}{2}$ we have that $0 \le \rho < 1$.

Corollary 2.2. *If $f_i \in \beta - \mathcal{UCV}(\alpha)$, for all $i \in \{1, \ldots, n\}$ with $-1 \le \alpha < 1$, $\beta > 0$, $\sum_{i=1}^n \gamma_i \le \frac{1}{2}$ then $F_{\gamma_1,\gamma_2,\ldots,\gamma_n} \in \mathcal{K}(\rho)$, where $\rho = 1 + (\alpha - 1)\sum_{i=1}^n \gamma_i$.*

Proof. In Theorem 2.1. we consider $\alpha_1 = \alpha_2 = \cdots = \alpha_n = \alpha$, $\beta_1 = \beta_2 = \cdots = \beta_n = \beta$. □

Corollary 2.3. *If $f \in \beta - \mathcal{UCV}(\alpha)$, with $-1 \le \alpha < 1$, $\beta > 0$, $\gamma \le \frac{1}{2}$ then $F_\gamma \in \mathcal{K}(\rho)$, where $\rho = 1 + (\alpha - 1)\gamma$ and $F_\gamma(z) = \int_0^z (f'(t))^\gamma \, \mathrm{d}t$.*

Proof. We consider $n = 1$ in Theorem 2.1. □

Theorem 2.4. *If $f_i \in \beta_i - \mathcal{S}_p(\alpha_i)$, $-1 \le \alpha_i < 1$, $\beta_i > 0$ for all $i \in \{1, \ldots, n\}$, and $\sum_{i=1}^n \gamma_i \le \frac{1}{2}$, then $G_{\gamma_1,\gamma_2,\ldots,\gamma_n} \in \mathcal{K}(\rho)$, where $\rho = 1 + \sum_{i=1}^n \gamma_i(\alpha_i - 1)$.*

Proof. We have

$$\frac{zG''_{\gamma_1,\gamma_2,\ldots,\gamma_n}(z)}{G'_{\gamma_1,\gamma_2,\ldots,\gamma_n}(z)} = \sum_{i=1}^n \gamma_i \left(\frac{zf_i'(z)}{f_i(z)} - 1\right)$$

$$\begin{aligned}\mathrm{Re}\sum_{i=1}^n \gamma_i \left(\frac{zf_i'(z)}{f_i(z)} - 1\right) &= \mathrm{Re}\gamma_1 \frac{zf_1'(z)}{f_1(z)} - \gamma_1 + \cdots + \mathrm{Re}\gamma_n \frac{zf_n'(z)}{f_n(z)} - \gamma_n \\ &= \mathrm{Re}\gamma_1 \left(\frac{zf_1'(z)}{f_1(z)} - \alpha_1\right) + (\gamma_1\alpha_1 - \gamma_1) + \cdots + \mathrm{Re}\gamma_n \left(\frac{zf_n'(z)}{f_n(z)} - \alpha_n\right) + (\gamma_n\alpha_n - \gamma_n) \\ &\ge \gamma_1 \cdot \beta_1 \left|\frac{zf_1'(z)}{f_1(z)} - 1\right| + \cdots + \gamma_n \cdot \beta_n \left|\frac{zf_n'(z)}{f_n(z)} - 1\right| + \sum_{i=1}^n \gamma_i(\alpha_i - 1) \ge \sum_{i=1}^n \gamma_i(\alpha_i - 1).\end{aligned}$$

So, we have:

$$\mathrm{Re}\left(\frac{zG''_{\gamma_1,\gamma_2,\ldots,\gamma_n}(z)}{G'_{\gamma_1,\gamma_2,\ldots,\gamma_n}(z)}+1\right)\geq 1+\sum_{i=1}^{n}\gamma_i(\alpha_i-1)$$

which implies that $G_{\gamma_1,\gamma_2,\ldots,\gamma_n}\in\mathcal{K}(\rho)$, where $\rho=1+\sum_{i=1}^{n}\gamma_i(\alpha_i-1)$.

Since $-1\leq\alpha_i<1$ for all $i\in\{1,\ldots,n\}$ and $\sum_{i=1}^{n}\gamma_i\leq\frac{1}{2}$ we have that $0\leq\rho<1$. □

Corollary 2.5. *If $f_i\in\beta-\mathcal{S}_p(\alpha)$, for all $i\in\{1,\ldots,n\}$, $-1\leq\alpha<1$, $\beta>0$ and $\sum_{i=1}^{n}\gamma_i\leq\frac{1}{2}$ then $G_{\gamma_1,\gamma_2,\ldots,\gamma_n}\in\mathcal{K}(\rho)$, where $\rho=1+(\alpha-1)\sum_{i=1}^{n}\gamma_i$.*

Proof. In Theorem 2.4. we consider $\alpha_1=\alpha_2=\cdots=\alpha_n=\alpha$, $\beta_1=\beta_2=\cdots=\beta_n=\beta$. □

Corollary 2.6. *If $f\in\beta-\mathcal{S}_p(\alpha)$, $-1\leq\alpha<1$, $\beta>0$ and $\gamma\leq\frac{1}{2}$ then $G_\gamma\in\mathcal{K}(\rho)$, where $\rho=1+\gamma(\alpha-1)$ and $G_\gamma=\int_0^z\left(\frac{f(t)}{t}\right)^\gamma\mathrm{d}t$.*

Proof. In Theorem 2.4, we consider $n=1$. □

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