

# ON FUNCTIONS OF BOUNDED BOUNDARY ROTATION I

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## 1. Introduction

Let  $V_k$  denote the class of functions

$$f(z) = z + a_2 z^2 + \dots \tag{1.1}$$

which map  $U = \{|z| < 1\}$  conformally onto an image domain  $f(U)$  of boundary rotation at most  $k\pi$  (see (7) for the definition and basic properties of the class  $V_k$ ). In this note we discuss the valency of functions in  $V_k$ , and also their Maclaurin coefficients.

In (8) it was shown that functions in  $V_k$  are close-to-convex in  $U$  if  $2 \leq k \leq 4$ . Here we show that  $V_k$  is a subclass of the class  $K(\alpha)$  of close-to-convex functions of order  $\alpha$  (10) for  $\alpha = \frac{1}{2}k - 1$ , and we give an upper bound for the valency of functions in  $V_k$  for  $k > 4$ .

In the third section we derive an upper bound for the integral means of  $f'(z)$ , and consequently for the coefficients of functions  $f(z)$  in  $V_k$ ; this improves a result in (3). We conclude with various estimates for the Maclaurin coefficients of functions in  $V_k$  when  $f(U)$  is bounded or of finite area.

## 2. Valency

**Theorem 2.1.** *Suppose that  $f(z)$  belongs to  $V_k$ , and assumes some value in  $f(U)$   $p$  times. Then  $p = 1$  if  $k = 2$ , and  $p < \frac{1}{2}k$  if  $k > 2$ .*

**Proof.** If  $k = 2$ ,  $f(z)$  is convex in  $U$ , and so is univalent. Hence we need only consider  $k > 2$ .

Suppose that  $w = f(z)$  assumes some value  $v$   $p$  times in  $U$ , at the distinct points  $z_1, z_2, \dots, z_p$ . Then there is an  $r_0$  such that  $|z_k| < r_0 < 1$  for  $1 \leq k \leq p$ , and  $f(z) \neq v$  on  $|z| = r_0$ . Let  $C(r_0) = f(|z| = r_0)$ . Then the winding number of  $C(r_0)$  round  $v$  is

$$\frac{1}{2\pi i} \int_{C(r_0)} \frac{dw}{w-v} = \frac{1}{2\pi i} \int_{|z|=r_0} \frac{f'(z)dz}{f(z)-v} = p,$$

since  $f'(z) \neq 0$  in  $U$ . Consequently the tangent rotation round  $C(r_0)$  is at least  $2p\pi$ , and so

$$2p\pi \leq \int_0^{2\pi} \left| \operatorname{Re} (1 + zf''/f') \right| d\theta \equiv I(r_0) < \limsup_{r_0 \rightarrow 1} \int_0^{2\pi} \left| \operatorname{Re} (1 + zf''/f') \right| d\theta \leq k\pi,$$

since, for some  $t$  between 0 and 1,  $I(r_0)$  is strictly increasing on  $(t, 1)$ . Thus  $p < \frac{1}{2}k$ , as required.

**Note.** The function

$$f(z) = \frac{1}{k} \left\{ \left( \frac{1+z}{1-z} \right)^{\frac{1}{2}k} - 1 \right\} \tag{2.1}$$

belongs to  $V_k$  (6), its valency is 1 if  $k = 2$ ,  $[\frac{1}{2}k]$  if  $k > 2$  and  $k$  is not an even integer, and  $\frac{1}{2}k - 1$  if  $k$  is an even integer. This shows that the bounds of the theorem cannot be improved in general.

**Theorem 2.2.** *Suppose that  $f(z)$  belongs to  $V_k$ , where  $2 \leq k \leq 4$ . Then  $f(z)$  belongs to  $K(\frac{1}{2}k - 1)$ .*

**Proof.** Choose any  $r$ ,  $0 < r < 1$ , and let  $C(r) = f(|z| = r)$ . It is clear, geometrically, that, since the tangent to  $C(r)$  cannot turn through more than  $k\pi$  radians, the tangent cannot bend back on itself more than  $(\frac{1}{2}k - 1)\pi$  radians. Since  $r$  is arbitrary, the result follows at once.

If we are given a bound for the rate of growth of the derivative of a function in  $V_k$ , integration gives a bound for the rate of growth of the function itself. However we now establish a result in the opposite direction, using Theorem 2.1 and the theory of multivalent functions.

**Theorem 2.3.** *Suppose that  $f(z)$  belongs to  $V_k$ , and  $M(r) = \max_{|z|=r} |f(z)|$ . Then*

$$|f'(z)| \leq 2k(1-r^2)^{-1} \{1 + M(r)\} \quad (|z| = r). \tag{2.2}$$

**Proof.** Since  $f'(z) \neq 0$  in  $U$ , it follows from (4, Theorem 217) that, unless  $f(z) \equiv z$ , there is a number  $w_0$ ,  $|w_0| < 1$ , such that  $f(z) - w_0$  does not vanish in  $U$ .

However  $f(z) - w_0$  is also at most  $\frac{1}{2}k$  valent in  $U$ . Consequently, by (2, Theorem 5.1), we have

$$|f'(z)| \leq 2k(1-r^2)^{-1} |f(z) - w_0|, \tag{2.3}$$

from which (2.2) follows at once.

### 3. The coefficient problem for $V_k$

One of our principal tools here will be

**Theorem 3.1.** *The function  $f(z)$ , of the form (1.1), belongs to  $V_k$  if and only if there are two functions  $s_1(z)$  and  $s_2(z)$ , normalized and starlike in  $U$ , such that*

$$f'(z) = \frac{(s_1/z)^{\frac{1}{2}k + \frac{1}{2}}}{(s_2/z)^{\frac{1}{2}k - \frac{1}{2}}}. \tag{3.1}$$

**Proof.** This follows at once from Paatero's integral representation for functions in  $V_k$  (7).

From this we obtain

**Theorem 3.2.** *Suppose that  $f(z)$  belongs to  $V_k$ , and*

$$I_\lambda(r) = \frac{1}{2\pi} \int_0^{2\pi} |f'(re^{i\theta})|^\lambda d\theta, \tag{3.2}$$

where  $0 < r < 1$ , and  $(\frac{1}{2}k + 1)\lambda > 1$ . Then

$$\limsup_{r \rightarrow 1} (1-r)^{(\frac{1}{2}k-1)\lambda-1} I_\lambda(r) \leq A(k, \lambda), \tag{3.3}$$

where

$$A(k, \lambda) = \frac{2^{(\frac{1}{2}k-1)\lambda} \Gamma(\frac{1}{4}k\lambda + \frac{1}{2}\lambda + \frac{1}{2})}{\pi^{\frac{1}{2}}(\frac{1}{2}k\lambda + \frac{1}{2}\lambda - 1) \Gamma(\frac{1}{4}k\lambda + \frac{1}{2}\lambda)}. \tag{3.4}$$

Furthermore the constant  $A(k, \lambda)$  cannot be improved over the whole class  $V_k$ .

**Proof.** By Theorem 3.1, we may suppose that  $f'(z)$  is given by (3.1). Then  $|s_2(z)/z| \geq (1+|z|)^2$  by the Koebe distortion theorem, and  $s_1(z)/z$  is subordinate to  $(1-z)^{-2}$  in  $U(5)$ . Consequently, on integrating (3.1), we have

$$\begin{aligned} I_\lambda(r) &\leq \frac{1}{2\pi} (1+r)^{(\frac{1}{2}k-1)\lambda} \int_0^{2\pi} |1+re^{i\theta}|^{-(\frac{1}{2}k+1)\lambda} d\theta \\ &\equiv (1+r)^{(\frac{1}{2}k-1)\lambda} J_{(\frac{1}{2}k+1)\lambda}(r), \text{ say.} \end{aligned} \tag{3.5}$$

In fact, Pommerenke (9) has shown that

$$\begin{aligned} J_m(r) &\sim \frac{\Gamma(m-1)}{2^{m-1} \Gamma^2(\frac{1}{2}m)} \cdot \frac{1}{(1-r)^{m-1}} \quad (m > 1, r \rightarrow 1) \\ &= \frac{\Gamma(\frac{1}{2}m + \frac{1}{2})}{\pi^{\frac{1}{2}}(m-1) \Gamma(\frac{1}{2}m)} \cdot \frac{1}{(1-r)^{m-1}}, \end{aligned} \tag{3.6}$$

using the recurrence and duplication formulae for the Gamma function. Substituting (3.6) into (3.5) with  $m = (\frac{1}{2}k + 1)\lambda$ , we get (3.3) and (3.4).

The constant  $A(k, \lambda)$ . Choosing  $s_1(z) = z(1-z)^{-2}$ ,  $s_2(z) = z(1+z)^{-2}$  (so that  $f(z)$  is given by (2.1)), and any constant  $B(k, \lambda) < A(k, \lambda)$ , it is easy to show that

$$I_\lambda(r) > B(k, \lambda)(1-r)^{1-(\frac{1}{2}k+1)\lambda}$$

for  $r$  sufficiently near to 1 (intuitively because  $s_1$  is large only near  $z = 1$ , where  $s_2$  is near  $\frac{1}{4}$ ).

Using the standard inequality (2, p. 11)

$$|a_n| < \frac{e}{n} I_1 \left( 1 - \frac{1}{n} \right), \tag{3.7}$$

we deduce

**Corollary 3.3.** *Suppose that  $f(z)$  is of the form (1.1), and belongs to  $V_k$ . Then*

$$\limsup_{n \rightarrow \infty} (n^{1-\frac{1}{2}k} |a_n|) \leq \frac{e 2^{\frac{1}{2}k} \Gamma(k/4 + 1)}{\pi^{\frac{1}{2}}(k-1) \Gamma(k/4 + \frac{1}{2})}. \tag{3.8}$$

Since, for positive  $x$ ,

$$\log \Gamma(x) = (2\pi)^{\frac{1}{2}} + (x - \frac{1}{2}) \log x - x + \theta(x)/12x,$$

where  $0 < \theta(x) < 1$  (6, p. 153), we see that

$$\frac{\Gamma(k/4 + 1)}{(k - 1)\Gamma(k/4 + \frac{1}{2})} \sim k^{-\frac{1}{2}} \text{ as } k \rightarrow \infty.$$

In the opposite direction, we have

**Theorem 3.4.** *Suppose that  $f(z)$  is given by (2.1). Then*

$$a_n \sim \frac{2^{\frac{1}{2}k}}{k\Gamma(\frac{1}{2}k)} n^{\frac{1}{2}k-1} \text{ as } n \rightarrow \infty. \tag{3.9}$$

This is verified by an argument similar to that of (4, p. 93), and improves the estimate in (3).

Now let us observe that, with very little technical effort, it is possible to obtain a coefficient estimate for functions in  $V_k$ . This is based on

**Theorem 3.5.** *Suppose that  $f(z)$  belongs to  $V_k$ ,  $M(r) = \max_{|z|=r} |f(z)|$ , and*

$$L(r) = \int_0^{2\pi} r |f'(re^{i\theta})| d\theta$$

*is the length of  $f(|z| = r)$ . Then*

$$2M(r) < L(r) < 2^{\frac{1}{2}}(2k + 1)M(r). \tag{3.10}$$

**Proof.** The left inequality of (3.10) is a consequence of the fact that  $f(|z| = r)$  is a closed curve round the origin, and the right inequality is Theorem 3.3 of (1) (whose proof was totally elementary).

**Corollary 3.6.** *A function in  $V_k$  is bounded if and only if its derivative belongs to the Hardy class  $H_1$ .*

We now have

**Theorem 3.7.** *Suppose that  $f(z)$  is of the form (1.1), and belongs to  $V_k$ . Then, if  $M(r) = \max_{|z|=r} |f(z)|$ ,*

$$|a_n| < \frac{e}{n\pi} 2^{\frac{1}{2}}(2k + 1)M \left(1 - \frac{1}{n}\right) \quad (n > 1). \tag{3.11}$$

This follows by applying (3.7) to (3.10), using the fact that  $L(r) = rI_1(r)$ .

It has been conjectured (7) that, if  $f(z)$  belongs to  $V_k$ , the moduli of its coefficients do not exceed the corresponding coefficients of the function (2.1). In this direction we have

**Theorem 3.8.** *Suppose that  $f(z)$  is of the form (1.1), belongs to  $V_k$ , and is given by (3.1). Then*

$$a_n = o(n^{\frac{1}{2}k-1}) \text{ as } n \rightarrow \infty, \tag{3.12}$$

unless

$$s_1(z) = z(1-tz)^{-2} \text{ for some } |t| = 1. \tag{3.13}$$

**Proof.** It follows from (10, Theorem 1) that, if  $s_1(z)$  is normalized and starlike in  $U$  and is not of the form (3.13),

$$\lim_{r \rightarrow 1} (1-r)^2 \max_{|z|=r} |s_1(z)| = 0.$$

Then, from (3.1),

$$\max_{|z|=r} |f'(z)| = o\{(1-r)^{-\frac{1}{2}k-1}\}.$$

Integrating (3.14) we get

$$\max_{|z|=r} |f(z)| = o\{(1-r)^{-\frac{1}{2}k}\},$$

so that, by Theorem 3.7,

$$a_n = o(n^{\frac{1}{2}k-1}).$$

Furthermore we observe that, if  $1/s_2(z)$  is continuous near the point  $z = 1/t$ , the coefficient conjecture is certainly true for sufficiently large indices; this is easily verified by applying the techniques of (4, p. 93) to Theorem 3.8.

Finally we note the following result, which seems rather interesting in view of the conjecture.

**Theorem 3.9.** *Suppose that  $g(z)$  belongs to  $V_k$ ,  $f(z)$  is of the form (1.1), and*

$$f'(z) = g'(z) \left( \frac{1+z}{1-z} \right)^m$$

for some  $m \geq 0$ . Then  $f(z)$  belongs to  $V_{k+2m}$ .

**Proof.** For any  $z = re^{i\theta}$  and  $0 \leq r < 1$ , we have

$$\int_0^{2\pi} \left| \operatorname{Re} \frac{2z}{1-z^2} \right| d\theta \leq \frac{1}{2} \int_0^{2\pi} \operatorname{Re} \frac{1+z}{1-z} d\theta + \frac{1}{2} \int_0^{2\pi} \operatorname{Re} \frac{1-z}{1+z} d\theta = 2\pi.$$

Consequently

$$\int_0^{2\pi} \left| \operatorname{Re} (1+zf''/f') \right| d\theta \leq \int_0^{2\pi} \left| \operatorname{Re} (1+zg''/g') \right| d\theta + m \int_0^{2\pi} \left| \operatorname{Re} \frac{2z}{1-z^2} \right| d\theta \leq k\pi + 2m\pi;$$

thus  $f(z)$  belongs to  $V_{k+2m}$  as required.

**Note.** This theorem can also be proved using Theorem 3.1.

#### 4. More coefficient results

We now consider the connection between the coefficients of functions  $f(z)$  in  $V_k$ , the area  $A(r)$  of  $f(|z| < r)$  (taking account of multiplicity), and the maximum modulus  $M(r) = \max_{|z|=r} |f(z)|$ ,  $0 < r < 1$ .

**Theorem 4.1.** *Suppose that  $f(z)$  belongs to  $V_k$ , and is of the form (1.1) and that  $f(U)$  has finite area  $A$  (taking account of multiplicity). Then*

$$|a_n| < \frac{k}{n} \left(1 + \frac{1}{2n}\right)^{\frac{1}{2}} \left(\frac{A}{\pi}\right)^{\frac{1}{2}}. \tag{4.1}$$

**Proof.** By the Paatero representation theorem, we have

$$1 + zf''/f' = (\frac{1}{2}k + \frac{1}{2})p_1(z) - (\frac{1}{2}k - \frac{1}{2})p_2(z), \tag{4.2}$$

where  $p_i(0) = 1$  and  $\text{Re } p_i(z) > 0$  in  $U$ ,  $i = 1, 2$ . Then

$$f' + zf'' = (\frac{1}{2}k + \frac{1}{2})p_1 f' - (\frac{1}{2}k - \frac{1}{2})p_2 f'; \tag{4.3}$$

hence, if  $z = re^{i\theta}$ , multiplying both sides of (4.3) by  $e^{-i(n-1)\theta}$ , integrating from 0 to  $2\pi$ , and using the triangle inequality, we obtain

$$2\pi n^2 r^{n-1} |a_n| \leq (\frac{1}{2}k + \frac{1}{2}) \int |p_1 f'| d\theta + (\frac{1}{2}k - \frac{1}{2}) \int |p_2 f'| d\theta. \tag{4.4}$$

Applying Schwarz's inequality for  $i = 1, 2$ , we obtain

$$\begin{aligned} \left(\int_0^{2\pi} |p_i f'| d\theta\right)^2 &\leq \int |p_i|^2 d\theta \int |f'|^2 d\theta \\ &\leq \int_0^{2\pi} \left|\frac{1+re^{i\theta}}{1-re^{i\theta}}\right|^2 d\theta \int_0^{2\pi} |f'|^2 d\theta \\ &\leq 2\pi \frac{1+r}{1-r} \int_0^{2\pi} |f'|^2 d\theta, \end{aligned} \tag{4.5}$$

using the fact that each  $p_i(z)$  is subordinate in  $U$  to  $\frac{1+z}{1-z}$ , and that

$$\int_0^{2\pi} \frac{1-r^2}{|1-re^{i\theta}|^2} d\theta = 2\pi$$

(as the integrand is the Poisson kernel). From (4.4) and (4.5) we have

$$\begin{aligned} 4\pi^2 n^4 |a_n|^2 r^{2n-2} &\leq \frac{1}{2}k^2 \cdot 2\pi \cdot \frac{1+r}{1-r} \int_0^{2\pi} |f'(re^{i\theta})|^2 d\theta \\ &< k^2 \pi (1-r)^{-1} \int_0^{2\pi} |f'|^2 d\theta. \end{aligned} \tag{4.6}$$

Multiplying both sides of (4.6) by  $r(1-r)$ , and integrating from 0 to 1, we get (4.1).

This leads at once to

**Theorem 4.2.** *Suppose that  $f(z)$  belongs to  $V_k$ , and is of the form (1.1). Then if  $A(r)$  is the area of  $f(|z| < r)$ , and  $M(r) = \max_{|z|=r} |f(z)|$ ,*

$$|a_n| < \frac{ek}{n} \left(\frac{A\left(1 - \frac{1}{n}\right)}{\pi}\right)^{\frac{1}{2}} \tag{4.7}$$

and

$$|a_n| < \frac{ek}{2^{\frac{1}{2}n}} M \left(1 - \frac{1}{n}\right). \tag{4.8}$$

**Proof.** Applying Theorem 4.1 to the function  $\frac{1}{r}f(rz)$ , we deduce that

$$r^{n-1} |a_n| < \frac{k}{n} \left(1 + \frac{1}{2n}\right)^{\frac{1}{2}} \left(\frac{A(r)}{\pi}\right)^{\frac{1}{2}}.$$

Substituting  $r = 1 - \frac{1}{n}$ , we obtain (4.7).

Since the valency of  $f(z)$  is at most  $\frac{1}{2}k$ , by Theorem 2.1, we have

$$A(r) \leq \pi M^2(r) \cdot \frac{1}{2}k; \tag{4.9}$$

then (4.8) follows from (4.7) and (4.9).

**5. Special subclasses of  $V_k$**

We now examine the coefficient problem for functions in  $V_k$  which are bounded or of finite area.

If  $f(U)$  is bounded, then  $f'(z)$  belongs to  $H_1$ , by Corollary 3.6; thus, if  $f(z)$  is of the form (1.1),

$$a_n = o(n^{-1}) \text{ as } n \rightarrow \infty \tag{5.1}$$

(see, for example, (11, p. 112)). Although we are unable to show that (5.1) is best possible, we can at least show that the exponent of  $n$  cannot be reduced.

**Theorem 5.1.** *Choose any  $\varepsilon > 0$ , and any  $k \geq 6 + 4\varepsilon$ . Then there is a bounded function  $f(z)$  in  $V_k$ , of the form (1.1), such that*

$$a_n \sim 1/n(\log n)^{1+\varepsilon} \text{ as } n \rightarrow \infty. \tag{5.2}$$

**Note.** In our proof, we use Theorem 3.1 and the fact that, if  $g(z)$  is normalized and starlike in  $U$ , then so is  $z(g/z)^t$  for  $0 < t < 1$ .

**Proof.** The function  $s_1(z) = z(1-z)^{-4/(k+2)}$  is starlike in  $U$ ; also, since  $\log(1-z)^{-1}$  is starlike in  $U$ , so is

$$s_2(z) = z \left(\frac{1}{z} \log \frac{1}{1-z}\right)^{4(1+\varepsilon)/(k-2)},$$

as  $4(1+\varepsilon) \leq k-2$ . Thus the function  $f(z)$ , of the form (1.1), belongs to  $V_k$ , where

$$f'(z) = (1-z)^{-1} \left(\frac{1}{z} \log \frac{1}{1-z}\right)^{-1-\varepsilon},$$

by Theorem 3.1. Hence

$$na_n \sim (\log n)^{-1-\varepsilon} \text{ as } n \rightarrow \infty,$$

using the coefficient estimates in (4, p. 93).

In the case that  $f(U)$  is not necessarily bounded, but does have finite area, Theorem 4.1 shows that

$$a_n = O(n^{-1}) \text{ as } n \rightarrow \infty. \tag{5.3}$$

We cannot show that (5.3) is best possible, but can establish

**Theorem 5.2.** *Choose any  $\varepsilon > 0$ , and any  $k \geq 4 + 2\varepsilon$ . Then there is a function  $f(z)$  in  $V_k$ , of the form (1.1), such that  $f(U)$  has finite area, and*

$$na_n \sim (\log n)^{-\frac{1}{2}-\frac{1}{2}\varepsilon} \text{ as } n \rightarrow \infty. \tag{5.4}$$

**Proof.** The functions

$$s_1(z) = z(1-z)^{-4/(k+2)} \text{ and } s_2(z) = z \left( \frac{1}{z} \log \frac{1}{1-z} \right)^{2(1+\varepsilon)/(k-2)}$$

are normalized starlike functions in  $U$  so long as  $k \geq 4 + 2\varepsilon$ . Hence, by Theorem 3.1, the function  $f(z)$ , of the form (1.1), belongs to  $V_k$ , where

$$f'(z) = (1-z)^{-1} \left( \frac{1}{z} \log \frac{1}{1-z} \right)^{-\frac{1}{2}-\frac{1}{2}\varepsilon}.$$

Then the area of  $f(U)$  is

$$\begin{aligned} A(1) &= \int_0^1 \int_0^{2\pi} |f'(re^{i\theta})|^2 r dr d\theta \\ &= \int_0^1 r^{-\varepsilon} dr \int_0^{2\pi} \left| (1-re^{i\theta})^2 \left( \log \frac{1}{1-re^{i\theta}} \right)^{-1-\varepsilon} \right|^{-1} d\theta \\ &< A(\varepsilon) \int_0^1 (1-r)^{-1} \left( \frac{1}{r} \log \frac{1}{1-r} \right)^{-1-\varepsilon} dr \\ &< +\infty; \end{aligned}$$

here we have used the results on integral means in (4, p. 96). (5.4) then follows from the coefficient estimates in (4, p. 93).

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