

ON UNIVALENT POLYNOMIALS

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1. Introduction. Let P_n be the class of normalised polynomials of the form

$$p_n(z) = z + a_2 z^2 + \dots + a_n z^n \quad (1.1)$$

of degree n which are univalent in $U = \{|z| < 1\}$. In this note we discuss the coefficients of polynomials in P_n and in some of its subclasses.

Our principal tools will be

LEMMA 1.1. (*Dieudonné criterion*) [6]. *The polynomial $p_n(z)$, of the form (1.1), is univalent in U if and only if the associated equation of $p_n(z)$,*

$$\phi(x, \theta) = 1 + \sum_{k=2}^n a_k x^{k-1} \sin(k-1)\theta / \sin \theta = 0,$$

has no roots in $|x| < 1$, for any θ with $0 \leq \theta \leq \frac{1}{2}\pi$.

LEMMA 1.2. (*Cohn rule*) [9]. *Suppose that*

$$f(x) = a_0 + a_1 x + \dots + a_n x^n$$

is a polynomial of degree n , and

$$f^*(x) = \bar{a}_n + \bar{a}_{n-1} x + \dots + \bar{a}_0 x^n.$$

Then, if $|a_0| \geq |a_n|$, the polynomial

$$f_1(x) = \bar{a}_0 f(x) - \bar{a}_n f^*(x)$$

has the same number of zeros in $|x| < 1$ as has $f(x)$.

Finally we recall the definitions of two classes of univalent functions which will appear later. The analytic function $f(z)$ is said to be *starlike* in U if $f(0) = 0$, and the segment $[0, f(z_0)]$ lies in $f(U)$ for any z_0 in U [7]; analytically this may be expressed by the condition

$$\operatorname{Re}(zf'/f) > 0 \quad (z \text{ in } U).$$

Further, the analytic function $g(z)$ is said to be *close-to-convex* in U if $g(0) = 0$, and

$$\operatorname{Re}(zg'/f) > 0 \quad (z \text{ in } U)$$

for some starlike function $f(z)$ [8].

I would like to thank Professor J. Clunie for introducing me to the class P_n , and for his help and encouragement over a long period in this work.

2. A particular subclass of P_n . If the polynomial $p_n(z)$, of the form (1.1), is univalent in U , $p'_n(z)$ cannot vanish in U ; consequently $|a_n| \leq 1/n$. In this section we consider polynomials in P_n where $a_n = 1/n$.

Suppose that

$$p_n(z) = z + a_2 z^2 + \dots + a_{n-1} z^{n-1} + z^n/n. \tag{2.1}$$

It was shown in [3] that, if $p_n(z)$ belongs to P_n , then

$$(n-k)a_{n-k} = (k+1)\bar{a}_{k+1} \quad (1 \leq k \leq n-2).$$

In the opposite direction we have

THEOREM 2.1. Suppose that $p_n(z)$ is of the form (2.1). Then, if

$$(n-k)a_{n-k} = (k+1)\bar{a}_{k+1} \quad (1 \leq k \leq n-2) \tag{2.2}$$

and each a_{k+1} is sufficiently small, $p_n(z)$ belongs to P_n .

Proof. The polynomial $p_n(z)$ belongs to P_n if the equation

$$1 + \sum_{k=1}^{n-2} a_{k+1} \frac{\sin(k+1)\theta}{\sin \theta} x^k + \frac{\sin n\theta}{n \sin \theta} x^{n-1} = 0$$

has no roots in $|x| < 1$ for $0 \leq \theta \leq \frac{1}{2}\pi$. Applying the Cohn rule to this equation, since $|\sin n\theta/n \sin \theta| \leq 1$ for $0 \leq \theta \leq \frac{1}{2}\pi$, we see that $p_n(z)$ belongs to P_n if

$$\begin{aligned} 0 &= 1 - \left(\frac{\sin n\theta}{n \sin \theta}\right)^2 + \sum_{k=1}^{n-2} x^k \left(a_{k+1} \frac{\sin(k+1)\theta}{\sin \theta} - \bar{a}_{n-k} \frac{\sin n\theta \sin(n-k)\theta}{n \sin \theta \sin \theta} \right) \\ &= 1 - \left(\frac{\sin n\theta}{n \sin \theta}\right)^2 + \sum_{k=1}^{n-2} x^k a_{k+1} \left(\frac{\sin(k+1)\theta}{\sin \theta} - \frac{(k+1) \sin n\theta \sin(n-k)\theta}{(n-k)n \sin \theta \sin \theta} \right) \end{aligned}$$

has no roots in $|x| < 1$ for $0 \leq \theta \leq \frac{1}{2}\pi$. Now each coefficient of x^r ($0 \leq r \leq n-2$) has a double zero at $\theta = 0$, and the constant term is always positive otherwise. Hence, if all the coefficients of x^r are chosen sufficiently small, this equation has no roots in $|x| < 1$, and $p_n(z)$ belongs to P_n .

In a much underestimated paper [1] Alexander showed that the polynomials $\sum_{k=1}^n z^k/k$ and $\sum_{k=0}^n z^{2k+1}/(2k+1)$ are univalent in U . We can put this result in a more general setting in

THEOREM 2.2. Suppose that

$$p_n(z) = z + \sum_{k=2}^n a_k z^k, \quad \text{and} \quad q_n(z) = z + \sum_{k=1}^n b_{2k+1} z^{2k+1},$$

where ka_k and $(2k+1)b_{2k+1}$ decrease as k increases. Then $p_n(z)$ and $q_n(z)$ are close-to-convex univalent functions in U .

Proof. We have, for $z \in U$,

$$\begin{aligned} \operatorname{Re} \left\{ \frac{z p'_n(z)}{z/(1-z)} \right\} &= \operatorname{Re} \left\{ 1 + (1-2a_2)z + \sum_{k=2}^{n-1} (ka_k - (k+1)a_{k+1})z^k - na_n z^n \right\} \\ &\geq 1 - (1-2a_2) - \sum_{k=2}^{n-1} (ka_k - (k+1)a_{k+1}) - na_n \\ &= 0. \end{aligned}$$

Hence $p_n(z)$ is close-to-convex in U . Similarly for $q_n(z)$.

However, in contrast with Theorems 2.1 and 2.2, we have the following surprising result for starlike polynomials

THEOREM 2.3. *Suppose that $p_n(z)$, of the form (2.1), belongs to P_n . Then $p_n(z)$ is starlike in U if and only if $a_k = 0$ for $2 \leq k \leq n-1$.*

Proof. If $a_2 = a_3 = \dots = a_{n-1} = 0$, it is easy to show that $p_n(z)$ is starlike in U . We therefore assume that $p_n(z)$ is starlike in U , and then show that this implies that $a_k = 0$ for $2 \leq k \leq n-1$. Then

$$\frac{1 + 2a_2 z + \dots + z^{n-1}}{1 + a_2 z + \dots + z^{n-1}/n} = \frac{p'_n(z)}{h(z)}$$

(where $h(z) = p_n(z)/z$ has positive real part in U . Since $p_n(z)$ belongs to P_n , we have that $(k+1)a_{k+1} = (n-k)\bar{a}_{n-k}$ for $1 \leq k \leq n-2$. Consequently, on $|z| = 1$, we may define

$$\begin{aligned} \alpha(\theta) &= p'_n(z^2)/z^{n-1} \quad (z = e^{i\theta}) \\ &= 2[\cos(n-1)\theta + 2|a_2| \cos\{(n-3)\theta - \phi_2\} + \dots], \end{aligned}$$

where $\phi_k = \arg a_k$. Furthermore, we may define $\beta(\theta)$ and $\gamma(\theta)$ by

$$\begin{aligned} \frac{p'_n(z^2)}{h(z^2)} &= \frac{p'_n(z^2)}{z^{n-1}} \bigg/ \frac{h(z^2)}{z^{n-1}} \\ &= \frac{\alpha(\theta)}{\beta(\theta) + i\gamma(\theta)} = \frac{\alpha(\theta)\beta(\theta) - i\alpha(\theta)\gamma(\theta)}{\beta^2(\theta) + \gamma^2(\theta)}, \end{aligned}$$

where $z = e^{i\theta}$. No difficulty arises from the denominator, since the univalence of $p_n(z)$ ensures that

$$\beta^2(\theta) + \gamma^2(\theta) = |h(e^{2i\theta})|^2 = |p_n(e^{2i\theta})|^2 > 0$$

for $0 \leq \theta \leq 2\pi$. We now show that $\alpha(\theta)$ can have only simple zeros for $0 \leq \theta \leq 2\pi$. Let ϕ be a zero of $\alpha(\theta)$. Now, with $z = e^{i\theta}$, we have

$$\alpha'(\theta) = \frac{\partial}{\partial \theta} \left[\frac{p'_n(z^2)}{z^{n-1}} \right] = iz \left[\frac{2z p''_n(z^2)}{z^{n-1}} - (n-1) \frac{p'_n(z^2)}{z^n} \right].$$

Now $\alpha(\phi) = 0$, so that $p'_n(z^2) = 0$ when $z = e^{i\phi}$. Hence, if $\alpha'(\phi)$ is also zero, we see that $p''_n(z^2)$ is zero at $z = e^{i\phi}$ as well. But then $p''_n(z)$ is zero at $z = e^{2i\phi}$. This, however, is impossible, since the existence of a double zero of $p'_n(z)$ on $|z| = 1$ is ruled out by the univalence of $p_n(z)$ in U .

Now, the condition $\text{Re}(zp'_n/p_n) \geq 0$ in U may be written in the form

$$\alpha(\theta)\beta(\theta) \geq 0 \quad \text{for } 0 \leq \theta \leq 2\pi;$$

since the zeros of $\alpha(\theta)$ are simple, this in turn shows that, whenever $\alpha(\theta) = 0$, necessarily $\beta(\theta) = 0$. Now all of its $2(n-1)$ zeros lie in $0 \leq \theta \leq 2\pi$ (corresponding to the $n-1$ zeros of $p'_n(z)$ all on $|z| = 1$) in the case of $\alpha(\theta)$, and hence the same must be true of $\beta(\theta)$ since it is also a trigonometric polynomial of degree $n-1$. Since a polynomial which has its maximum number of zeros is determined by these zeros to within a constant factor, it follows that, for some constant C , we have

$$\alpha(\theta) = C\beta(\theta),$$

or

$$p'_n(z^2)/z^{n-1} = C \text{Re} \{p_n(z^2)/z^{n-1}\} \quad \text{on } |z| = 1.$$

Expanding both sides of this equation, and equating the highest terms, with $z = e^{i\theta}$, we find that $C = 2n/(n+1)$. Substituting this value of C , and equating the other terms of the expansion in turn, we find that $a_k = 0$ for $2 \leq k \leq n-1$. This completes the proof.

Note. In the case $n = 3$, this result also appears in [4].

3. Some coefficient bounds for P_n . First we give bounds for the central coefficient of particular trinomials in P_{2n+1} .

THEOREM 3.1. *The polynomial*

$$p_{2n+1}(z) = z + az^{n+1} + z^{2n+1}/(2n+1)$$

belongs to P_{2n+1} if and only if a is real and

$$|a| \leq \text{Min}_{(0, \frac{1}{2}\pi)} \left\{ \frac{1 + [\sin(2n+1)\theta/(2n+1)\sin\theta]}{|\sin(n+1)\theta/\sin\theta|} \right\} = \pi/4n\{1 + o(1)\} \quad \text{for large } n.$$

Note. By (2.2), a must be real, and so we may assume that $a \geq 0$.

Proof. Applying the Cohn rule to the associated Dieudonné equation for $p_{2n+1}(z)$, and using the fact that $|\sin(2n+1)\theta/(2n+1)\sin\theta| < 1$ for $0 < \theta \leq \frac{1}{2}\pi$, the first inequality follows. Since $(2/\pi)x \leq \sin x \leq x$ for $0 \leq x \leq \frac{1}{2}\pi$, we can show that the above minimum occurs in $0 \leq \theta \leq 4\pi/(2n+1)$; by elementary differentiation, it must occur at $\pi/(2n+1)\{1 + o(1)\}$ for large n . This gives the last inequality.

COROLLARY. *The polynomials $z + az^2 + \frac{1}{3}z^3$ and $z + bz^3 + \frac{1}{5}z^5$ are univalent in U if and only if a and b are real, and $|a| \leq 8/9$ and $|b| \leq 3/5$.*

We now turn to the estimation of the $(n-1)$ th coefficients of polynomials in P_n .

THEOREM 3.2. *Suppose that $p_n(z)$, of the form (1.1), belongs to P_n . Then*

$$(n-1)|a_{n-1}| \leq 1 + 2n|a_2 a_n| - n^2|a_n|^2 < 4; \tag{3.1}$$

in particular,

$$(n-1)|a_{n-1}| \leq \begin{cases} 1 + |a_2|^2, & \text{if } |a_2| \geq 1, \\ 2|a_2|, & \text{if } |a_2| < 1. \end{cases}$$

Proof. By the Dieudonné criterion, since $p_n(z)$ belongs to P_n , the equation

$$1 + \sum_{k=1}^{n-1} a_{k+1} \frac{\sin(k+1)\theta}{\sin \theta} x^k = 0$$

has no roots in $|x| < 1$ for $0 \leq \theta \leq \frac{1}{2}\pi$, and $|a_n| \leq 1/n$. Applying the Cohn rule, we deduce that the equation

$$1 - |a_n|^2 \left(\frac{\sin n\theta}{\sin \theta} \right)^2 + \sum_{k=1}^{n-2} x^k \left(a_{k+1} \frac{\sin(k+1)\theta}{\sin \theta} - a_n \bar{a}_{n-k} \frac{\sin n\theta \sin(n-k)\theta}{\sin \theta \sin \theta} \right) = 0$$

has no roots in $|x| < 1$ for $0 \leq \theta \leq \frac{1}{2}\pi$. Consequently

$$\begin{aligned} 1 - |a_n|^2 \left(\frac{\sin n\theta}{\sin \theta} \right)^2 &\geq \left| a_{n-1} \frac{\sin(n-1)\theta}{\sin \theta} - a_n \bar{a}_2 \frac{\sin n\theta \sin 2\theta}{\sin \theta \sin \theta} \right| \\ &\geq \left| a_{n-1} \frac{\sin(n-1)\theta}{\sin \theta} \right| - \left| a_n a_2 \frac{\sin n\theta \sin 2\theta}{\sin \theta \sin \theta} \right|; \end{aligned}$$

substituting $\theta = 0$, we obtain

$$1 - n^2|a_n|^2 \geq (n-1)|a_{n-1}| - 2n|a_2 a_n|.$$

This gives the first inequality. The next three follow at once by considering the behaviour of the expression $1 + (n|a_n|)(2|a_2|) - (n|a_n|)^2$ where $|a_2| \leq 2$ and $|a_n| \leq 1/n$.

Note 1. Suffridge [10] has shown that the polynomial

$$\sum_{k=1}^n \frac{n-k+1}{n} \cdot \frac{\sin(k\pi/n+1)}{\sin(\pi/n+1)} z^k$$

belongs to P_n . Consequently the constant 4 in (3.1) cannot be improved independently of n .

Note 2. Recent work has determined the coefficient regions for P_3 [3, 5], for starlike polynomials in P_3 [4], and for the subclass of P_n with real Maclaurin coefficients [10]. However, much work remains to be done on the general coefficient problem for P_n ($n > 3$).

Note 3. Results similar to Theorems 2.1, 2.3, 3.1, and 3.2 were obtained in [2] for the class of "pseudo-polynomials"

$$\mu_n(z) = z^{-1} + \sum_{k=1}^n a_k z^k$$

analytic and univalent in $0 < |z| < 1$.

Note 4. Using the fact that the class of linearly-accessible functions of Biernacki is exactly the class of close-to-convex functions of Kaplan, we may observe that Alexander [1] showed that the polynomial

$$p(z) = \int_0^z (1 - e^{i\theta_1} t) \dots (1 - e^{i\theta_{n-1}} t) dt,$$

where $0 \leq \theta_1 < \theta_2 < \dots < \theta_{n-1} < 2\pi \leq \theta_n = \theta_1 + 2\pi$, is close-to-convex in U if and only if

$$\theta_{j+1} - \theta_j \geq 2\pi/(n+1) \quad (1 \leq j \leq n-1).$$

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