Applied Mathematics Letters 24 (2011) 416–419

Contents lists available at ScienceDirect

Applied Mathematics Letters

journal homepage: www.elsevier.com/locate/aml

The univalence conditions for a general integral operator

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ARTICLE INFO

Article history:
Received 11 April 2010
Accepted 19 October 2010

Keywords:
Integral operator
Univalence
Modulus
Integer part

ABSTRACT

In this paper we extend a general integral operator which was introduced in the paper (Breaz, 2010) [3]. We denote this operator by $H_{\gamma_1,\gamma_2,\ldots,\gamma_{[|\eta|]},\beta,\eta}$. For this integral operator we show some conditions of univalence on the class of analytical functions.

1. Introduction

Let $\mathcal{A}$ be the class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the open unit disk $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$. Let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of all univalent functions f in $\mathcal{U}$.

Lemma 1.1 ([1]). *Let α be a complex number, $\operatorname{Re}\alpha > 0$ and $f \in \mathcal{A}$. If*

$$\frac{1 - |z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha} \left| \frac{zf''(z)}{f'(z)} \right| \leq 1, \tag{1.1}$$

for all $z \in \mathcal{U}$, then for any complex number β, $\operatorname{Re}\beta \geq \operatorname{Re}\alpha$ the function

$$F_\beta(z) = \left[\beta \int_0^z u^{\beta-1} f'(u)\mathrm{d}u \right]^{\frac{1}{\beta}}, \tag{1.2}$$

is in the class $\mathcal{S}$.

Lemma 1.2 (Schwarz [2]). *Let f be the function regular in the disk $\mathcal{U}_R = \{z \in \mathbb{C} : |z| < R\}$ with $|f(z)| < M$, M fixed. If $f(z)$ has in $z = 0$ one zero with order of multiplicity bigger than m, then*

$$|f(z)| \leq \frac{M}{R^m} |z|^m, \quad z \in \mathcal{U}_R, \tag{1.3}$$

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doi:10.1016/j.aml.2010.10.025

the equality (in the inequality (1.3) for $z \neq 0$) can hold only if

$$f(z) = \mathrm{e}^{\mathrm{i}\theta} \frac{M}{R^m} z^m,$$

where θ is constant.

We introduce the general integral operator

$$H_{\gamma_1,\gamma_2,\ldots,\gamma_{[|\eta|]},\beta,\eta}(z) = \left\{ \eta\beta \int_0^z u^{\eta\beta-1} \left(\frac{f_1(u)}{u}\right)^{\frac{1}{\gamma_1}} \cdots \left(\frac{f_{[|\eta|]}(u)}{u}\right)^{\frac{1}{\gamma_{[|\eta|]}}} \mathrm{d}u \right\}^{\frac{1}{\eta\beta}} \tag{1.4}$$

for $f_j \in \mathcal{A}$, γ_j, η, β complex numbers, $\gamma_j \neq 0$, $|\eta| \notin [0, 1)$, $j = \overline{1, [|\eta|]}$, $\beta \neq 0$, $[|\eta|]$ is the integer part of the modulus of η.

2. Main results

Theorem 2.1. *Let γ_j, β, η be complex numbers, $\beta \neq 0$, $|\eta| \notin [0, 1)$, $j = \overline{1, [|\eta|]}$, $a = \sum_{j=1}^{[|\eta|]} \operatorname{Re} \frac{1}{\gamma_j} > 0$ and $f_j \in \mathcal{A}$, $f_j(z) = z + b_{2j}z^2 + b_{3j}z^3 + \cdots$, $j = \overline{1, [|\eta|]}$.*

If

$$\left| \frac{zf_j'(z)}{f_j(z)} - 1 \right| \leq \frac{(2a+1)^{\frac{2a+1}{2a}}}{2\,[|\eta|]} \left|\gamma_j\right|, \quad j = \overline{1, [|\eta|]}, \tag{2.1}$$

for all $z \in \mathcal{U}$, and $\operatorname{Re} \eta\beta \geq a$, then the function $H_{\gamma_1,\gamma_2,\ldots,\gamma_{[|\eta|]},\beta,\eta}$ defined in (1.4) is in the class $\mathcal{S}$.

Proof. We consider the function

$$g(z) = \int_0^z \left(\frac{f_1(u)}{u}\right)^{\frac{1}{\gamma_1}} \cdots \left(\frac{f_{[|\eta|]}(u)}{u}\right)^{\frac{1}{\gamma_{[|\eta|]}}} \mathrm{d}u. \tag{2.2}$$

The function g is regular in $\mathcal{U}$. We define the function $p(z) = \frac{zg''(z)}{g'(z)}$, $z \in \mathcal{U}$ and we obtain

$$p(z) = \frac{zg''(z)}{g'(z)} = \sum_{j=1}^{[|\eta|]} \left[\frac{1}{\gamma_j} \left(\frac{zf_j'(z)}{f_j(z)} - 1 \right) \right], \quad z \in \mathcal{U}. \tag{2.3}$$

From (2.1) and (2.3) we have

$$|p(z)| \leq \frac{(2a+1)^{\frac{2a+1}{2a}}}{2},$$

for all $z \in \mathcal{U}$ and applying Lemma 1.2 we get

$$|p(z)| \leq \frac{(2a+1)^{\frac{2a+1}{2a}}}{2} |z|, \quad z \in \mathcal{U}. \tag{2.4}$$

From (2.3) and (2.4) we have

$$\frac{1-|z|^{2a}}{a} \left| \frac{zg''(z)}{g'(z)} \right| \leq \frac{(1-|z|^{2a})|z|}{a} \cdot \frac{(2a+1)^{\frac{2a+1}{2a}}}{2}, \tag{2.5}$$

for all $z \in \mathcal{U}$.

Since

$$\max_{|z|\leq 1} \frac{(1-|z|^{2a})|z|}{a} = \frac{2}{(2a+1)^{\frac{2a+1}{2a}}},$$

from (2.5) we have

$$\frac{1-|z|^{2a}}{a} \left| \frac{zg''(z)}{g'(z)} \right| \leq 1,$$

for all $z \in \mathcal{U}$. So, by the Lemma 1.1, the integral operator $H_{\gamma_1,\gamma_2,\ldots,\gamma_{[|\eta|]},\beta,\eta}$ belongs to class $\mathcal{S}$. □

Corollary 2.2 ([3]). *Let γ_j, β, η be complex numbers, $\beta \neq 0, \operatorname{Re}\eta \notin [0, 1), j = \overline{1, |[\operatorname{Re}\eta]|}$, $a = \sum_{j=1}^{|[\operatorname{Re}\eta]|} \operatorname{Re}\frac{1}{\gamma_j} > 0$ and $f_j \in \mathcal{A}, f_j(z) = z + b_{2j}z^2 + b_{3j}z^3 + \cdots, j = \overline{1, |[\operatorname{Re}\eta]|}$.*

If

$$\left|\frac{zf_j'(z)}{f_j(z)} - 1\right| \leq \frac{(2a+1)^{\frac{2a+1}{2a}}}{2\,|[\operatorname{Re}\eta]|}\left|\gamma_j\right|, \quad j = \overline{1, |[\operatorname{Re}\eta]|},$$

for all $z \in \mathcal{U}$, and $\operatorname{Re}\eta\beta \geq a$, then the function

$$H_{\gamma_1,\gamma_2,\ldots,\gamma_{|[\operatorname{Re}\eta]|},\beta,\eta}(z) = \left\{\eta\beta\int_0^z u^{\eta\beta-1}\left(\frac{f_1(u)}{u}\right)^{\frac{1}{\gamma_1}}\cdots\left(\frac{f_{|[\operatorname{Re}\eta]|}(u)}{u}\right)^{\frac{1}{\gamma_{|[\operatorname{Re}\eta]|}}}\mathrm{d}u\right\}^{\frac{1}{\eta\beta}}, \tag{2.6}$$

is in the class $\mathcal{S}$.

Corollary 2.3. *Let α, η be complex numbers $a = [|\eta|]\cdot\operatorname{Re}\frac{1}{\alpha}$, $|\eta| \notin [0, 1), a \in (0, 1]$ and $f_j \in \mathcal{A}, f_j(z) = z + b_{2j}z^2 + \cdots, j = \overline{1, [|\eta|]}$. If*

$$\left|\frac{zf_j'(z)}{f_j(z)} - 1\right| \leq \frac{(2a+1)^{\frac{2a+1}{2a}}}{2\,[|\eta|]}|\alpha|, \quad j = \overline{1, [|\eta|]},$$

for all $z \in \mathcal{U}$, then the function

$$L_\alpha(z) = \int_0^z \left(\frac{f_1(u)}{u}\right)^{\frac{1}{\alpha}}\cdots\left(\frac{f_{[|\eta|]}(u)}{u}\right)^{\frac{1}{\alpha}}\mathrm{d}u, \tag{2.7}$$

is in the class $\mathcal{S}$.

Proof. For $\eta\beta = 1, \gamma_1 = \gamma_2 = \cdots = \gamma_{[|\eta|]} = \alpha$ from Theorem 2.1. we obtain the Corollary 2.3. □

Theorem 2.4. *Let $\gamma_j, \alpha, \beta, \eta$ be complex numbers, $\gamma_j \neq 0, |\eta| \notin [0, 1), \beta \neq 0, a = \operatorname{Re}\alpha > 0, j = \overline{1, [|\eta|]}$ and $f_j \in \mathcal{S}, f_j(z) = z + \sum_{k=2}^\infty b_{kj}z^k, j = \overline{1, [|\eta|]}$.*

If

$$\sum_{j=1}^{[|\eta|]}\frac{1}{|\gamma_j|} \leq \frac{a}{2}, \quad \textit{for } 0 < a < \frac{1}{2} \tag{2.8}$$

or

$$\sum_{j=1}^{[|\eta|]}\frac{1}{|\gamma_j|} \leq \frac{1}{4}, \quad \textit{for } a \geq \frac{1}{2}, \tag{2.9}$$

then for $\operatorname{Re}\eta\beta \geq a$, the integral operator $H_{\gamma_1,\gamma_2,\ldots,\gamma_{[|\eta|]},\beta,\eta}$ given by (1.4) is in the class $\mathcal{S}$.

Proof. We consider the function

$$g(z) = \int_0^z \left(\frac{f_1(u)}{u}\right)^{\frac{1}{\gamma_1}}\cdots\left(\frac{f_{[|\eta|]}(u)}{u}\right)^{\frac{1}{\gamma_{[|\eta|]}}}\mathrm{d}u.$$

The function g is regular in $\mathcal{U}$. We have

$$\frac{1-|z|^{2a}}{a}\left|\frac{zg''(z)}{g'(z)}\right| \leq \frac{1-|z|^{2a}}{a}\sum_{j=1}^{[|\eta|]}\left[\frac{1}{|\gamma_j|}\left|\frac{zf_j'(z)}{f_j(z)} - 1\right|\right]. \tag{2.10}$$

Since $f_j \in \mathcal{S}, j = \overline{1, [|\eta|]}$ we have

$$\left|\frac{zf_j'(z)}{f_j(z)}\right| \leq \frac{1+|z|}{1-|z|}, \quad z \in \mathcal{U},\ j = \overline{1, [|\eta|]}. \tag{2.11}$$

From (2.10) and (2.11) we obtain

$$\frac{1-|z|^{2a}}{a}\left|\frac{zg''(z)}{g'(z)}\right| \leq \frac{1-|z|^{2a}}{a}\frac{2}{1-|z|}\sum_{j=1}^{[|\eta|]}\frac{1}{|\gamma_j|}, \tag{2.12}$$

for all $z \in \mathcal{U}$.

For $0 < a < \frac{1}{2}$ we have

$$\max_{|z|\le 1} \frac{1-|z|^{2a}}{1-|z|} = 1$$

and from (2.8) and (2.12) we get

$$\frac{1-|z|^{2a}}{a} \left| \frac{zg''(z)}{g'(z)} \right| \le 1, \quad z \in \mathcal{U}. \tag{2.13}$$

For $a \ge \frac{1}{2}$ we have

$$\max_{|z|\le 1} \frac{1-|z|^{2a}}{1-|z|} = 2a$$

and from (2.9) and (2.12) we obtain (2.13).

From (2.13) and Lemma 1.1 it results that the integral operator $H_{\gamma_1,\gamma_2,\ldots,\gamma_{[|\eta|]},\beta,\eta}$ belongs to class $\mathcal{S}$. □

Corollary 2.5 ([3]). *Let $\gamma_j, \alpha, \beta, \eta$ be complex numbers, $\gamma_j \ne 0$, $\operatorname{Re}\eta \notin [0,1)$, $\beta \ne 0$, $a = \operatorname{Re}\alpha > 0$, $j = \overline{1, |[\operatorname{Re}\eta]|}$ and $f_j \in \mathcal{S}$, $f_j(z) = z + \sum_{k=2}^{\infty} b_{kj} z^k$, $j = \overline{1, |[\operatorname{Re}\eta]|}$.*

If

$$\sum_{j=1}^{|[\operatorname{Re}\eta]|} \frac{1}{|\gamma_j|} \le \frac{a}{2}, \quad \text{for } 0 < a < \frac{1}{2},$$

or

$$\sum_{j=1}^{|[\operatorname{Re}\eta]|} \frac{1}{|\gamma_j|} \le \frac{1}{4}, \quad \text{for } a \ge \frac{1}{2},$$

then for $\operatorname{Re}\eta\beta \ge a$, the integral operator $H_{\gamma_1,\gamma_2,\ldots,\gamma_{|[\operatorname{Re}\eta]|},\beta,\eta}$ given by (2.6) is in the class $\mathcal{S}$.

Corollary 2.6. *Let α, η, γ be complex numbers, $|\eta| \notin [0,1)$, $a = \operatorname{Re}\gamma \in (0,1]$, $f_j \in \mathcal{S}$, $f_j(z) = z + \sum_{k=2}^{\infty} b_{kj} z^k$, $j = \overline{1, [|\eta|]}$.*

If

$$\frac{[|\eta|]}{|\alpha|} \le \frac{a}{2}, \quad \text{for } 0 < a < \frac{1}{2},$$

or

$$\frac{[|\eta|]}{|\alpha|} \le \frac{1}{4}, \quad \text{for } a \ge \frac{1}{2},$$

then the integral operator L_α given by (2.7) is in the class $\mathcal{S}$.

Proof. For $\eta\beta = 1$, $|\eta| \notin [0,1)$, $\gamma_1 = \gamma_2 = \cdots = \gamma_{[|\eta|]} = \alpha$, from Theorem 2.4 we obtain the Corollary 2.6. □

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