FROM DISCRETE TO CONTINUUM: A VARIATIONAL APPROACH

The one-dimensional case

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Abstract These Lecture Notes cover the course given at SISSA by AB in Spring 2000 (Chapters 1 and 2) and some general results just hinted at in the course (Chapter 3). They include mainly results by the authors and by Chambolle, Dal Maso, Garroni and Truskinovsky, but some results are new; e.g., Sections 1.4–1.8 (except 1.4.1 and 1.7.2), and Section 3.3.

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INTRODUCTION

In these lecture notes we treat the problem of the description of variational limits of discrete problems in a one-dimensional setting. Given $n \in \mathbf{N}$ we consider energies of the general form

$$E_n(\{u_i\}) = \sum_{j=1}^{n} \sum_{i=0}^{n-j} \lambda_n \psi_n^j \Big(\frac{u_{i+j} - u_i}{j\lambda_n} \Big)$$

defined on $(n+1)$-tuples $\{u_i\}$. We may view $\{u_i\}$ as a *discrete function* defined on a lattice covering a fixed interval $[0, L]$ by introducing points $x_i^n = i\lambda_n$ ($\lambda_n = L/n$ is the *lattice spacing*) If we picture the set $\{x_i^n\}$ as the reference configuration of an array of material points interacting through some forces, and let u_i represent the displacement of the i-th point, then ψ_n^j can be thought as the energy density of the interaction of points with distance $j\lambda_n$ (j lattice spacings) in the reference lattice. Note that the only assumption we make is that ψ_n^j depends on $\{u_i\}$ through the differences $u_{i+j} - u_i$, but we find it more convenient to highlight its dependence on the 'discrete difference quotients'

$$\frac{u_{i+j} - u_i}{j\lambda_n}.$$

One must not be distracted from this notation and should note the generality of the approach.

Our goal is to describe the behaviour of problems of the form

$$\min\Big\{ E_n(\{u_i\}) - \sum_{i=0}^{n} u_i f_i : \ u_0 = U_0, \ u_n = U_L \Big\}$$

(and similar), and to show that for a quite general class of energies these problems have a limit continuous counterpart. Here $\{f_i\}$ represents the external forces and U_0, U_L are the boudary conditions at the endpoints of the interval $(0, L)$. More general statement and different problems can be also obtained. To make this asymptotic analysis precise, we use the notation and methods of Γ-convergence, for which we refer to the lecture notes by A. Braides Γ-*Convergence for Beginners* (a more complete theoretical introduction can be found in the book by G. Dal Maso *An Introduction to* Γ-*convergence*). We will show that, upon suitably identifying discrete functions $\{u_i\}$ with suitable (posssibly discontinuous) interpolations, the free energies E_n 'Γ-*converge*' to a limit energy F. As a consequence we obtain that minimizers of the problem above are 'very close' to minimizers of

$$\min\Big\{F(u)-\int_0^L fu\,dt:\ u(0)=U_0,\ u(L)=U_L\Big\}.$$

The energies F can be explicitly identified by a series of operations on the functions ψ_n^j. In order to give an idea of how F can be described, we first consider the case when only nearest-neighbour interactions are taken into account:

$$E_n(\{u_i\})=\sum_{i=0}^{n-1}\lambda_n\psi_n\Big(\frac{u_{i+1}-u_i}{\lambda_n}\Big).$$

In this case, the limit functional F can be described by introducing for each n a 'threshold' T_n such that $T_n\to+\infty$ and $\lambda_nT_n\to 0$, and defining a limit *bulk energy density*

$$f(z)=\lim_n(\text{convex envelope of }\tilde\psi_n(z)),$$

and a limit *interfacial energy density*

$$g(z)=\lim_n(\text{subadditive envelope of }\lambda_n\tilde{\tilde\psi}_n\Big(\frac{z}{\lambda_n}\Big)),$$

where

$$\tilde\psi_n(z)=\begin{cases}\psi_n(z) & \text{if } |z|\le T_n\\ +\infty & \text{otherwise,}\end{cases}\qquad \tilde{\tilde\psi}_n(z)=\begin{cases}\psi_n(z) & \text{if } |z|\ge T_n\\ +\infty & \text{otherwise.}\end{cases}$$

Note the crucial *separation of scales* argument: essentially, the limit behaviour of $\psi_n(z)$ defines the bulk energy density, while $\lambda_n\,\psi_n(z/\lambda_n)$ determines the interfacial energy. The limit F is defined (up to passing to its lower semicontinuity envelope) on piecewise-Sobolev functions as

$$F(u)=\int_{(0,L)}f(u')\,dt+\sum_{S(u)}g(u(t+)-u(t-)),$$

where $S(u)$ denotes the set of *discontinuity points* of u. Hence, we have a limit energy with two competing contributions of a bulk part and of an interfacial energy. In this form we can recover *fracture* and *softening* phenomena.

The description of the limit energy gets more complex when not only nearest-neighbour interactions come into play. We first examine the case when interactions up to a fixed order K are taken into account:

$$E_n(\{u_i\})=\sum_{j=1}^{K}\sum_{i=0}^{n-j}\lambda_n\psi_n^j\Big(\frac{u_{i+j}-u_i}{j\lambda_n}\Big)$$

(or, equivalently, $\psi_n^j=0$ if $j>K$). The main idea is to show that (upon some controllable errors) we can find a lattice spacing η_n (possibly much larger than

λ_n) such that E_n is 'equivalent' (as Γ-convergence is concerned) to a nearest-neighbour interaction energy on a lattice of step size η_n, of the form

$$\overline{E}_n(\{u_i\}) = \sum_{i=0}^{m-1} \eta_n \overline{\psi}_n\Big(\frac{u_{i+1}-u_i}{\eta_n}\Big),$$

and to which then the recipe above can be applied.

The crucial points here are the computation of $\overline{\psi}_n$ and the choice of the scaling η_n. In the case of *next-to-nearest neighbours* this computation is particularly simple, as it consists in choosing $\eta_n = 2\lambda_n$ and in 'integrating out the contribution of first neighbours': in formula,

$$\overline{\psi}_n(z) = \psi^2_n(z) + \frac{1}{2}\min\{\psi^1_n(z_1) + \psi^1_n(z_2) : z_1 + z_2 = 2z\}.$$

In a sense this is a formula of *relaxation type*. If $K > 2$ then the formula giving ψ_n resembles more a *homogenization formula*, and we have to choose $\eta_n = K_n\lambda_n$ with K_n large. In this case the reasoning that leads from E_n to $\overline{E}_n$ is that the overall behaviour of a system of interacting point will behave as *clusters* of large arrays of neighbouring points interacting through their 'extremities'

When the number of interaction orders we consider is not bounded the description becomes more complex. In particural additional *non-local* terms may appear in F.

Note that *first order* Γ-limits may not capture completely the behaviour of minimizers for variational problems as above. Additional information, as *phase transitions*, *boundary layer effects* and *multiple cracking*, may be extracted from the study of *higher order* Γ-limit.

1

DISCRETE PROBLEMS WITH LIMIT ENERGIES DEFINED ON SOBOLEV SPACES

1.1 Discrete functionals

We will consider the limit of energies defined on one-dimensional discrete systems of n points as n tends to $+\infty$. In order to define a limit energy on a continuum we parameterize these points as a subset of a single interval $(0, L)$. Set

$$\lambda_n = \frac{L}{n}, \qquad x_i^n = \frac{i}{n}L = i\lambda_n, \qquad i = 0, 1, \ldots, n. \tag{1.1}$$

We denote $I_n = \{x_0^n, \ldots, x_n^n\}$ and by $\mathcal{A}_n(0, L)$ the set of functions $u : I_n \to \mathbf{R}$. If n is fixed and $u \in \mathcal{A}_n(0, L)$ we equivalently denote

$$u_i = u(x_i^n).$$

Given $K \in \mathbf{N}$ with $1 \leq K \leq n$ and functions $f^j : \mathbf{R} \to [0, +\infty]$, with $j = 1, \ldots, K$, we will consider the related functional $E : \mathcal{A}_n(0, L) \to [0, +\infty]$ given by

$$E(u) = \sum_{j=1}^{K} \sum_{i=0}^{n-j} f^j(u_{i+j} - u_i). \tag{1.2}$$

Note that E can be viewed simply as a function $E : \mathbf{R}^n \to [0, +\infty]$.

An interpretation with a physical flavour of the energy E is as the internal interaction energy of a chain of $n+1$ material points each one interacting with its K-nearest neighbours, under the assumption that the interaction energy densities depend only on the order j of the interaction and on the distance between the two points $u_{i+j} - u_i$ in the reference configuration. If $K = 1$ then each point interacts with its nearest neighbour only, while if $K = n$ then each pair of points interacts.

Remark 1.1 From elementary calculus we have that E is lower semicontinuous if each f^j is lower semicontinuous, and that E is coercive on bounded sets of $\mathcal{A}_n(0, L)$.

1.2 Equivalent energies on Sobolev functions

We will describe the limit as $n \to +\infty$ of sequences (E_n) with $E_n : \mathcal{A}_n(0, L) \to [0, +\infty]$ of the general form

$$E_n(u) = \sum_{j=1}^{K_n} \sum_{i=0}^{n-j} f_n^j(u_{i+j} - u_i). \tag{1.3}$$

Since each functional E_n is defined on a different space, the first step is to identify each $\mathcal{A}_n(0,L)$ with a subspace of a common space of functions defined on $(0,L)$. In order to identify each discrete function with a continuous counterpart, we extend u by $\tilde{u} : (0,L) \to \mathbf{R}$ as the piecewise-affine function defined by

$$\tilde{u}(s) = u_{i-1} + \frac{u_i - u_{i-1}}{\lambda_n}(s - x_{i-1}) \qquad \text{if } s \in (x_{i-1}, x_i). \tag{1.4}$$

In this case, $\mathcal{A}_n(0,L)$ is identified with those continuous $u \in \mathrm{W}^{1,1}(0,L)$ (actually, in $\mathrm{W}^{1,\infty}(0,L)$) such that u is affine on each interval (x_{i-1}, x_i). Note moreover that we have

$$\tilde{u}' = \frac{u_i - u_{i-1}}{\lambda_n} \tag{1.5}$$

on (x_{i-1}, x_i). If no confusion is possible, we will simply write u in place of $\tilde{u}_n$.

As we will treat limit functionals defined on Sobolev spaces, it is convenient to rewrite the dependence of the energy densities in (1.3) with respect to difference quotients rather than the differences $u_{i+j} - u_i$. We then write

$$E_n(u) = \sum_{j=1}^{K_n} \sum_{i=0}^{n-j} \lambda_n \psi_n^j\Big(\frac{u_{i+j} - u_i}{j\lambda_n}\Big), \tag{1.6}$$

where

$$\psi_n^j(z) = \frac{1}{\lambda_n} f_n^j(j\lambda_n z).$$

With the identification of u with $\tilde{u}$, E_n may be viewed as an integral functional defined on $\mathrm{W}^{1,1}(0,L)$. In fact, for fixed $j \in \{0, \dots, K-1\}$, $k \in \{0, \dots, n-1\}$ and i such that $i \le k < i + j$ we have

$$\frac{u_{i+j} - u_i}{j\lambda_n} = \frac{1}{j} \sum_{m=i-k}^{i-k+j-1} \frac{u_{k+m+1} - u_{k+m}}{\lambda_n} = \frac{1}{j} \sum_{m=i-k}^{i-k+j-1} \tilde{u}'(x + m\lambda_n)$$

for all $x \in (x_k^n, x_{k+1}^n)$, so that

$$\lambda_n \psi^j\Big(\frac{u_{i+j} - u_i}{j\lambda_n}\Big) = \frac{1}{j} \sum_{k=i}^{i+j-1} \int_{x_k^n}^{x_{k+1}^n} \psi^j\Big(\frac{1}{j} \sum_{m=i-k}^{i-k+j-1} \tilde{u}'(x + m\lambda_n)\Big)\, dx.$$

We then get

$$\sum_{i=0}^{n-j} \lambda_n \psi_n^j\Big(\frac{u_{i+j} - u_i}{j\lambda_n}\Big) = \frac{1}{j} \sum_{l=0}^{j-1} \int_{l\lambda_n}^{L-(j-1-l)\lambda_n} \psi_n^j\Big(\frac{1}{j} \sum_{k=-l}^{j-1-l} \tilde{u}'(x + k\lambda_n)\Big) dx.$$

and the equality

$$E_n(u) = F_n(\tilde{u}), \tag{1.7}$$

where

$$F_n(v) = \begin{cases} \displaystyle\sum_{j=1}^{K_n} \sum_{l=0}^{j-1} \frac{1}{j} \int_{l\lambda_n}^{L-(j-1-l)\lambda_n} \psi_n^j \Big(\frac{1}{j} \sum_{k=-l}^{j-1-l} v'(x + k\lambda_n) \Big) dx & \\ & \text{if } v \in \mathcal{A}_n(0,L) \\ +\infty & \text{otherwise.} \end{cases} \tag{1.8}$$

Note that in the particular case $K_n = 1$ we have (set $\psi_n = \psi_n^1$)

$$F_n(v) = \begin{cases} \displaystyle\int_0^L \psi_n(v')dx & \text{if } v \in \mathcal{A}_n(0,L) \\ +\infty & \text{otherwise.} \end{cases} \tag{1.9}$$

Definition 1.2. (Convergence of discrete functions and energies) With the identifications above we will say that u_n *converges to* u (respectively, in L^1, in measure, in $W^{1,1}$, etc.) if $\tilde{u}_n$ converge to u (respectively, in L^1, in measure, weakly in $W^{1,1}$, etc.), and we will say that E_n Γ-*converges to* F (respectively, with respect to the convergence in L^1, in measure, weakly in $W^{1,1}$, etc.) if F_n Γ-converges to F (respectively, with respect to the convergence in L^1, in measure, weakly in $W^{1,1}$, etc.).

1.3 Convex energies

We first treat the case when the energies ψ_n^j are convex. We will see that in the case of nearest neighbours, the limit is obtained by simply replacing sums by integrals, while in the case of long-range interactions a superposition principle holds.

For simplicity we suppose that the energy densities do not depend on n; i.e.,

$$\psi_n^j = \psi^j.$$

1.3.1 *Nearest-neighbour interactions*

We start by considering the case $K = 1$, so that the functionals E_n are given by

$$E_n(u) = \sum_{i=0}^{n-1} \lambda_n \psi\Big(\frac{u_{i+1} - u_i}{\lambda_n}\Big). \tag{1.10}$$

The integral counterpart of E_n is given by

$$F_n(v) = \begin{cases} \displaystyle\int_0^L \psi(v')dx & \text{if } v \in \mathcal{A}_n(0,L) \\ +\infty & \text{otherwise.} \end{cases} \tag{1.11}$$

Note that F_n depends on n only through its domain $\mathcal{A}_n(0,L)$.

The following result states that as n approaches ∞ the identification of E_n with its continuous analog is complete.

Theorem 1.3 *Let $\psi : \mathbf{R} \to [0,+\infty)$ be convex and let E_n be given by (1.10).*

(i) *The Γ-limit of E_n with respect to the weak convergence in $\mathrm{W}^{1,1}(0,L)$ is given by F defined by*

$$F(u) = \int_{(0,L)} \psi(u')\,dx. \tag{1.12}$$

(ii) *If*

$$\lim_{|z|\to\infty} \frac{\psi(z)}{|z|} = +\infty \tag{1.13}$$

then the Γ-limit of E_n with respect to the convergence in $L^1(0,L)$ is given by F defined by

$$F(u) = \begin{cases} \displaystyle\int_{(0,L)} \psi(u')\,dx & \text{if } u \in \mathrm{W}^{1,1}(0,L) \\ +\infty & \text{otherwise} \end{cases} \tag{1.14}$$

on $L^1(0,L)$.

Proof (i) The functional F defines a weakly lower semicontinuous functional on $\mathrm{W}^{1,1}(0,L)$ and clearly $F_n \geq F$; hence also we have Γ-$\liminf_j F_j(u) \geq F(u)$. Conversely, fixed $u \in \mathrm{W}^{1,1}(0,L)$ let $u_n \in \mathcal{A}_n(0,L)$ be such that $u_n(x_i^n) = u(x_i^n)$. By convexity we have

$$\int_{x_i^n}^{x_{i+1}^n} \psi(u')\,dt \geq \lambda_n \psi\Big(\frac{1}{\lambda_n}\int_{x_i^n}^{x_{i+1}^n} u'\,dt\Big) = \lambda_n \psi\Big(\frac{u(x_{i+1}^n) - u(x_i^n)}{\lambda_n}\Big);$$

hence, summing up,

$$\int_0^L \psi(u')\,dt \geq E_n(u_n).$$

This shows that (u_n) is a recovery sequence for F.

(ii) If (1.13) holds then the sequence (E_n) is equi-coercive on bounded sets of $L^1(0,L)$ with respect to the weak convergence in $\mathrm{W}^{1,1}(0,L)$, from which the thesis is easily deduced. □

1.3.2 *Long-range interactions*

Let now $K \in N$ be fixed. The energies E_n take the form

$$E_n(u) = \sum_{j=1}^{K}\sum_{i=0}^{n-j} \lambda_n \psi_n^j\Big(\frac{u_{i+j} - u_i}{j\lambda_n}\Big). \tag{1.15}$$

Theorem 1.4 *Let $\psi^j : \mathbf{R} \to [0,+\infty)$ be convex and let E_n be given by* (1.15)*. Let ψ^1 satisfy*

$$\lim_{|z|\to\infty} \frac{\psi^1(z)}{|z|} = +\infty \tag{1.16}$$

then the Γ-limit of E_n with respect to the convergence in $L^1(0,L)$ is given by F defined by

$$F(u) = \begin{cases} \displaystyle\int_{(0,L)} \psi(u')\,dx & \text{if } u \in \mathrm{W}^{1,1}(0,L) \\ +\infty & \text{otherwise} \end{cases} \tag{1.17}$$

on $L^1(0,L)$, where

$$\psi = \sum_{j=1}^{K} \psi^j. \tag{1.18}$$

Proof Note that (E_n) is equi-coercive on bounded set of $L^1(0,L)$ as in the proof of Theorem 1.3. Then it suffices to check the Γ-limit on $\mathrm{W}^{1,1}(0,L)$.

To prove the Γ-liminf inequality let $u_n \rightharpoonup u$ weakly in $\mathrm{W}^{1,1}(0,L)$. Then, for every $j \in \{0,\dots,K\}$ and $l \in \{0,\dots,j-1\}$, also the convex combination

$$u_n^{j,l} = \frac{1}{j} \sum_{k=-l}^{j-1-l} \tilde{u}_n(x + k\lambda_n)$$

converge weakly to u in $\mathrm{W}^{1,1}_{\mathrm{loc}}(0,L)$. By (1.7) then we have, for all fixed $\eta > 0$,

$$\begin{aligned} \liminf_n E_n(u) &\geq \liminf_n \sum_{j=1}^{K} \frac{1}{j} \sum_{l=0}^{j-1} \int_\eta^{L-\eta} \psi_n^j\Big((u_n^{j,l})'\Big)\,dx \\ &\geq \sum_{j=1}^{K} \sum_{l=0}^{j-1} \frac{1}{j} \liminf_n \int_\eta^{L-\eta} \psi_n^j\Big((u_n^{j,l})'\Big)\,dx \\ &\geq \sum_{j=1}^{K} \sum_{l=0}^{j-1} \frac{1}{j} \int_\eta^{L-\eta} \psi^j(u')dx = \int_\eta^{L-\eta} \psi(u')\,dt. \end{aligned}$$

The liminf inequality follows by the arbitrariness of η.

Again, fixed $u \in \mathrm{W}^{1,1}(0,L)$ let $u_n \in \mathcal{A}_n(0,L)$ be such that $u_n(x_i^n) = u(x_i^n)$. By Jensen's inequality,

$$\begin{aligned} E_n(u_n) &= \sum_{j=1}^{K} \sum_{i=0}^{n-j} \lambda_n \psi^j\Big(\frac{1}{j\lambda_n} \int_{x_i^n}^{x_{i+j}^n} u'\,dt\Big) \leq \sum_{j=1}^{K} \sum_{i=0}^{n-j} \frac{1}{j} \int_{x_i^n}^{x_{i+j}^n} \psi^j(u')dt \\ &= \sum_{j=1}^{K} \frac{1}{j} \sum_{i=0}^{n-j} \int_{x_i^n}^{x_{i+j}^n} \psi^j(u')dt \leq \sum_{j=1}^{K} \int_0^L \psi^j(u')dt, \end{aligned}$$

which implies the limsup inequality. □

1.4 Energies with superlinear growth

We now investigate the effects of the lack of convexity, always in the framework of limits defined on Sobolev spaces. Again we suppose that the energy densities do not depend on n; i.e.,

$$\psi_n^j = \psi^j,$$

but are not necessarily convex.

1.4.1 *Nearest-neighbour interactions*

We consider the case $K = 1$. In this case the only effect of the passage from the discrete setting to the continuum is a convexification of the integrand.

Theorem 1.5 *Let $\psi : \mathbf{R} \to [0, +\infty)$ be a Borel function satisfying* (1.13). *Let E_n be given by* (1.10)*; then the Γ-limit of E_n with respect to the convergence in $L^1(0, L)$ is given by F defined by*

$$F(u) = \begin{cases} \displaystyle\int_{(0,L)} \psi^{**}(u')\,dx & \text{if } u \in \mathrm{W}^{1,1}(0, L) \\ +\infty & \text{otherwise} \end{cases} \tag{1.19}$$

on $L^1(0, L)$.

Proof The Γ-liminf inequality immediately follows as in the proof of Theorem 1.3(i).

As for the limsup inequality, first note that if $u \in \mathrm{W}^{1,1}(0, L)$ and $\psi(u') = \psi^{**}(u')$ a.e. then we may simply take u_n as in the proof of Theorem 1.3(i), so that for such u we have $\Gamma\text{-}\lim_n E_n(u) = F(u)$. If ψ is lower semicontinuous and u is affine with $u' = z$, let $z_1, z_2 \in \mathbf{R}$ and $\lambda \in [0, 1]$ be such that

$$z = \lambda z_1 + (1-\lambda) z_2, \qquad \psi(z_1) = \psi^{**}(z_1), \psi(z_2) = \psi^{**}(z_2)$$

and

$$\psi^{**}(z) = \lambda \psi(z_1) + (1-\lambda)\psi(z_2).$$

Then there exists u_j weakly converging to u such that $u_j' \in \{z_1, z_2\}$ and $F(u) = \lim_j F(u_j)$. By the lower semicontinuity of the Γ-limsup we then have

$$\Gamma\text{-}\limsup_n E_n(u) \le \liminf_j \Gamma\text{-}\limsup_n E_n(u_j) = \liminf_j F(u_j) = F(u),$$

as desired. If ψ is not lower semicontinuous then suitable $z_{1,j}$ and $z_{2,j}$ must be chosen such that u_j weakly converges to u such that $u_j' \in \{z_{1,j}, z_{2,j}\}$ and $F(u) = \lim_j F(u_j)$.

To conclude the proof it remains to suitably approximate any function $u \in \mathrm{W}^{1,1}(0, L)$ by some its affine interpolations (u_k) and remark that by the convexity of F we have $F(u) = \lim_k F(u_k)$. □

1.4.2 *Next-to-nearest neighbour interactions*

In the non-convex setting, the case $K = 2$ offers an interesting way of describing the two-level interactions between first and second neighbours. Such description is more difficult in the case $K \geq 3$. Essentially, the way the limit continuum theory is obtained is by first integrating-out the contribution due to nearest neighbours by means of an inf-convolution procedure and then by applying the previous results to the resulting functional.

Theorem 1.6 *Let $\psi^1, \psi^2 : \mathbf{R} \to [0, +\infty)$ be Borel functions such that*

$$\lim_{|z|\to\infty} \frac{\psi^1(z)}{|z|} = +\infty, \tag{1.20}$$

and let $E_n(u) : \mathcal{A}_n(0, L) \to [0, +\infty)$ be given by

$$E_n(u) = \sum_{i=0}^{n-1} \lambda_n \psi^1\Big(\frac{u_{i+1} - u_i}{\lambda_n}\Big) + \sum_{i=0}^{n-2} \lambda_n \psi^2\Big(\frac{u_{i+2} - u_i}{2\lambda_n}\Big) \tag{1.21}$$

Let $\tilde{\psi} : \mathbf{R} \to [0, +\infty)$ be defined by

$$\begin{aligned}\tilde{\psi}(z) &= \psi^2(z) + \frac{1}{2}\inf\{\psi^1(z_1) + \psi^1(z_2)) : z_1 + z_2 = 2z\} \\ &= \inf\Big\{\psi^2(z) + \frac{1}{2}(\psi^1(z_1) + \psi^1(z_2)) : z_1 + z_2 = 2z\Big\},\end{aligned} \tag{1.22}$$

and let

$$\psi = \tilde{\psi}^{**}. \tag{1.23}$$

Then the Γ-limit of E_n with respect to the convergence in $L^1(0, L)$ is given by F defined by

$$F(u) = \begin{cases} \displaystyle\int_{(0,L)} \psi(u')\,dx & \text{if } u \in \mathrm{W}^{1,1}(0, L) \\ +\infty & \text{otherwise} \end{cases} \tag{1.24}$$

on $L^1(0, L)$.

Remark 1.7 (i) The growth conditions on ψ^2 can be weakened, by requiring that $\psi^2 : \mathbf{R} \to \mathbf{R}$ and

$$-c_1\psi^1 \leq \psi^2 \leq c_2(1 + \psi^1)$$

provided that we still have

$$\lim_{|z|\to\infty} \frac{\psi(z)}{|z|} = +\infty.$$

(ii) If ψ^1 is convex then $\tilde{\psi} = \psi^1 + \psi^2$. If also ψ^2 is convex then we recover a particular case of Theorem 1.4.

Proof Let $u \in \mathcal{A}_n(0,L)$. We have, regrouping the terms in the summation,

$$
\begin{aligned}
E_n(u) = & \sum_{\substack{i=0 \\ i \text{ even}}}^{n-2} \lambda_n \Big(\psi^2\Big(\frac{u_{i+2}-u_i}{2\lambda_n}\Big) + \frac{1}{2}\psi^1\Big(\frac{u_{i+2}-u_{i+1}}{2\lambda_n}\Big) + \frac{1}{2}\psi^1\Big(\frac{u_{i+2}-u_{i+1}}{2\lambda_n}\Big)\Big) \\
& + \sum_{\substack{i=0 \\ i \text{ odd}}}^{n-2} \lambda_n \Big(\psi^2\Big(\frac{u_{i+2}-u_i}{2\lambda_n}\Big) + \frac{1}{2}\psi^1\Big(\frac{u_{i+2}-u_{i+1}}{2\lambda_n}\Big) + \frac{1}{2}\psi^1\Big(\frac{u_{i+1}-u_i}{2\lambda_n}\Big)\Big) \\
& + \frac{\lambda_n}{2}\psi^1\Big(\frac{u_n-u_{n-1}}{2\lambda_n}\Big) + \frac{1}{2}\psi^1\Big(\frac{u_1-u_0}{2\lambda_n}\Big) \\
\geq & \frac{1}{2}\left(\sum_{\substack{i=0 \\ i \text{ even}}}^{n-2} 2\lambda_n \tilde{\psi}\Big(\frac{u_{i+2}-u_i}{2\lambda_n}\Big) + \sum_{\substack{i=0 \\ i \text{ odd}}}^{n-2} 2\lambda_n \tilde{\psi}\Big(\frac{u_{i+2}-u_i}{2\lambda_n}\Big)\right) \\
\geq & \frac{1}{2}\left(\sum_{\substack{i=0 \\ i \text{ even}}}^{n-2} 2\lambda_n \psi\Big(\frac{u_{i+2}-u_i}{2\lambda_n}\Big) + \sum_{\substack{i=0 \\ i \text{ odd}}}^{n-2} 2\lambda_n \psi\Big(\frac{u_{i+2}-u_i}{2\lambda_n}\Big)\right) \\
= & \frac{1}{2}\Big(\int_0^{2\lambda_n[n/2]} \psi(\tilde{u}_1')\,dt + \int_{\lambda_n}^{(1+2[n-1/2])\lambda_n} \psi(\tilde{u}_2')\,dt\Big), \qquad (1.25)
\end{aligned}
$$

where $\tilde{u}_k$, respectively, with $k=1,2$, are the continuous piecewise-affine functions such that

$$\tilde{u}_k' = \frac{u_{i+2}-u_i}{2\lambda_n} \qquad \text{on } (x_i^n, x_{i+2}^n) \qquad (1.26)$$

for i, respectively, even or odd.

Let now $u_n \to u$ in $L^1(0,L)$ and $\sup_n E_n(u_n) < +\infty$; then $u_n \rightharpoonup u$ in $\mathrm{W}^{1,1}(0,L)$. Let $u_{k,n}$ be defined as in (1.26); as in the proof of Theorem 1.4, we deduce $u_{k,n} \rightharpoonup u$ as $n \to +\infty$, for $k=1,2$. For every fixed $\eta > 0$ by (1.25) we obtain

$$
\begin{aligned}
\liminf_n E_n(u_n) &\geq \frac{1}{2}\left(\liminf_n \int_\eta^{L-\eta} \psi(u_{1,n}')\,dt + \liminf_n \int_\eta^{L-\eta} \psi(u_{2,n}')\,dt\right) \\
&\geq \int_\eta^{L-\eta} \psi(u')\,dt,
\end{aligned}
$$

and the liminf inequality follows by the arbitrariness of $\eta > 0$.

Now we prove the limsup inequality. By arguing as in the proof of Theorem 1.5, note that it suffices to treat the case when $\tilde{\psi}$ is lower semicontinuous, $u(x) = zx$ and $\psi(z) = \tilde{\psi}(z)$. With fixed $\eta > 0$ let z_1, z_2 be such that $z_1 + z_2 = 2z$ and

$$\psi^2(z) + \frac{1}{2}(\psi^1(z_1) + \psi^2(z_2)) \leq \tilde{\psi}(z) + \eta.$$

We define the recovery sequence u_n as

$$u_n(x_i^n) = \begin{cases} zx_i^n & \text{if } i \text{ is even} \\ \\ z(i-1)\lambda_n + z_1\lambda_n & \text{if } i \text{ is odd.} \end{cases}$$

We then have

$$\begin{aligned} E_n(u_n) &= \sum_{i=0}^{n-1} \lambda_n \psi^1\Big(\frac{u_n(x_{i+1}^n) - u_n(x_i^n)}{\lambda_n}\Big) \\ &\quad + \sum_{i=0}^{n-2} \lambda_n \psi^2\Big(\frac{u_n(x_{i+2}^n) - u_n(x_i^n)}{2\lambda_n}\Big) \\ &\leq \frac{L}{2}(\psi^1(z_1) + \psi^1(z_2)) + L\psi^2(z) \\ &\leq L\tilde{\psi}(z) = L\psi(z) = F(u) \end{aligned}$$

as desired. □

Remark 1.8. (Multiple-scale effects) The formula defining ψ highlights a double-scale effect. The operation of inf-convolution highlights oscillations on the scale λ_n, while the convexification of $\tilde{\psi}$ acts at a much larger scale.

1.4.3 *Long-range interactions*

We consider now the case of a general $K \geq 1$. In this case the effective energy density will be given by a homogenization formula. We suppose for the sake of simplicity that $\psi^j : \mathbf{R} \to [0, +\infty)$ are lower semicontinuous and there exists $p > 1$ such that

$$\psi^1(z) \geq c_0(|z|^p - 1), \qquad \psi^j(z) \leq c_j(1 + |z|^p). \tag{1.27}$$

for all $j = 1, \ldots, K$. Before stating the convergence result we define some energy densities.

Let $N \in \mathbf{N}$. We define $\psi_N : \mathbf{R} \to [0, +\infty)$ as follows:

$$\begin{aligned} \psi_N(z) = \min\Big\{ &\frac{1}{N} \sum_{j=1}^{K} \sum_{i=0}^{N-j} \psi^j\Big(\frac{u(i+j) - u(i)}{j}\Big) \\ &u : \{0, \ldots, N\} \to \mathbf{R},\ u(i) = zi \text{ for } i \leq K \text{ or } i \geq N - K\Big\}. \end{aligned} \tag{1.28}$$

Proposition 1.9 *For all $z \in \mathbf{R}$ there exists the limit $\psi(z) = \lim_N \psi_N(z)$.*

Proof With fixed $z \in \mathbf{R}$, let $N, M \in \mathbf{N}$ with $M > N$, and let u_N be a minimizer for $\psi_N(z)$. We define $u_M : \{0, \ldots, M\} \to \mathbf{R}$ as follows:

$$u_M(i) = \begin{cases} u_N(i - lN) + lNz & \text{if } lN \leq i \leq (l+1)N \ \ (0 \leq l \leq \frac{M}{N} - 1) \\ \\ zi & \text{otherwise.} \end{cases}$$

Then we can estimate

$$\begin{aligned}
\psi_M(z) &\le \frac{1}{M}\sum_{j=1}^{K}\sum_{i=0}^{M-j}\psi^j\Big(\frac{u_M(i+j)-u_M(i)}{j}\Big)\\
&\le \frac{1}{N}\sum_{j=1}^{K}\sum_{i=0}^{N-j}\psi^j\Big(\frac{u_N(i+j)-u_N(i)}{j}\Big)\\
&\quad+\frac{1}{N}\sum_{j=1}^{K}(2K-j)\psi^j(z)+\sum_{j=1}^{K}\frac{M-[M/N]N+k-j}{M}\psi^j(z)\\
&\le \psi_N(z)+\frac{2K}{N}\sum_{j=1}^{K}\psi_j(z)+\frac{N+K}{M}\sum_{j=1}^{K}\psi_j(z)\\
&\le \psi_N(z)+c\Big(\frac{2K}{N}+\frac{N+K}{M}\Big)(1+|z|^p).
\end{aligned} \tag{1.29}$$

Taking first the limsup in M and then the liminf in N we deduce that

$$\limsup_M \psi_M(z) \le \liminf_N \psi_N(z)$$

as desired □

Remark 1.10 (i) $c_0(|z|^p-1)\le\psi^1(z)\le\psi(z)\le c(1+|z|^p)$;
(ii) ψ is lower semicontinuous;
(iii) ψ is convex;
(iv) for all $N\in\mathbf{N}$ we have $\psi(z)\le\psi_N(z)+\frac{c}{N}(1+|z|^p)$.

We can state the convergence theorem.

Theorem 1.11 *Let ψ^j be as above and let E_n be defined by* (1.15). *Then the Γ-limit of E_n with respect to the convergence in $L^1(0,L)$ is given by F defined by*

$$F(u)=\begin{cases}\displaystyle\int_{(0,L)}\psi(u')\,dx & \text{if } u\in \mathrm{W}^{1,p}(0,L)\\ +\infty & \text{otherwise}\end{cases} \tag{1.30}$$

on $L^1(0,L)$, where ψ is given by Proposition 1.9.

Proof We begin by establishing the liminf inequality. Let $u_n\to u$ in $L^1(0,L)$ be such that $\sup_n E_n(u_n)<+\infty$. Note that this implies that

$$\sup_n\int_0^L |u_n'|^p\,dt<+\infty,$$

so that indeed $u_n\rightharpoonup u$ weakly in $\mathrm{W}^{1,p}(0,L)$ and hence also $u_n\to u$ in $L^\infty(0,L)$.

For all $k \in \{0, \ldots, N-1\}$ let

$$\Phi_n(k) = \sum_{l \in \mathbf{N}} \int_{((k+Nl-2K)\lambda_n, (k+Nl+2K)\lambda_n) \cap (0,L)} |u_n'|^p \, dt.$$

We have

$$\sum_{k=0}^{N-1} \Phi_n(k) \leq 2K \int_0^L |u_n'|^p \, dt \leq c,$$

so that, upon choosing a subsequence if necessary, there exists k such that

$$\Phi_n(k) \leq \frac{c}{N}.$$

For the sake of notational simplicity we will suppose that this holds with $k = 0$, and also that $n = MN$ with $M \in \mathbf{N}$, so that the inequality above reads

$$\sum_{l=0}^{M-1} \int_{((Nl-2K)\lambda_n, (Nl+2K)\lambda_n) \cap (0,L)} |u_n'|^p \, dt. \tag{1.31}$$

We may always suppose so, upon first reasoning in slightly smaller intervals than $(0, L)$ and then let those intervals invade $(0, L)$.

Let v_n^N be the piecewise-affine function defined on $(0, L)$ such that

$$\begin{aligned}
&v_n^N(0) = u_n(0)\\
&(v_n^N)' = u_n' \qquad \text{on } (x_i^n, x_{i+1}^n),\ nl + K \leq i \leq Nl + l - K - 1\\
&(v_n^N)' = \frac{u_n((Nl + N - K)\lambda_n) - u_n((Nl + K)\lambda_n)}{(N - 2K)\lambda_n} =: z_{n,l}^N\\
&\qquad\qquad \text{on } (Nl\lambda_n, (Nl + K)\lambda_n) \cup ((N(l + 1) - K)\lambda_n, N(l + 1)\lambda_n).
\end{aligned}$$

The construction of v_n^N deserves some words of explanation. The function v_n^N is constructed on each interval $(Nl\lambda_n, (N + 1)\lambda_n)$ as equal to the function u_n (up to an additive constant) in the middle interval $((Nl + K)\lambda_n, (N(l + 1) - K)\lambda_n)$, and as the affine function of slope $z_{n,l}^N$ in the remaining two intervals. Note that the construction implies that the function

$$v_{n,l}^N : \{0, \ldots, N\} \to \mathbf{R}$$

defined by

$$v_{n,l}^N(i) = \frac{1}{\lambda_n} v_n^N((lN + i)\lambda_n)$$

is a test function for the minimum problem defining $\psi_N(z_{n,l}^N)$, and that

$$\sum_{j=1}^{K} \sum_{i=Nl}^{N(l+1)-j} \lambda_n \psi^j \Big(\frac{v_n^N(x_{i+j}^n) - v_n^N(x_i^n)}{j\lambda_n} \Big)$$

$$= \sum_{j=1}^{K} \sum_{i=Nl}^{N(l+1)-j} \lambda_n \psi^j \Big(\frac{v^N_{n,l}((i+j)) - v^N_{n,l}(i)}{j} \Big) \geq N \lambda_n \psi_N(z^N_{n,l}). \qquad (1.32)$$

Note moreover that, by Hölder's inequality, we have

$$\int_{(0,L)} |(v^N_n)' - u'_n| \, dt \leq \Big(\frac{2K}{N} L \Big)^{1-1/p} \|u'_n\|_{L^p(0,L)} + \frac{2K}{N-2K} \|u'_n\|_{L^1(0,L)},$$

so that, since $u_n(0) = v^N_n(0)$ we have a uniform bound

$$\|v^N_n - u_n\|_{L^\infty(0,L)} \leq \frac{C}{N}. \qquad (1.33)$$

We have that

$$\begin{aligned}
E_n(u_n) &\geq \sum_{l=0}^{M-1} \sum_{j=1}^{K} \sum_{i=Nl+K}^{N(l+1)-K-j} \lambda_n \psi^j \Big(\frac{u_n(x^n_{i+j}) - u_n(x^n_i)}{j\lambda_n} \Big) \\
&= \sum_{l=0}^{M-1} \sum_{j=1}^{K} \sum_{i=Nl+K}^{N(l+1)-K-j} \lambda_n \psi^j \Big(\frac{v^N_n(x^n_{i+j}) - v^N_n(x^n_i)}{j\lambda_n} \Big) \\
&= \sum_{l=0}^{M-1} \sum_{j=1}^{K} \sum_{i=Nl}^{N(l+1)-j} \lambda_n \psi^j \Big(\frac{v^N_n(x^n_{i+j}) - v^N_n(x^n_i)}{j\lambda_n} \Big) \\
&\qquad - \sum_{l=0}^{M-1} \sum_{j=1}^{K} \sum_{i=Nl}^{Nl+K} \lambda_n \psi^j \Big(\frac{v^N_n(x^n_{i+j}) - v^N_n(x^n_i)}{j\lambda_n} \Big) \\
&\qquad - \sum_{l=1}^{M} \sum_{j=1}^{K} \sum_{i=Nl-K-j}^{Nl-j} \lambda_n \psi^j \Big(\frac{v^N_n(x^n_{i+j}) - v^N_n(x^n_i)}{j\lambda_n} \Big) \\
&=: \sum_{l=0}^{M-1} \sum_{j=1}^{K} \sum_{i=Nl}^{N(l+1)-j} \lambda_n \psi^j \Big(\frac{v^N_n(x^n_{i+j}) - v^N_n(x^n_i)}{j\lambda_n} \Big) - I^1_n - I^2_n \\
&\geq \sum_{l=0}^{M-1} \sum_{j=1}^{K} N \lambda_n \psi_N(z^N_{n,l}) - I^1_n - I^2_n, \qquad (1.34)
\end{aligned}$$

the last estimate being given by (1.32).

We give an estimate of the term I^1_n; the term I^2_n can be dealt with similarly. Let $i < Nl + K \leq i + j$; by the growth conditions on ψ^j and the convexity of $z \mapsto |z|^p$ we have

$$\psi^j \Big(\frac{v^N_n(x^n_{i+j}) - v^N_n(x^n_i)}{j\lambda_n} \Big)$$

$$
\begin{aligned}
&\le c\Big(1+\Big|\frac{v_n^N(x_{i+j}^n)-v_n^N(x_i^n)}{j\lambda_n}\Big|^p\Big)\\
&\le c\Big(1+\frac{1}{j}\sum_{k=i}^{i+j-1}\Big|\frac{v_n^N(x_{i+1}^n)-v_n^N(x_i^n)}{\lambda_n}\Big|^p\Big)\\
&\le c\Big(1+K|z_{n,l}^N|^p+\frac{1}{\lambda_n}\int_{((Nl-2K)\lambda_n,(Nl+2K)\lambda_n)\cap(0,L)}|u_n'|^p\,dt\Big)
\end{aligned}
$$

We then deduce by (1.31) and the fact that

$$
|z|^p \le c(1+\psi_N(z))
$$

that

$$
\begin{aligned}
I_n^1 &\le \sum_{l=0}^{M-1}\sum_{j=1}^{K}\sum_{i=Nl}^{Nl+K}\lambda_n c\Big(1+\psi_N(z_{n,l}^N)+\frac{1}{\lambda_n}\int_{((Nl+K)\lambda_n,(Nl+2K)\lambda_n)}|u_n'|^p\,dt\Big)\\
&\le \frac{c}{N}+\frac{c}{N}\sum_{l=0}^{M-1}N\lambda_n\psi_N(z_{n,l}^N). \qquad (1.35)
\end{aligned}
$$

Plugging this estimate and the analog for I_n^2 into (1.34) we get

$$
E_n(u_n) \ge \Big(1-\frac{c}{N}\Big)\sum_{l=0}^{M-1}N\lambda_n\psi_N(z_{n,l}^N)-\frac{c}{N}. \qquad (1.36)
$$

By Remark 1.10(iv) we have

$$
\psi_N(z) \ge \psi(z)-\frac{c}{N}(1+|z|^p) \ge \Big(1-\frac{c}{N}\Big)\psi(z)-\frac{c}{N}.
$$

From (1.36) we then have

$$
E_n(u_n) \ge \Big(1-\frac{c}{N}\Big)\sum_{l=0}^{M-1}N\lambda_n\psi(z_{n,l}^N)-\frac{c}{N} \qquad (1.37)
$$

Now, note that the piecewise-affine functions u_n^N defined by

$$
u_n^N(0)=u_n(0) \qquad \text{and} \qquad (u_n^N)'=z_{n,l}^N \text{ on } (Nl\lambda_n, N(l+1)\lambda_n)
$$

are weakly precompact in $\mathrm{W}^{1,p}(0,L)$, so that we may suppose that $u_n^N \rightharpoonup u^N$. Then by Theorem 1.3 we have

$$
\liminf_n \sum_{l=0}^{M-1}N\lambda_n\psi(z_{n,l}^N) = \liminf_n \int_0^L \psi((u_n^N)')\,dt \ge \int_0^L \psi((u^N)')\,dt, \qquad (1.38)
$$

so that

$$\liminf_n E_n(u_n) \geq \Big(1 - \frac{c}{N}\Big) \int_0^L \psi((u^N)')\,dt - \frac{c}{N} \tag{1.39}$$

By (1.33) and the uniform convergence of u_n to u we have

$$\|u^N - u\|_{L^\infty(0,L)} \leq \frac{c}{N}. \tag{1.40}$$

By letting $N \to +\infty$ we then obtain the thesis by the lower semicontinuity of $\int \psi(u')\,dt$.

To prove the limsup inequality it suffices to deal with the case $u(x) = zx$ since from this construction we easily obtain a recovery sequence for piecewise-affine functions and then reason by density. To exhibit a recovery sequence for such u it suffices to fix $N \in \mathbf{N}$, consider v^N a minimum point for the problem defining $\psi_N(z)$ and define

$$u_n(x_i^n) = v^N(i - Nl)\lambda_n + zNl\lambda_n \qquad \text{if } Nl \leq i \leq N(l+1).$$

We then have

$$\limsup_n E_n(u_n) \leq \psi_N(z) + \frac{c}{N}\sum_{j=1}^K \psi^j(z),$$

and the thesis follows by the arbitrariness of N. □

1.5 A general convergence theorem

By slightly modifying the proof of Theorem 1.11 we can easily state a general Γ-convergence result, allowing a dependence also on n for the energy densities.

Theorem 1.12 *Let $K \geq 1$. Let $\psi_n^j : \mathbf{R} \to [0, +\infty)$ be lower semicontinuous functions and let $p > 1$ exists such that*

$$\psi_n^1(z) \geq c_0(|z|^p - 1), \qquad \psi_n^j(z) \leq c_j(1 + |z|^p). \tag{1.41}$$

for all $j \in \{1, \dots, K\}$ and $n \in \mathbf{N}$. For all $N, n \in \mathbf{N}$ let $\psi_{N,n} : \mathbf{R} \to [0, +\infty)$ be defined by

$$\psi_{N,n}(z) = \min\Big\{ \frac{1}{N} \sum_{j=1}^K \sum_{i=0}^{N-j} \psi_n^j\Big(\frac{u(i+j) - u(i)}{j}\Big) \\ u : \{0, \dots, N\} \to \mathbf{R},\ u(i) = zi \text{ for } i \leq K \text{ or } i \geq N - K \Big\} \tag{1.42}$$

Suppose that $\psi : \mathbf{R} \to [0, +\infty)$ exists such that

$$\psi(z) = \lim_N \lim_n \psi_{N,n}^{**}(z) \qquad \text{for all } z \in \mathbf{R} \tag{1.43}$$

(note that this is not restrictive upon passing to a subsequence of n and N). Let E_n be defined on $\mathcal{A}_n(0,L)$ by

$$E_n(u) = \sum_{j=1}^{K}\sum_{i=0}^{n-j} \lambda_n \psi_n^j\Big(\frac{u_{i+j}-u_i}{j\lambda_n}\Big). \tag{1.44}$$

Then the Γ-limit of E_n with respect to the convergence in $L^1(0,L)$ is given by F defined by

$$F(u) = \begin{cases} \displaystyle\int_{(0,L)} \psi(u')\,dx & \text{if } u \in \mathrm{W}^{1,p}(0,L) \\ +\infty & \text{otherwise} \end{cases} \tag{1.45}$$

on $L^1(0,L)$.

Proof Let $u_n \to u$ in $L^1(0,L)$. We can repeat the proof for the liminf inequality for Theorem 1.11, substituting ψ^j by ψ_n^j and ψ_N by $\psi_{N,n}$. We then deduce as in (1.38)–(1.39) that

$$\begin{aligned}\liminf_n E_n(u_n) &\geq \Big(1-\frac{c}{N}\Big)\liminf_n \int_0^L \psi_{N,n}((u_n^N)')\,dt - \frac{c}{N}\\ &\geq \Big(1-\frac{c}{N}\Big)\int_0^L \psi_N((u^N)')\,dt - \frac{c}{N},\end{aligned}$$

where $\psi_N = \lim_n \psi^{**}_{N,n}$ and the thesis by letting $N \to +\infty$.

To prove the limsup inequality it suffices to deal with the case $u(x) = zx$ since from this construction we easily obtain a recovery sequence for piecewise-affine functions and then reason by density. To exhibit a recovery sequence for such u it suffices to fix $N \in \mathbf{N}$, consider $z_{1,n}, z_{2,n}$ and $\eta_n \in [0,1]$ such that

$$\psi^{**}_{N,n}(z) = \eta_n\psi_{N,n}(z_{1,n}) + (1-\eta_n)\psi_{N,n}(z_{2,n}), \qquad z = \eta_n z_{1,n} + (1-\eta_n)z_{2,n}.$$

Let $v^N_{1,n}$, $v^N_{2,n}$ be minimum points for the problem defining $\psi_{N,n}(z_{1,n})$, $\psi_{N,n}(z_{2,n})$, respectively. For the sake of simplicity assume that there exists m such that $mN\eta_n \in \mathbf{N}$ for all n. Define

$$u_n(x_i^n) = \begin{cases} v^N_{1,n}(i-Nl)\lambda_n + zmNl\lambda_n & \text{if } mNl \leq i \leq mNl + mN\eta_n \\ v^N_{2,n}(i-Nl-mN\eta_n)\lambda_n + zmNl + z_{1,n}mN\eta_n\lambda_n & \\ \qquad\qquad \text{if } mNl + mN\eta_n \leq i \leq mN(l+1). \end{cases}$$

By the growth conditions on ψ_n^j it is easily seen that $(z_{k,n})$ are equi bounded and that

$$\sup\{v^N_{k,n}(i) - z_{k,n}i : \ i \in \{0,\dots,N\}, n \in \mathbf{N}\} < +\infty,$$

so that u_n converges to zx uniformly. We then have

$$\limsup_n E_n(u_n) \le L \limsup_n \psi^{**}_{N,n}(z)$$

and the thesis follows by the arbitrariness of N. □

1.6 Convergence of minimum problems

We first give a general convergence theorem, and subsequently state a finer theorem for next-to-nearest neighbour interactions.

1.6.1 *Limit continuum minimum problems*

From Theorem 1.12 we immediately deduce the following theorem.

Theorem 1.13 *Let E_n and F be given by Theorem 1.12, let $f \in L^1(0,L)$ and $d > 0$. Then the minimum values*

$$m_n = \min\Big\{E_n(u) + \int_0^L f u\,dt : \ u(0) = 0,\ u(L) = d\Big\} \tag{1.46}$$

converge to

$$m = \min\Big\{F(u) + \int_0^L f u\,dt : \ u(0) = 0,\ u(L) = d\Big\}, \tag{1.47}$$

and from each sequence of minimizers of (1.46) we can extract a subsequence converging to a minimizer of (1.47).

Proof Since the sequence of functionals (E_n) is equi-coercive, it suffices to show that the boundary conditions do not change the form of the Γ-limit; i.e., that for all $u \in \mathrm{W}^{1,p}(0,L)$ such that $u(0) = 0$ and $u(L) = d$ and for all $\varepsilon > 0$ there exists a sequence u_n such that $u_n(0) = 0$, $u_n(L) = d$ and $\limsup_n E_n(u_n) \le F(u) + \varepsilon$.

Let $v_n \to u$ in $L^\infty(0,L)$ be such that $\lim_n E_n(v_n) = F(u)$. With fixed $\eta > 0$ and $N \in \mathbf{N}$ let $K_n \in \mathbf{N}$ be such that

$$\lim_n K_n\lambda_n = \frac{\eta}{N}.$$

For all $l \in \{1,\dots,N\}$ let $\phi_n^{N,l} : [0,L] \to [0,1]$ be the piecewise-affine function defined by $\phi_n^{N,l}(0) = 0$,

$$\phi_n^{N,l\,\prime} = \begin{cases} 1/(K_n\lambda_n) & \text{on } ((l-1)K_n\lambda_n, lK_n\lambda_n) \\ -1/(K_n\lambda_n) & \text{on } ((n-lK_n)\lambda_n, (n-lK_n+K-n)\lambda_n) \\ 0 & \text{otherwise.} \end{cases}$$

Let

$$u_n^{N,l} = \phi_n^{N,l} v_n + (1-\phi_n^{N,l})u.$$

We have

$$E_n(u_n^{N,l}) \le E_n(u_n) + c\Big(\int_0^{\eta+K\lambda_n} (1+|u'|^p)\,dt + \int_{L-\eta-K\lambda_n}^{L} (1+|u'|^p)\,dt\Big)$$

$$
\begin{aligned}
&+c\Big(\int_{((l-1)K_n-K)\lambda_n,(lK_n+K)\lambda_n)\cap(0,L)}|u_n'|^p\,dt\\
&+\int_{((n-lK_n-K)\lambda_n,(n-lK_n+K_n+K)\lambda_n)\cap(0,L)}|v_n'|^p\,dt\\
&+\int_0^L\frac{1}{(K_n\lambda_n)^p}|v_n-u|^p\Big)\\
\le\ &E_n(u_n)+c\Big(\int_0^{2\eta}(1+|u'|^p)\,dt+\int_{L-2\eta}^L(1+|u'|^p)\,dt\Big)\\
&+c\Big(\int_{(((l-2)\eta/N,((l+1)\eta/N)\cup(L-(l+1)\eta/N,L-(l-2)\eta/N))\cap(0,L)}|v_n'|^p\,dt\Big)\\
&+c\frac{N^p}{\eta^p}\|v_n-u\|^p_{L^\infty(0,L)}
\end{aligned}
$$

for n large enough. Since

$$
\begin{aligned}
&\sum_{l=1}^N\int_{((l-2)\eta/N,((l+1)\eta/N)\cup(L-(l+1)\eta/N,L-(l-2)\eta/N)\cap(0,L)}|u_n'|^p\,dt\\
\le\ &2\int_0^L(1+|v_n'|^p)\,dt\le c,
\end{aligned}
$$

for all n there exists $l_n\in\{1,\dots,N\}$ such that

$$
\begin{aligned}
E_n(u_n^{N,l_n})\le E_n(v_n)&+c\Big(\int_0^{2\eta}(1+|u'|^p)\,dt+\int_{L-2\eta}^L(1+|u'|^p)\,dt\Big)\\
&+\frac{c}{N}+c\frac{N^p}{\eta^p}\|v_n-u\|^p_{L^\infty(0,L)}
\end{aligned}
$$

Setting $u_n=u_n^{N,l_n}$ we then have

$$
\limsup_n E_n(u_n)\le F(u)+c\Big(\int_0^{2\eta}(1+|u'|^p)\,dt+\int_{L-2\eta}^L(1+|u'|^p)\,dt\Big)+\frac{c}{N},
$$

and the desired inequality by the arbitrariness of η and N. □

1.6.2 *Next-to-nearest interactions: phase transitions and boundary layers*

If the function ψ giving the limit energy density in Theorem 1.12 is not strictly convex, converging sequences of minimizers of problems of the type (1.46) may converge to particular minimizers of (1.47). This happens in the case of next-to-nearest interactions, where the formula giving ψ is of particular help.

We examine the case when $\tilde\psi$ in (1.22) is not convex and of minimum problems (1.46) with $f=0$. Upon some change of coordinates it is not restrictive to examine problems of the form

$$m_n = \min\{E_n(u):\ u(0)=0,\ u(L)=0\}, \tag{1.48}$$

and to suppose

(H1) we have

$$\min \tilde{\psi} = \tilde{\psi}(1) = \tilde{\psi}(-1). \tag{1.49}$$

For the sake of simplicity we make the additional assumptions

(H2) we have

$$\tilde{\psi}(z) > 0 \text{ if } |z| \neq 1; \tag{1.50}$$

(H3) there exist unique z_1^+, z_2^+ and z_1^-, z_2^- such that

$$\psi^2(\pm 1) + \frac{1}{2}\Big(\psi^1(z_1^\pm) + \psi^1(z_2^\pm)\Big) = \min \tilde{\psi}, \qquad z_1^\pm, z_2^\pm = \pm 2;$$

We set

$$\mathbf{M}^+ = \{(z_1^+, z_2^+), (z_2^+, z_1^+)\}, \qquad \mathbf{M}^- = \{(z_1^-, z_2^-), (z_2^-, z_1^-)\} \tag{1.51}$$

$$\mathbf{M} \ = \ \mathbf{M}^+ \cup \mathbf{M}^-. \tag{1.52}$$

(H4) we have $z_i^+ \neq z_j^-$ for all $i, j \in \{1,2\}$;

(H5) all functions are C^1.

Under hypotheses (H1)–(H2) Theorem 1.12 simply gives that $m_n \to 0$ and that the limits u of minimizers satisfy $|u'| \leq 1$ a.e. We will see that indeed they are 'extremal' solutions to the problem

$$\min\{F(u):\ u(0)=0,\ u(L)=0\}. \tag{1.53}$$

The effect of the non validity of hypotheses (H3)–(H5) is explained in Remark 1.18.

The key idea is that it is energetically convenient for discrete minimizer to remain close to the two states minimizing $\tilde{\psi}$, and that every time we have a transition from one of the two minimal configurations to the other a fixed amount of energy is spent (independent of n). To exactly quantify this fact we introduce some functions and quantities.

Definition 1.14. (Minimal energy configurations) *Let* $\mathbf{z} = (z_1, z_2) \in \mathbf{M}$*; we define* $u^{\mathbf{z}} : \mathbf{Z} \to \mathbf{R}$ *by*

$$u^{\mathbf{z}}(i) = \Big[\frac{i}{2}\Big] z_2 + \Big(i - \Big[\frac{i}{2}\Big]\Big) z_1, \tag{1.54}$$

and $u_n^{\mathbf{z}} : \lambda_n \mathbf{Z} \to \mathbf{R}$ *by*

$$u_n^{\mathbf{z}}(x_i^n) = u^{\mathbf{z}}(i)\lambda_n \tag{1.55}$$

Definition 1.15. (Crease and boundary-layer energies) Let $v:\mathbf{Z}\to\mathbf{R}$. The *right-hand side boundary layer energy of* v is

$$B_+(v) = \inf_{N\in\mathbf{N}} \min\Big\{\sum_{i\geq 0}\Big(\psi^2\Big(\frac{u(i+2)-u(i)}{2}\Big) + \psi^1(u(i+1)-u(i)) - \min\tilde\psi\Big)$$
$$: u:\mathbf{N}\cup\{0\}\to\mathbf{R},\ u(i)=v(i) \text{ if } i\geq N\Big\},$$

The *left-hand side boundary layer energy of* v is

$$B_-(v) = \inf_{N\in\mathbf{N}} \min\Big\{\sum_{i\leq 0}\Big(\psi^2\Big(\frac{u(i)-u(i-2)}{2}\Big) + \psi^1(u(i)-u(i-1)) - \min\tilde\psi\Big)$$
$$: u:-\mathbf{N}\cup\{0\}\to\mathbf{R},\ u(i)=v(i) \text{ if } i\leq -N\Big\},$$

Let $v^\pm:\mathbf{Z}\to\mathbf{R}$. The *transition energy between* v^- *and* v^+ is

$$C(v^-,v^+) = \inf_{N\in\mathbf{N}} \min\Big\{\sum_{i\in\mathbf{Z}}\Big(\psi^2\Big(\frac{u(i+2)-u(i)}{2}\Big) + \psi^1(u(i+1)-u(i)) - \min\tilde\psi\Big)$$
$$: u:\mathbf{Z}\to\mathbf{R},\ c^\pm\in\mathbf{R}, u(i)=v^\pm(i)+c^\pm \text{ if } \pm i\geq N\Big\}.$$

Remark 1.16 Condition (H4) implies that

$$C(u^{\mathbf{z}^+},u^{\mathbf{z}^-})>0, \qquad C(u^{\mathbf{z}^-},u^{\mathbf{z}^+})>0$$

if $\mathbf{z}^\pm\in\mathbf{M}^\pm$.

We can now describe the behaviour of minimizing sequences for (1.46).

Theorem 1.17 *Suppose that* (H1)–(H5) *hold. We then have:*

(Case n even) *The minimizers* (u_n) *of* (1.46) *for* n *even converge, up to subsequences, to one of the functions*

$$\overline{u}_+(x) = \begin{cases} x & \text{if } 0\leq x\leq L/2 \\ L-x & \text{if } L/2\leq x\leq L,\end{cases} \qquad \overline{u}_-(x) = \begin{cases} -x & \text{if } 0\leq x\leq L/2 \\ -(L-x) & \text{if } L/2\leq x\leq L\end{cases}.$$

Let

$$D := \min\Big\{B_+(u^{\mathbf{z}^+}) + C(u^{\mathbf{z}^+},u^{\mathbf{z}^-}) + B_-(u^{\mathbf{z}^-}),$$
$$B_+(u^{\mathbf{z}^-}) + C(u^{\mathbf{z}^-},u^{\mathbf{z}^+}) + B_-(u^{\mathbf{z}^+}):\ \mathbf{z}^+\in\mathbf{M}^+,\ \mathbf{z}^-\in\mathbf{M}^-\}.$$

If (u_n) *converges (up to subsequences) to* $\overline{u}_\pm$ *then there exist* $\mathbf{z}^+\in\mathbf{M}^+$, *and* $\mathbf{z}^-\in\mathbf{M}^-$ *such that*

$$D = B_+(u^{\mathbf{z}^+}) + C(u^{\mathbf{z}^+},u^{\mathbf{z}^-}) + B_-(u^{\mathbf{z}^-}) \tag{1.56}$$

and

$$E_n(u_n) = D\,\lambda_n + o(\lambda_n). \tag{1.57}$$

(Case n odd) *In the case n odd the same conclusions hold, upon substituting terms of the form*

$$B_+(u^{\mathbf{z}^\pm}) + C(u^{\mathbf{z}^\pm}, u^{\mathbf{z}^\mp}) + B_-(u^{\mathbf{z}^\mp})$$

by terms of the form

$$B_+(u^{\mathbf{z}^\pm}) + C(u^{\mathbf{z}^\pm}, u^{\mathbf{z}^\mp}) + B_-(u^{\overline{\mathbf{z}^\mp}}),$$

where we have set $\overline{(z_1, z_2)} = (z_2, z_1)$.

Proof We only deal with the case n even, as the case n odd is dealt with similarly.

Let u_n be a minimizer for (1.46). We may assume that u_n converge in $\mathrm{W}^{1,p}(0,L)$ and uniformly. By comparison with $E_n(\overline{u})$ we have

$$E_n(u_n) \le L \min\tilde{\psi} + c\lambda_n. \tag{1.58}$$

We can consider the scaled energies

$$E^1_n(u) = \frac{1}{\lambda_n}(E_n(u) - L\min\tilde{\psi}). \tag{1.59}$$

Note that we have

$$\begin{aligned} E^1_n(u) = \sum_{i=0}^{n-2}\Big(&\psi^2\Big(\frac{u_{i+2}-u_i}{2\lambda_n}\Big)\\ &+\frac{1}{2}\Big(\psi^1\Big(\frac{u_{i+2}-u_{i+1}}{\lambda_n}\Big) + \psi^1\Big(\frac{u_{i+1}-u_i}{\lambda_n}\Big)\Big) - \min\tilde{\psi}\Big)\\ &+\frac{1}{2}\Big(\psi^1\Big(\frac{u_n-u_{n-1}}{\lambda_n}\Big) + \psi^1\Big(\frac{u_1-u_0}{\lambda_n}\Big)\Big) - \min\tilde{\psi}. \end{aligned} \tag{1.60}$$

From (1.58) and (1.60) we deduce that

$$\begin{aligned} \sum_{i=0}^{n-2}\Big(&\psi^2\Big(\frac{u_n(x^n_{i+2})-u_n(x^n_i)}{2\lambda_n}\Big)\\ &+\frac{1}{2}\Big(\psi^1\Big(\frac{u_n(x^n_{i+2})-u_n(x^n_{i+1})}{\lambda_n}\Big) + \psi^1\Big(\frac{u_n(x^n_{i+1})-u_n(x^n_i)}{\lambda_n}\Big)\Big) - \min\tilde{\psi}\Big) \le c. \end{aligned}$$

We infer that for every $\eta > 0$ we have that if we denote by $I_n(\eta)$ the set of indices i such that

$$\psi^2\Big(\frac{u_n(x^n_{i+2})-u_n(x^n_i)}{2\lambda_n}\Big)$$

$$+\frac{1}{2}\Big(\psi^1\Big(\frac{u_n(x^n_{i+2})-u_n(x^n_{i+1})}{\lambda_n}\Big)+\psi^1\Big(\frac{u_n(x^n_{i+1})-u_n(x^n_{i})}{\lambda_n}\Big)\Big)\le \min\tilde\psi+\eta$$

then

$$\sup_n I_n(\eta)<+\infty.$$

Let $\varepsilon=\varepsilon(\eta)$ be defined so that if

$$\psi^2\Big(\frac{z_1+z_2}{2}\Big)+\frac{1}{2}\Big(\psi^1(z_1)+\psi^1(z_2)\Big)-\min\tilde\psi\le\eta$$

then

$$\mathrm{dist}\,((z_1,z_2),\mathbf{M})\le\varepsilon(\eta).$$

Choose $\eta>0$ so that

$$2\varepsilon(\eta)<\min\{|\mathbf{z}^+-\mathbf{z}^-|,\ \mathbf{z}^+\in\mathbf{M}^+,\ \mathbf{z}^-\in\mathbf{M}^-\}.$$

We then deduce that if $i-1,i\notin I_n(\eta)$ then there exists $\mathbf{z}\in\mathbf{M}$ such that

$$\Big|\Big(\frac{u_n(x^n_{i+1})-u_n(x^n_{i})}{\lambda_n},\frac{u_n(x^n_{i+2})-u_n(x^n_{i+1})}{\lambda_n}\Big)-\mathbf{z}\Big|\le\varepsilon$$

and

$$\Big|\Big(\frac{u_n(x^n_{i})-u_n(x^n_{i-1})}{\lambda_n},\frac{u_n(x^n_{i+1})-u_n(x^n_{i})}{\lambda_n}\Big)-\overline{\mathbf{z}}\Big|\le\varepsilon$$

Hence, there exist a finite number of indices $0=i_0<i_1<i_2<\cdots<i_{N_n}=n$ such that for all $j=1,\dots,N_n$ there exists $\mathbf{z}^n_j\in\mathbf{M}$ such that for all $i\in\{i_{j-1}+1,\dots,i_j-1\}$ we have

$$\Big|\Big(\frac{u_n(x^n_{i+1})-u_n(x^n_{i})}{\lambda_n},\frac{u_n(x^n_{i+2})-u_n(x^n_{i+1})}{\lambda_n}\Big)-\mathbf{z}^n_j\Big|\le\varepsilon.$$

Let $\{j_0,j_1,\dots,j_{M_n}\}$ be the maximal subset of $\{i_0,i_1,\dots,i_{N_n}\}$ defined by the requirement that if $z^n_{j_k}\in\mathbf{M}^\pm$ then $z^n_{j_k+1}\in\mathbf{M}^\mp$. Note that in this case we deduce that $E_n(u_n)\ge cM_n$, so that M_n are equi-bounded. Upon choosing a subsequence we may then suppose $M_n=M$ independent of n, and also that $x^n_{j_k}\to x_k\in[0,L]$ and $\mathbf{z}^n_{j_k}=\mathbf{z}_k$. By the arbitrariness of η we deduce that $\lim_n u_n=u$, and u is characterized by $u(0)=u(L)=L$ and $u'=\pm1$ on (x_{k-1},x_k), the sign determined by whether $\mathbf{z}_k\in\mathbf{M}^+$ or $\mathbf{z}_k\in\mathbf{M}^+$. Let $y_0=0,y_1,\dots,y_N=L$ be distinct ordered points such that $\{y_i\}=\{x_k\}$ (the set of indices may be different if $x_k=x_{k+1}$ for some k). Choose indices $k_1,\dots,k_N$ such that $x^n_{k_j}\to(y_{j-1}+y_j)/2$. Let $\mathbf{z}_j$ be the limit of $\mathbf{z}^n_{j_k}$ related to the interval (y_j,y_{j+1}). We then have, for a suitable continuous $\omega:[0,+\infty)\to[0,+\infty)$,

$$\sum_{i=0}^{k_1-2}\Big(\psi^2\Big(\frac{u_n(x^n_{i+2})-u_n(x^n_{i})}{2\lambda_n}\Big)$$

$$
+\frac{1}{2}\Big(\psi^1\Big(\frac{u_n(x^n_{i+2})-u_n(x^n_{i+1})}{\lambda_n}\Big)+\psi^1\Big(\frac{u_n(x^n_{i+1})-u_n(x^n_{i})}{\lambda_n}\Big)\Big)-\min\tilde\psi\Big)
$$
$$
\geq B_+(u^{\mathbf{z}_1})-\omega(\varepsilon)
$$

$$
\sum_{i=k_j}^{k_{j+1}-2}\Big(\psi^2\Big(\frac{u_n(x^n_{i+2})-u_n(x^n_{i})}{2\lambda_n}\Big)
$$
$$
+\frac{1}{2}\Big(\psi^1\Big(\frac{u_n(x^n_{i+2})-u_n(x^n_{i+1})}{\lambda_n}\Big)+\psi^1\Big(\frac{u_n(x^n_{i+1})-u_n(x^n_{i})}{\lambda_n}\Big)\Big)-\min\tilde\psi\Big)
$$
$$
\geq C(u^{\mathbf{z}_j},u^{\mathbf{z}_{j+1}})-\omega(\varepsilon) \text{ for all } j\in\{1,\dots,N-1\},
$$

$$
\sum_{i=k_N}^{n-2}\Big(\psi^2\Big(\frac{u_n(x^n_{i+2})-u_n(x^n_{i})}{2\lambda_n}\Big)
$$
$$
+\frac{1}{2}\Big(\psi^1\Big(\frac{u_n(x^n_{i+2})-u_n(x^n_{i+1})}{\lambda_n}\Big)+\psi^1\Big(\frac{u_n(x^n_{i+1})-u_n(x^n_{i})}{\lambda_n}\Big)\Big)-\min\tilde\psi\Big)
$$
$$
\geq B_-(u^{\mathbf{z}_N})-\omega(\varepsilon).
$$

By the arbitrariness of ε and the definition of D we easily get $\liminf_n E^1_n(u_n)\geq D$, and by Remark 1.16 that if $u\neq\overline{u}_\pm$ then $\liminf_n E^1_n(u_n)>D$.

It remains to show that $\limsup_n E^1_n(u_n)\leq D$; i.e., for every fixed $\eta>0$ to exhibit a sequence $\overline{u}_n$ such that $\overline{u}_n(0)=\overline{u}_n(L)=0$ and $\limsup_n E^1_n(\overline{u}_n)\leq D+c\eta$. Suppose that

$$
D=B_+(u^{\mathbf{z}^+})+C(u^{\mathbf{z}^+},u^{\mathbf{z}^-})+B_-(u^{\mathbf{z}^-}),
$$

with $\mathbf{z}^+=(z^+_1,z^+_2)$, $\mathbf{z}^-=(z^-_1,z^-_2)$, the other cases being dealt with in the same way. Let $\eta>0$ be fixed and let $N\in\mathbf{N}$, $v_+,v_-,v:\mathbf{Z}\to\mathbf{R}$ be such that

$$
v_+(i)=u^{\mathbf{z}^+}(i)\qquad \text{for } i\geq N,
$$
$$
v_-(i)=u^{\mathbf{z}^-}(i)\qquad \text{for } i\leq -N,
$$
$$
v(i)=\begin{cases} u^{\mathbf{z}^+}(i) & \text{for } i\leq -N\\ u^{\mathbf{z}^-}(i) & \text{for } i\geq N\end{cases},
$$

and

$$
\sum_{i\geq 0}\Big(\psi^2\Big(\frac{v_+(i+2)-v_+(i)}{2}\Big)+\psi^1(u(i+1)-u(i))-\min\tilde\psi\Big)\leq B_+(u^{\mathbf{z}^+})+\eta
$$
$$
\sum_{i\leq 0}\Big(\psi^2\Big(\frac{v_-(i)-v_-(i-2)}{2}\Big)+\psi^1(u(i)-u(i-1))-\min\tilde\psi\Big)\leq B_-(u^{\mathbf{z}^-})+\eta
$$

$$\sum_{i\in\mathbf{Z}}\Big(\psi^2\Big(\frac{v(i+2)-v(i)}{2}\Big)+\psi^1(v(i+1)-v(i))-\min\tilde\psi\Big)\le C(u^{\mathbf{z}^+},u^{\mathbf{z}^-})+\eta.$$

We then set

$$\overline{u}(x_i^n)=\begin{cases}(v_+(i)-v_+(0))\lambda_n & \text{if } i\le N\\ u_n^{\mathbf{z}^+}(x_i^n)-v_+(0)\lambda_n+z_n^1(x_i^n-x_N^n) & \text{if } N\le i\le \frac{n}{2}-N\\ v\Big(i-\frac{n}{2}\Big)\lambda_n-\frac{L}{2} & \text{if } \frac{n}{2}-N\le i\le\frac{n}{2}+N\\ u_n^{\mathbf{z}^-}(x_{n-i}^n)-v_-(0)\lambda_n+z_n^2(x_i^n-x_{n-N}^n) & \text{if } \frac{n}{2}+N\le i\le n-N\\ (v_-(n-i)-v_-(0))\lambda_n & \text{if } n-N\le i\le n,\end{cases}$$

where

$$z_n^1=\frac{u^{\mathbf{z}^+}(\frac{n}{2})\lambda_n-\frac{L}{2}+v_+(0)\lambda_n}{(\frac{n}{2}-2N)\lambda_n}$$

$$z_n^2=\frac{u^{\mathbf{z}^-}(\frac{n}{2})\lambda_n+\frac{L}{2}+v_-(0)\lambda_n}{(\frac{n}{2}-2N)\lambda_n}.$$

Note that $\lim_n z_n^1=\lim_n z_n^2=0$. Using (H5) we easily get the desired inequality. □

Remark 1.18 From the proof above it can be easily seen that hypotheses (H3)–(H5) may be relaxed at the expense of a heavier notation and some changes in the results. Clearly, if (H3) does not hold then the sets of minimal pairs $\mathbf{M}^+$, $\mathbf{M}^-$ are larger, and the definition of D must be changed accordingly, possibly taking into account also more than one transition.

If hypothesis (H4) does not hold then $C(u^{\mathbf{z}^+},u^{\mathbf{z}^-})=C(u^{\mathbf{z}^-},u^{\mathbf{z}^+})=0$ for some $\mathbf{z}^+\in\mathbf{M}^+$, $\mathbf{z}^-\in\mathbf{M}^-$. In this case the energetic analysis of E_n^1 is not sufficient to characterize the minimizers, as we have no control on the number of transitions between $u'=1$ and $u'=-1$.

Condition (H5) has been used to construct the recovery sequence $(\overline{u}_n)$. It can be relaxed to assuming that $\tilde\psi$ is smooth at ±1; more precisely, it suffices to suppose that

$$\lim_{z\to\pm1}\frac{\tilde\psi(z)-\min\tilde\psi}{|z\mp1|}=0. \tag{1.61}$$

If this condition does not hold the value D is given by a more complex formula, where we take into account also the values at 0 of the solutions of the boundary layer terms.

The proof of Theorem 1.17 easily yields the corresponding Γ-limit result for E_n^1. We leave the details to the reader.

1.7 More examples

In this section we examine some situations when some of the hypotheses considered hitherto are relaxed. Namely,

(i) (*weak nearest-neighbour interactions*) when the condition

$$\psi_n^1(z) \geq c_0(|z|^p - 1)$$

does not hold. In this case, the limit energy may be defined on a set of vector functions;

(ii) (*very-long-range interactions*) when the energy E_n takes into account interactions up to the order K_n with $K_n \to +\infty$. In this case, the limit energy may be non-local;

(iii) (*non spatially homogeneous interactions*) when the interaction between u_i and u_{i+j} may depend also on i. In this case a homogenization process may take place.

For the sake of presentation we will explicitly treat only the case of quadratic energies, of the form

$$E_n(u) = \sum_{j=1}^{K_n} \sum_{i=0}^{n-j} \lambda_n \rho_n^{j,i} \Big(\frac{u_{i+j} - u_i}{j\lambda_n}\Big)^2, \tag{1.62}$$

with $\rho_n^{j,i} > 0$ and $1 \leq K_n \leq n$.

Remark 1.19 In the case when $\rho_n^{j,i} = \rho^j$, $K_n = K$ and $\rho^1 > 0$ then the Γ-limit in Theorem 1.4 of E_n is given by

$$F(u) = \rho \int_0^L |u'|^2\, dt, \qquad \text{where } \rho = \sum_{j=1}^{K} \rho^j.$$

The same conclusion holds if $\rho_n^{j,i} = \rho^j$, $K_n = n$, $\rho^1 > 0$, and $\rho = \sum_{j=1}^\infty \rho^j$.

1.7.1 *Weak nearest-neighbour interactions: multiple-density limits*

We only treat the case of next-to-nearest neighbour interactions with weak nearest-neighbour interactions; i.e., in (1.62) we take $K_n = 2$, $\rho_n^{2,i} = c_2$, and $\rho_n^{1,i} = a_n$ with

$$\lim_n \frac{a_n}{\lambda_n^2} = c_1.$$

The energies we consider take the form

$$E_n(u) = c_2 \sum_{i=0}^{n-2} \lambda_n \Big(\frac{u_{i+2} - u_i}{2\lambda_n}\Big)^2 + \sum_{i=0}^{n-1} \lambda_n a_n \Big(\frac{u_{i+1} - u_i}{\lambda_n}\Big)^2. \tag{1.63}$$

For all n $u_n \in \mathcal{A}_n(0,L)$, we consider the functions $u_{n,\mathrm{e}}, u_{n,\mathrm{o}} : \{0,\dots,[n/2]\} \to \mathbf{R}$, defined by

$$u_{n,\mathrm{e}}(i) = u_n(2i\lambda_n), \qquad u_{n,\mathrm{o}}(i) = u_n((2i+1)\lambda_n)$$

(for simplicity, $u_n(x_i^n) = u_n(L)$ if $i > n$), which take into account the values of u_n on even and odd points, respectively. Note that the energy $E_n(u_n)$ can be identified with an energy $E_n(u_{n,\mathrm{e}}, u_{n,\mathrm{o}})$ defined by

$$\begin{aligned} E_n(u_{n,\mathrm{e}}, u_{n,\mathrm{o}}) = c_2 & \sum_{i=0}^{[n/2]-1} \lambda_n \Big(\frac{u_{n,\mathrm{e}}(i+1) - u_{n,\mathrm{e}}(i)}{2\lambda_n}\Big)^2 \\ & + c_2 \sum_{i=0}^{[n/2]-1} \lambda_n \Big(\frac{u_{n,\mathrm{o}}(i+1) - u_{n,\mathrm{o}}(i)}{2\lambda_n}\Big)^2 \\ & + \sum_{i=0}^{[n/2]-1} \lambda_n a_n \Big(\frac{u_{n,\mathrm{o}}(i) - u_{n,\mathrm{e}}(i)}{\lambda_n}\Big)^2 \\ & + \sum_{i=0}^{[n/2]-1} \lambda_n a_n \Big(\frac{u_{n,\mathrm{o}}(i) - u_{n,\mathrm{e}}(i+1)}{\lambda_n}\Big)^2 . \end{aligned} \tag{1.64}$$

We say that the sequence (u_n) converges (in $L^1(0,L)$) to u to the pair $(u_\mathrm{e}, u_\mathrm{o})$ if the piecewise-affine interpolates $\tilde u_{n,\mathrm{e}}, \tilde u_{n,\mathrm{o}}$ defined by

$$\tilde u'_{n,\mathrm{e}} = \frac{u_{n,\mathrm{e}}(i+1) - u_{n,\mathrm{e}}(i)}{2\lambda_n} \qquad \text{on } (x_n^{2i}, x_n^{2i+2}),$$

$$\tilde u'_{n,\mathrm{o}} = \frac{u_{n,\mathrm{o}}(i+1) - u_{n,\mathrm{o}}(i)}{2\lambda_n} \qquad \text{on } (x_n^{2i}, x_n^{2i+2}),$$

respectively, converge to $(u_\mathrm{e}, u_\mathrm{o})$, respectively. We then have the following result.

Theorem 1.20 *The energies E_n Γ-converge with respect to the convergence of u_n to $(u_\mathrm{e}, u_\mathrm{o})$, to the functional*

$$F(u_\mathrm{e}, u_\mathrm{o}) = \begin{cases} \frac{1}{2} c_2 \int_0^L |u'_\mathrm{e}|^2\,dt + \frac{1}{2} c_2 \int_0^L |u'_\mathrm{o}|^2\,dt + c_1 \int_0^L |u_\mathrm{e} - u_\mathrm{o}|^2\,dt & \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \textit{if } u_\mathrm{e}, u_\mathrm{o} \in H^1(0,L) & \\ +\infty & \textit{otherwise.} \end{cases}$$

If $c_1 = +\infty$ the formula above is understood to mean that $F(u_\mathrm{e}, u_\mathrm{o}) = +\infty$ if $u_\mathrm{e} \neq u_\mathrm{o}$, so that, having set $u = u_\mathrm{e} = u_\mathrm{o}$ we recover for F the form

$$F(u) = \begin{cases} c_2 \int_0^L |u'|^2\,dt & \textit{if } u \in H^1(0,L) \\ +\infty & \textit{otherwise.} \end{cases}$$

Proof It suffices to treat the case $a_n = c_1\lambda_n^2$ with $c_1 < +\infty$, as all the others are easily obtained from that by a comparison argument. To obtain the liminf inequality, it suffices to use Theorem 1.3 for the first two terms in (1.64) and note that each of the last two terms converges to

$$\frac{1}{2}c_1 \int_0^L |u_{\rm e} - u_{\rm o}|^2\, dt,$$

as the convergence of $\tilde u_{n,{\rm e}}, \tilde u_{n,{\rm o}}$ to $(u_{\rm e}, u_{\rm o})$, respectively, is uniform.

The limsup inequality is obtained by direct computations on piecewise-affine functions, and then reasoning by density as usual. □

1.7.2 *Very-long interactions: non-local limits*

For all $n \in \mathbf{N}$ let $\rho_n : \lambda_n\mathbf{Z} \to [0, +\infty)$. We consider the following form of the discrete energies

$$E_n(u) = \sum_{\substack{x,y\in\lambda_n\mathbf{Z}\cap[0,L]\\ x\neq y}} \lambda_n\rho_n(x-y)\Big(\frac{u(x)-u(y)}{x-y}\Big)^2 \tag{1.65}$$

defined for $u : \lambda_n\mathbf{Z} \to \mathbf{R}$. Note that we may assume that ρ_n is an even function, upon replacing $\rho_n(z)$ by $\tilde\rho_n(z) = (1/2)(\rho_n(z) + \rho_n(-z))$. We will tacitly make this simplifying assumption in the sequel.

We will consider the following hypotheses on ρ_n:

(H1) (*equi-coerciveness of nearest-neighbour interactions*) $\inf_n \rho_n(\lambda_n) > 0$;

(H2) (*local uniform summability of* ρ_n) for all $T > 0$ we have

$$\sup_n \sum_{x\in\lambda_n\mathbf{Z}\cap(0,T)} \rho_n(x) < +\infty.$$

Remark 1.21 Note that (H2) can be rephrased as a local uniform integrability property for $\lambda_n\rho_n$ on $\mathbf{R}^2$: for all $T > 0$

$$\sup_n \sum_{\substack{x,y\in\lambda_n\mathbf{Z}\\ x\neq y,|x|,|y|\leq T}} \lambda_n\rho_n(x-y) < +\infty.$$

As a consequence, if (H2) holds then, up to a subsequence, we can assume that the Radon measures

$$\mu_n = \sum_{x,y\in\lambda_n\mathbf{Z},\ x\neq y} \lambda_n\rho_n(x-y)\delta_{(x,y)}$$

(δ_z denotes the Dirac mass at z) locally converge weakly in $\mathbf{R}^2$ to a Radon measure μ_0, and that the Radon measures

$$\beta_n = \sum_{z \in \lambda_n \mathbf{Z}} \rho_n(z)\delta_z$$

locally converge weakly in $\mathbf{R}$ to a Radon measure β_0. These two limit measures are linked by the relation

$$\mu_0(A) = \frac{1}{\sqrt{2}} \int_{\mathbf{R}} |A_s| d\beta_0(s), \tag{1.66}$$

where $|A_s|$ is the Lebesgue measure of the set

$$A_s = \{t \in \mathbf{R} : (s(e_1 - e_2) + t(e_1 + e_2))/\sqrt{2} \in A\}.$$

If (H1) holds then we have the orthogonal decomposition

$$\beta_0 = \beta_1 + c_1\delta_0, \tag{1.67}$$

for some $c_1 > 0$ and a Radon measure β_1 on $\mathbf{R}$. We also denote

$$\mu = \mu_0 \llcorner (\mathbf{R}^2 \setminus \Delta) \tag{1.68}$$

(the restriction of μ_0 to $\mathbf{R}^2 \setminus \Delta$), where $\Delta = \{(x, x) : x \in \mathbf{R}\}$. By the decomposition above, we have

$$\mu_0 = \mu + \frac{1}{\sqrt{2}} c_1 \mathcal{H}^1 \llcorner \Delta,$$

where $\mathcal{H}^1$ stands for the 1-dimensional Hausdorff measure.

The main result of this section is the following.

Theorem 1.22. (Compactness and representation) *If conditions* (H1) *and* (H2) *hold, then there exist a subsequence (not relabeled), a Radon measure* μ *on* $\mathbf{R}^2$ *and a constant* $c_1 > 0$ *such that the energies* E_n Γ*-converge to the energy* F *defined on* $\mathrm{L}^1(0, L)$ *by*

$$F(u) = \begin{cases} c_1 \displaystyle\int_{(0,L)} |u'|^2\, dt + \int_{(0,L)^2} \Big(\frac{u(x) - u(y)}{x - y}\Big)^2 d\mu(x, y) & \text{if } u \in W^{1,2}(0, L) \\ +\infty & \text{otherwise,} \end{cases} \tag{1.69}$$

wit respect to convergence in measure and $L^1(0, L)$*, where the measure* μ *and* c_1 *are given by* (1.68) *and* (1.67)*, respectively.*

Proof Upon passing to a subsequence we may assume that the measures μ_n in Remark 1.21 converge to μ_0. Then, μ and c_1 given by (1.68) and (1.67) are well defined. Hence, it suffices to prove the representation for the Γ-limit along this sequence.

We begin by proving the liminf inequality. Let $u_n \to u$ in $\mathrm{L}^1(0,L)$ be such that $\sup_n E_n(u_n) < +\infty$. By hypothesis (H1), the sequence u_n converges weakly in $W^{1,2}((0,L)$.

With fixed $m \in \mathbf{N}$, we have the equality

$$\begin{aligned} E_n(u_n) \ &= \sum_{\substack{x,y\in\lambda_n\mathbf{Z}\cap[0,L]\\ |x-y|\le 1/m,\ x\ne y}} \rho_n(x-y)\lambda_n\Big(\frac{u_n(x)-u_n(y)}{x-y}\Big)^2 \\ &\quad + \sum_{\substack{x,y\in\lambda_n\mathbf{Z}\cap[0,L]\\ |x-y|> 1/m}} \rho_n(x-y)\lambda_n\Big(\frac{u_n(x)-u_n(y)}{x-y}\Big)^2 \\ &=: I^1_n(u_n) + I^2_n(u_n). \end{aligned} \tag{1.70}$$

We now estimate these two terms separately.

We first note that there exist positive α_n converging to 0 such that

$$\lim_n 2\sum_{k=1}^{[\alpha_n/\lambda_n]} \rho_n(\lambda_n k) \ge c_1 - \frac{1}{m}.$$

Let $(a,b)\subset(0,L)$. For all $N\in\mathbf{N}$ and for n large enough we then have

$$\begin{aligned} I^1_n(u_n) &\ge \sum_{\substack{x,y\in\lambda_n\mathbf{Z}\cap(a,b)\\ |x-y|\le\alpha_n,\ x\ne y}} \lambda_n\rho_n(x-y)\Big(\frac{u_n(x)-u_n(y)}{x-y}\Big)^2 \\ &\ge \sum_{i=1}^{N} 2\sum_{k=1}^{[\alpha_n/\lambda_n]} \sum_{\substack{x,y\in\lambda_n\mathbf{Z}\cap(y_{i-1},y_i)\\ |x-y|=\lambda_n k}} \lambda_n\rho_n(\lambda_n k)\Big(\frac{u_n(x)-u_n(y)}{x-y}\Big)^2 \\ &\ge \sum_{i=1}^{N} 2\sum_{k=1}^{[\alpha_n/\lambda_n]} \frac{(b-a)}{N}\rho_n(\lambda_n k)\Big(\frac{u(y_i)-u(y_{i-1})}{y_i-y_{i-1}}\Big)^2 + o(1) \end{aligned}$$

as $n\to\infty$, where we have set

$$y_i = a + \frac{i}{N}(b-a),$$

we have used the fact that $u_n \to u$ uniformly and the convexity of $z\mapsto z^2$. This shows that

$$\liminf_n I^1_n(u_n) \ge \Big(c_1 - \frac{1}{m}\Big)\int_{(a,b)} |u'|^2\,dt.$$

From this inequality we obtain that

$$\liminf_n I^1_n(u_n) \ge \Big(c_1 - \frac{1}{m}\Big)\int_{(0,L)} |u'|^2\,dt.$$

As for the second term, for all $\eta > 0$ let $\Delta_\eta = \{(x,y) \in \mathbf{R}^2 : \ |x-y| > \eta\}$. Note that the convergence

$$\frac{u_n(x) - u_n(y)}{x-y} \longrightarrow \frac{u(x)-u(y)}{x-y}$$

is uniform on $(0,L)^2 \setminus \Delta_\eta$, so that, by the weak convergence of μ_n we have

$$\liminf_n I_n^2(u_n) \geq \int_{(0,L)^2 \setminus \Delta_{1/m}} \Big(\frac{u(x)-u(y)}{x-y}\Big)^2 d\mu(x,y).$$

By summing up all these inequalities and letting $m \to +\infty$ we eventually get

$$\liminf_n E_n(u_n) \geq c_1 \int_{(0,L)} |u'|^2\,dt + \int_{(0,L)^2} \Big(\frac{u(x)-u(y)}{x-y}\Big)^2 d\mu(x,y).$$

To prove the limsup inequality it suffices to show it for piecewise-affine functions, since this set is strongly dense in the space of piecewise $W^{1,2}$ functions. In this case it suffices to take $u_n = u$. □

1.7.3 *Homogenization*

We only treat the case of nearest-neighbour interactions; i.e., in (1.62) we take $K_n = 1$ and $\rho_n^{1,i} = \rho_i$ with $i \mapsto \rho_i$ defining a M-periodic function $\mathbf{Z} \to \mathbf{R}$:

$$\rho_{i+M} = \rho_i.$$

The energies we consider take the form

$$E_n(u) = \sum_{i=0}^{n-1} \lambda_n \rho_i \Big(\frac{u_{i+1}-u_i}{\lambda_n}\Big)^2. \tag{1.71}$$

Theorem 1.23 *The energies E_n Γ-converge to the energy defined by*

$$F(u) = \begin{cases} \overline{\rho} \displaystyle\int_0^L |u'|^2\,dt & \textit{if } u \in H^1(0,L) \\ +\infty & \textit{otherwise,} \end{cases}$$

where

$$\overline{\rho} = M\Big(\sum_{i=1}^M \frac{1}{\rho_i}\Big)^{-1}.$$

Proof Note that

$$\overline{\rho} = \min\Big\{ M \sum_{i=1}^M \rho_i z_i^2 : \ \sum_{i=1}^M z_i = 1 \Big\},$$

so that

$$\overline{\rho} z^2 = \min\Big\{ M \sum_{i=1}^{M} \rho_i z_i^2 : \sum_{i=1}^{M} z_i = z \Big\}.$$

We then immediately have

$$\sum_{i=0}^{n-1} \lambda_n \rho_i \Big(\frac{u_{i+1} - u_i}{\lambda_n} \Big)^2 \geq \sum_{i=0}^{[n/M]-1} M \lambda_n \overline{\rho} \Big(\frac{u_{M(i+1)} - u_{Mi}}{M\lambda_n} \Big)^2,$$

which gives the liminf inequality.

The limsup inequality for the function $u(x) = zx$ is obtained by choosing u_n defined by

$$u_n(x_i^n) = \overline{\rho} z \lambda_n \sum_{k=0}^{i} \frac{1}{\rho_k}.$$

□

1.8 Energies depending on second difference quotients

We consider the case of energies

$$E_n(u) = \sum_{i=1}^{n-1} \lambda_n f\Big(\frac{u_{i+1} - 2u_i + u_{i-1}}{\lambda_n^2} \Big), \tag{1.72}$$

with f convex and such that $c_1(|z|^p - 1) \leq f(z) \leq c_2(1 + |z|^p)$ $(p > 1)$.

In this case we identify the discrete function u with a function in $W^{2,p}(0, L)$. Given the values u_{i-1}, u_i, u_{i+1} we define the function u on the interval

$$I_i^n = \Big(\frac{x_{i-1}^n + x_i^n}{2}, \frac{x_{i+1}^n + x_i^n}{2} \Big) = \Big(x_i^n - \frac{\lambda_n}{2}, x_i^n + \frac{\lambda_n}{2} \Big)$$

$(i \in \{1, \dots, n-1\})$ by

$$\begin{aligned} u(t) = & \frac{u_i + u_{i-1}}{2} + \frac{u_i - u_{i-1}}{\lambda_n} \Big(t - \frac{x_{i-1}^n + x_i^n}{2} \Big) \\ & + \frac{u_{i+1} - 2u_i + u_{i-1}}{2\lambda_n^2} \Big(t - \frac{x_{i-1}^n + x_i^n}{2} \Big)^2 \end{aligned} \tag{1.73}$$

Note that

$$u'' = \frac{u_{i+1} - 2u_i + u_{i-1}}{\lambda_n^2} \qquad \text{on } I_i^n,$$

and

$$u\Big(\frac{x_{i-1}^n + x_i^n}{2} \Big) = \frac{u_i + u_{i-1}}{2}, \qquad u'\Big(\frac{x_{i-1}^n + x_i^n}{2} \Big) = \frac{u_i - u_{i-1}}{\lambda_n}$$

$$u\Big(\frac{x_i^n + x_{i+1}^n}{2}\Big) = \frac{u_{i+1}+u_i}{2}, \qquad u'\Big(\frac{x_i^n + x_{i+1}^n}{2}\Big) = \frac{u_{i+1}-u_i}{\lambda_n}.$$

Finally, we set

$$u(t) = \frac{u_1+u_0}{2} + \frac{u_1-u_0}{\lambda_n}\Big(t - \frac{\lambda_n}{2}\Big)$$

on $(0, \lambda_n/2)$ and

$$u(t) = \frac{u_n+u_{n-1}}{2} + \frac{u_n-u_{n-1}}{\lambda_n}\Big(t - L - \frac{\lambda_n}{2}\Big)$$

on $(L-(\lambda_n/2), L)$. In this way $u \in C^1(0,L)$ and u'' is piecewise constant, so that $u \in W^{2,p}(0,L)$ (actually, $u \in W^{2,\infty}(0,L)$). Moreover,

$$E_n(u) = \int_0^L f(u'')\,dt. \tag{1.74}$$

We have the following result.

Theorem 1.24 *With the identification above, the energies E_n Γ-converge as $n \to +\infty$ to the functional*

$$F(u) = \begin{cases} \displaystyle\int_{(0,l)} f(u'')\,dt & \text{if } u \in W^{2,p}(0,L) \\ +\infty & \text{otherwise} \end{cases}$$

with respect to the convergence in $L^1(0,L)$ and weak in $W^{2,p}(0,L)$.

Proof Let $u_n \to u$ in $L^1(0,L)$ and $\sup_n E_n(u_n) < +\infty$. Then we have

$$\sup_n \Big(\int_0^L (|u_n| + |u_n''|^p)\,dt\Big) < +\infty.$$

By interpolation, we deduce that $\sup_n \|u_n\|_{W^{2,p}(0,L)} < +\infty$; hence $u_n \rightharpoonup u$ weakly in $u \in W^{2,p}(0,L)$. In particular $u_n'' \rightharpoonup u''$ in $L^p(0,L)$, so that

$$F(u) = \int_{(0,l)} f(u'')\,dt \le \liminf_n \int_{(0,l)} f(u_n'')\,dt = \liminf_n E_n(u_n).$$

If $u \in C^2([0,L])$ then, upon choosing $(u_n)_i = u(x_i^n)$ we have $u_n \to u$ and

$$E_n(u_n) = \int_{(0,l)} f(u'' + o(1))\,dt,$$

so that $\lim_n E_n(u_n) = F(u)$. For a general $u \in W^{2,p}(0,L)$ it suffices to use an approximation argument. □

2

LIMIT ENERGIES ON DISCONTINUOUS FUNCTIONS: TWO EXAMPLES

In this chapter we begin dealing with energy density which do not satisfy a growth condition of polynomial type. We explicitly treat two model situations.

2.1 The Blake Zisserman model

A finite-difference scheme proposed by Blake Zisserman to treat signal reconstruction problems takes into account (beside other terms of 'lower order') energies defined on discrete functions of the form

$$E_n(u) = \sum_{i=1}^{n} \lambda_n \psi_n\Big(\frac{u_i - u_{i-1}}{\lambda_n}\Big), \tag{2.1}$$

with

$$\psi_n(z) = \min\Big\{z^2, \frac{\alpha}{\lambda_n}\Big\}, \tag{2.2}$$

for some $\alpha > 0$. An interesting interpretation of the energy density ψ_n can be given also as relative to the energy between two neighbours in an array of material points connected by springs. In this case the springs are quadratic until a threshold, after which they bear no response to traction (broken springs).

Note that the energies above do not fit in the framework of the previous chapter, as they do not satisfy a growth condition of order p from below. Note moreover that no interesting result can be obtained by taking into account the convexifications ψ_n^{**} as they are trivially 0.

In this section we will treat the limit of energies modeled on E_n above. We first define the proper convergence under which such energies are equi-coercive.

2.1.1 *Coerciveness conditions*

We examine the coerciveness conditions for sequences of (piecewise-affine interpolations of) functions (u_n) such that

$$\sup_n E_n(u_n) < +\infty. \tag{2.3}$$

For such a sequence, denote by

$$I^n = \{i \in \{1, \dots, n\} : \ |u_n(x_i^n) - u_n(x_{i-1}^n)| > \alpha\lambda_n\} \tag{2.4}$$

the set of indices such that

$$\psi_n(u') \neq (u')^2 \qquad \text{on } (x_{i-1}^n, x_i^n), \tag{2.5}$$

and by

$$S_n = \bigcup_{i \in I^n} (x_{i-1}^n, x_i^n) \tag{2.6}$$

the union of the corresponding intervals.

Note that we have

$$E_n(u_n) = \int_{(0,L)\setminus S_n} (u_n')^2\,dt + \alpha\#(I^n), \tag{2.7}$$

so that by (2.3) we deduce that

$$\sup_n \#(I^n) \leq \frac{1}{\alpha} \sup_n E_n(u_n) < +\infty. \tag{2.8}$$

Upon extracting a subsequence, we may assume then that

$$\#(I^n) = N \qquad \text{for all } n \in \mathbf{N}, \tag{2.9}$$

with N independent of n. Let $t_0^n, \ldots, t_{N+1}^n$ be points in $[0, L]$ such that $t_0^n = 0$, $t_{N+1}^n = L$, $t_{i-1}^n < t_i^n$ and

$$\{t_i^n : i = 1, \ldots, N+1\} = (\lambda_n I^n) \cup \{0, L\}. \tag{2.10}$$

Upon further extracting a subsequence we may suppose that

$$t_i^n \to t_i \in [0, L] \qquad \text{for all } i. \tag{2.11}$$

Denote the set of these limit points by

$$S = \{t_i : i = 0, \ldots, N+1\}.$$

Let $\eta > 0$ be fixed; then for n large enough we have

$$S_n = \bigcup_{i \in I^n} \Big(x_i^n - (0, \lambda_n)\Big) \subset S + (-\eta, \eta). \tag{2.12}$$

Hence, from (2.7) and (2.11) we deduce that

$$\limsup_n \int_{(0,L)\setminus(S+(-\eta,\eta))} (u_n')^2\,dt$$
$$\leq \sup_n \int_{(0,L)\setminus S_n} (u_n')^2\,dt \leq \sup_n E_n(u_n) < +\infty. \tag{2.13}$$

We deduce that for every $\eta > 0$ $u_n \in W^{1,2}((0,L)\setminus(S+(-\eta,\eta)))$ and, if for every $i = 0, \ldots, N$ we have

$$\liminf_n \Big(\text{ess-inf}\{|u_n(t)| : t \in (t_i^n + \eta, t_{i+1}^n - \eta)\}\Big) < +\infty, \tag{2.14}$$

then (u_n) is weakly precompact in $W^{1,2}(t_i^n + \eta, t_{i+1}^n - \eta)$ by Poincaré's inequality. Let u be its limit defined separately on each $(t_i^n + \eta, t_{i+1}^n - \eta)$. By the arbitrariness of η we have that u can be defined on $(0, L) \setminus S$, and hence a.e. on $(0, L)$. By this construction $u \in \mathrm{W}^{1,p}_{\mathrm{loc}}((0, L) \setminus S)$. Moreover, by (2.13) we deduce that for all $\eta > 0$

$$\int_{(0,L)\setminus(S+(-\eta,\eta))} (u')^2\,dt \le \liminf_n \int_{(0,L)\setminus(S+(-\eta,\eta))} (u_n')^2\,dt \le \sup_n E_n(u_n), \tag{2.15}$$

which gives a bound independent of η, so that by the arbitrariness of $\eta > 0$ we deduce that $u \in \mathrm{W}^{1,p}((0, L) \setminus S)$.

We now introduce the following notation.

Definition 2.1 The space P-$\mathrm{W}^{1,p}(0, L)$ of *piecewise-Sobolev functions* on $(0, L)$ is defined as the set of functions $u \in L^1(0, L)$ such that a finite set $S \subset (0, L)$ exists such that $u \in \mathrm{W}^{1,p}((0, L) \setminus S)$. The minimal such set S is called the *set of discontinuity points* of u and denoted by $S(u)$. For such u we regard the derivative $u' \in L^p(0, L)$ as defined a.e. and coinciding with its usual definition outside S.

We then have the following compactness result.

Theorem 2.2 *Let (u_n) be a sequence of functions such that $\sup_n E_n(u_n) < +\infty$ and such that (u_n) is bounded in measure. Then there exists a function $u \in P$-$W^{1,2}(0, L)$ such that $u_n \to u$ in measure. Moreover there exists a finite set S such that $u_n \rightharpoonup u$ weakly in $\mathrm{W}^{1,p}_{\mathrm{loc}}((0, L) \setminus S)$.*

Proof The proof is contained in (2.4)–(2.15) above, once we remark that boundedness in measure implies (2.14). □

2.1.2 *Limit energies for nearest-neighbour interactions*

From the reasonings above we easily deduce a first convergence result.

Theorem 2.3 *Let E_n be given by (2.1)–(2.2). Then E_n converge with respect to the convergence in measure and in $L^1(0, L)$ to the energy*

$$F(u) = \begin{cases} \displaystyle\int_0^L |u'|^2\,dt + \alpha\,\#(S(u)) & \text{if } u \in P\text{-}W^{1,2}(0, L) \\ +\infty & \text{otherwise} \end{cases} \tag{2.16}$$

in $L^1(0, L)$.

Proof Let $u_n \to u$ in measure. Then by (2.7)–(2.15) it remains to show that $\#(S(u)) \le \liminf_n \#(I^n)$. This follows immediately from the facts that $S(u) \subset S$, and that, in the notation of (2.7)–(2.15), $\#(S) \le N = \lim_n \#(I^n)$.

As for the limsup inequality, it suffices to remark that if we take $u_n = u \in P$-$\mathrm{W}^{1,\infty}(0, L)$ then for n large $E_n(u_n) \le F(u)$. For a general u we may proceed by density. □

From the lower semicontinuity properties of Γ-limits we immediately have the following corollary.

Corollary 2.4 *The functional F in* (2.16) *is lower semicontinuous with respect to the convergence in measure and in* $L^1(0, L)$.

Remark 2.5 In Theorem 2.3 we can also consider the weak*-convergence of u_n.

2.1.3 *Equivalent energies on the continuum*

The first difference that meets the eye in Theorem 2.3 from the theory developed for energy densities with polynomial growth is that we have two different parts of the energy densities ψ_n that give rise to a bulk and a jump energy, respectively. In particular we cannot simply substitute the difference quotient by a derivative, or the function ψ_n by its convexification. A continuum counterpart of E_n is immediately obtained if we consider a different identification, other than the piecewise-affine one, for a discrete functions $u : \{x_0^n, \dots, x_n^n\} \to \mathbf{R}$: using the notation

$$I^n(u) = \{i \in \{1, \dots, n\} : \ |u(x_i^n) - u(x_{i-1}^n)| > \alpha\lambda_n\} \tag{2.17}$$

we may extend u to the whole $(0, L)$ by setting

$$u(t) = \begin{cases} u_{i-1} + \dfrac{u_i - u_{i-1}}{\lambda_n}(t - x_i^n) & \text{if } t \in (x_{i-1}^n, x_i^n),\ i \notin I^n(u) \\ u_{i-1} & \text{if } x_{i-1}^n \le x \le x_{i-1}^n + \frac{\lambda_n}{2},\ i \in I^n(u) \\ u_i & \text{if } x_i^n - \frac{\lambda_n}{2} \le x \le x_i^n,\ i \in I^n(u). \end{cases} \tag{2.18}$$

Note that such extension of u belongs to $P\text{-}W^{1,2}(0, L)$,

$$S(u) = \Big\{x_i^n - \frac{\lambda_n}{2} : i \in I^n(u)\Big\}, \tag{2.19}$$

and we have the identification

$$E_n(u) = F(u). \tag{2.20}$$

In this sense, F is the continuum counterpart of each E_n.

2.1.4 *Limit energies for long-range interactions*

We now investigate the limit of superpositions of energies of the form (2.1). Let (ρ_j) and (α_j) be given sequences of non-negative numbers. We suppose that if $\alpha_j\rho_j = 0$ then $\alpha_j = \rho_j = 0$. We define the energy densities

$$\psi_n^j(z) = \min\Big\{\rho_j z^2, \frac{\alpha_j}{\lambda_n}\Big\} \tag{2.21}$$

and the energies

$$E_n(u) = \sum_{j=1}^{n}\sum_{i=0}^{n-j} \lambda_n \psi_n^j\Big(\frac{u_{i+j} - u_i}{j\lambda_n}\Big) \tag{2.22}$$

Theorem 2.6 *Suppose that*

$$\rho_1 > 0, \qquad \alpha_1 > 0. \tag{2.23}$$

Let $\rho, \alpha \in (0, +\infty]$ be defined by

$$\rho = \sum_{j=1}^{\infty} \rho_j, \qquad \alpha = \sum_{j=1}^{\infty} j\alpha_j. \tag{2.24}$$

Then the energies E_n Γ-converge with respect to the convergence in measure and in $L^1(0, L)$ to the functional F given by

$$F(u) = \begin{cases} \rho \int_0^L |u'|^2\,dt + \alpha\,\#(S(u)) & \textit{if } u \in P\text{-}W^{1,2}(0, L) \\ +\infty & \textit{otherwise} \end{cases} \tag{2.25}$$

in $L^1(0, L)$, where it is understood that if $\alpha = +\infty$ then $F(u) = +\infty$ if $S(u) \neq \emptyset$, and that if $\rho = +\infty$ then $F(u) = +\infty$ if $u' \neq 0$ a.e.

Proof Preliminarily note that by (2.23) we have that the Γ-limit (exists and) is $+\infty$ outside P-$W^{1,2}(0, L)$.

With fixed $K \in \mathbf{N}$ consider for $n \geq K$ the energies

$$E_n^K(u) = \sum_{j=1}^{K} \sum_{i=0}^{n-j} \lambda_n \psi_n^j \Big(\frac{u_{i+j} - u_i}{j\lambda_n}\Big), \tag{2.26}$$

so that

$$E_n^K(u) \leq E_n(u). \tag{2.27}$$

For all $j = 1, \ldots, K$ and $k = 0, \ldots, j-1$ let

$$E_n^{j,k}(u) = \sum_{i=0}^{[n/j]-2} \lambda_n \psi_n^j \Big(\frac{u_{k+(i+1)j} - u_{k+ij}}{j\lambda_n}\Big), \tag{2.28}$$

so that

$$E_n^K(u) \geq \sum_{j=1}^{K} \sum_{k=0}^{j-1} E_n^{j,k}(u). \tag{2.29}$$

Note that proceeding as in the proof of Theorem 2.3 by interpreting $E_n^{j,k}$ as an energy on the lattice $j\lambda_n\mathbf{Z} + k\lambda_n$, we easily get that $E_n^{j,k}$ Γ-converge as $n \to +\infty$ to the functional F^j (independent of k) given by

$$F^j(u) = \begin{cases} \dfrac{\rho_j}{j} \int_0^L |u'|^2\,dt + \alpha_j\,\#(S(u)) & \text{if } u \in P\text{-}W^{1,2}(0, L) \\ +\infty & \text{otherwise.} \end{cases} \tag{2.30}$$

We then immediately get the following liminf inequality: if $u_n \to u$ then

$$\begin{aligned}\liminf_n E_n(u_n) &\geq \liminf_n E_n^K(u_n)\\ &\geq \sum_{j=1}^K \sum_{k=0}^{j-1} \liminf_n E_n^{j,k}(u_n)\\ &\geq \sum_{j=1}^K jF^j(u) = \sum_{j=1}^K \int_0^L |u'|^2\,dt + \sum_{j=1}^K j\alpha_j \#(S(u)).\end{aligned}$$

The desired inequality is obtained by letting $K \to +\infty$, and using the Monotone Convergence Theorem.

Let now $u \in P\text{-}W^{1,2}(0,L)$ be such that $F(u) < +\infty$. Consider first the case $\alpha < +\infty$, $\rho < +\infty$. By a density argument it suffices to consider the case $u \in P\text{-}\mathrm{W}^{1,\infty}(0,L)$. In this case we can choose $u_n = u$, and note that

$$\limsup_n E_n(u_n) \leq \sum_{j=1}^K j \lim_n F^j(u_n) + c \sum_{j=K+1}^{\infty} \Big(\rho_j \|u'\|_\infty^2 + j\alpha_j \#(S(u))\Big).$$

In the case when $\alpha = +\infty$ it suffices to compare with the convex case as $\psi_n^j(z) \leq \rho_j z^2$. When $\rho = +\infty$ F is finite only on piecewise-constant u, for which we take $u_n = u$ and the computation is straightforward. □

2.1.5 *Boundary value problems*

In contrast to what happened to functionals with limits defined on Sobolev spaces, the coerciveness conditions at our disposal do not guarantee that minimizers satisfying some boundary conditions converge to a minimizer satisfying the same boundary condition. We have thus to relax the notion of boundary values.

We consider boundary value problems given in two ways.

(I) **Interaction at the boundary**: we fix two values U_0 and U_L and add to the energy E_n the constraint $u(0) = U_0$, $u(L) = U_L$;

(II) **Interaction through the boundary**: we fix $\phi : \mathbf{R} \to \mathbf{R}$ and add to E_n the 'boundary value term'

$$\begin{aligned}B_n(u) = &\sum_{j=1}^{n+K_n} \sum_{i=\max\{-j,-K_n\}}^{-1} \lambda_n \psi_n^j\Big(\frac{u(x_{i+j}^n) - \phi(x_i^n)}{j\lambda_n}\Big)\\ &+ \sum_{j=1}^{n+K_n} \sum_{i=n+1}^{\min\{n+j,n+K_n\}} \lambda_n \psi_n^j\Big(\frac{\phi(x_i^n) - u(x_{i-j}^n)}{j\lambda_n}\Big),\end{aligned}$$

which corresponds to setting $u = \phi$ outside $[0, L]$ and to considering the energy of this extension on an enlarged interval with an addition of a 'layer' of size $K_n\lambda_n$ on both sides of the interval.

We first treat the case (II). For the sake of simplicity we consider the case when $\rho_j = 0$ for $j > K$, and we choose $K_n = K$. In this case our energy $E_n + B_n$ can be written as

$$\tilde{E}_n(u) = \sum_{j=1}^{K} \sum_{i=-j}^{n} \lambda_n \psi_n^j \Big(\frac{u_{i+j} - u_i}{j\lambda_n}\Big), \tag{2.31}$$

with the constraint

$$u_i = \phi(x_n^i) \qquad \text{for } i \in \{-K, \dots, -1\} \cup \{n+1, \dots, n+K\} \tag{2.32}$$

Theorem 2.7 *Let $\phi : \mathbf{R} \to \mathbf{R}$ be a continuous function. Let $\tilde{E}_n$ be given by (2.31)–(2.32). Then $\tilde{E}_n$ Γ- converges to the functional $\tilde{F}$ given by*

$$\tilde{F}(u) = \begin{cases} \rho \displaystyle\int_0^L |u'|^2\,dt + \alpha\#(\{x \in [0,L] : u(x+) \neq u(x-)\}) & \\ & \text{if } u \in P\text{-}W^{1,2}(0,L) \\ +\infty & \text{otherwise,} \end{cases} \tag{2.33}$$

where

$$\rho = \sum_{j=1}^{K} \rho_j, \qquad \alpha = \sum_{j=1}^{K} \alpha_j,$$

and we have set

$$u(0-) = \phi(0), \qquad u(L+) = \phi(L). \tag{2.34}$$

Proof Let $E_n(v, (-L, 2L))$ be defined by

$$E_n(v, (-L, 2L)) = \sum_{j=1}^{K} \sum_{i=-n}^{2n-j} \lambda_n \psi_n^j \Big(\frac{v_{i+j} - v_i}{j\lambda_n}\Big).$$

the choice of the interval $(-L, 2L)$ has been done only for convenience of notation; indeed any open interval containing $[0, L]$ would do. By the previous results $E_n(\cdot, (-L, 2L))$ Γ-converges to the functional $F(\cdot, (-L, 2L))$ with domain $P\text{-}W^{1,2}(-L, 2L)$ and defined there by

$$F(v, (-L, 2L)) = \rho \int_{-L}^{2L} |v'|^2\,dt + \alpha\#(S(v)).$$

Let $u_n \to u$ in measure on $(0, L)$. Let v_n be defined by

$$v_n(x_i^n) = \begin{cases} \phi(0) & \text{if } i < 0 \\ u_n(x_i^n) & \text{if } 0 \le i \le n \\ \phi(L) & \text{if } i > n, \end{cases}$$

and similarly define also v. Note that $v_n \to v$ in measure on $(-L, 2L)$. We then have

$$\liminf_n \tilde{E}_n(u_n) = \liminf_n E_n(v_n, (-L, 2L)) \geq F(v, (-L, 2L)) = F(u).$$

To obtain the limsup inequality it suffices to take $u_n = u$. □

In the case (I) we treat arbitrarily long-range interactions.

Theorem 2.8 *Let E_n be given by (2.22) and let $\tilde{E}_n$ be given by*

$$\tilde{E}_n(u) = \begin{cases} E_n(u) & \text{if } u(0) = U_0 \text{ and } u(L) = U_L \\ +\infty & \text{otherwise.} \end{cases} \tag{2.35}$$

Then $\tilde{E}_n$ Γ- converges to the functional $\tilde{F}$ given by

$$\tilde{F}(u) = \begin{cases} \rho \int_0^L |u'|^2\,dt + \alpha\#(S(u)) + \alpha_0\#(\{x \in \{0, L\} : u(x+) \neq u(x-)\}) \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } u \in P\text{-}W^{1,2}(0, L) \\ +\infty \qquad\qquad\qquad\qquad\qquad\quad \text{otherwise,} \end{cases} \tag{2.36}$$

where

$$\rho = \sum_{j=1}^{\infty} \rho_j, \qquad \alpha = \sum_{j=1}^{\infty} j\alpha_j, \qquad \alpha_0 = \sum_{j=1}^{\infty} \alpha_j, \tag{2.37}$$

and we have set

$$u(0-) = U_0, \qquad u(L+) = U_L. \tag{2.38}$$

Proof We begin with the case $j = 1$ and with a boundary condition on only one side (e.g. at 0). Consider the functional

$$E_n^0(u) = \begin{cases} E_n(u) & \text{if } u(0) = U_0 \\ +\infty & \text{otherwise.} \end{cases}$$

As in the proof of Theorem 2.7 we can write

$$E_n^0(u) = E_n(v, (-L, L)),$$

where

$$v(x_i^n) = \begin{cases} U_0 & \text{if } i \leq 0 \\ u(x_i^n) & \text{if } 0 < i \leq n \end{cases}$$

and

$$E_n(v, (-L, L)) = \sum_{i=-n}^{n-1} \lambda_n \psi_n^1\Big(\frac{u_{i+1} - u_i}{\lambda_n}\Big).$$

If $u_n \to u$ we then obtain

$$\liminf_n E_n(u_n) \geq \rho_1 \int_0^L |u'|^2\, dt + \alpha_1 \#(S(u)) + \alpha_1(1 - \chi_0(u(0+) - U_0)).$$

The limsup inequality is immediately obtained by taking $u_n = u$.

In the same way we treat the boundary condition at L and the boundary conditions at both sides.

With fixed K can repeat the same reasoning as above for all $j \in \{1, \ldots, K\}$ such that $\alpha_j \rho_j \neq 0$ (otherwise the limit is trivial) and obtain that the Γ-limit of

$$E_n^{0,j}(u) = \begin{cases} \displaystyle\sum_{i=0}^{n-j} \lambda_n \psi_n^j\Big(\frac{u_{i+j} - u_i}{\lambda_n}\Big) & \text{if } u_0 = U_0 \\ +\infty & \text{otherwise} \end{cases}$$

as $n \to +\infty$ is given by

$$\rho_j \int_0^L |u'|^2\, dt + j\alpha_j \#(S(u)) + \alpha_j(1 - \chi_0(u(0+) - U_0)).$$

Symmetrically we can treat the case $u_n = U_L$. The case of boundary condition on both sides gives that the Γ-limit of

$$E_n^{0,L,j}(u) = \begin{cases} \displaystyle\sum_{i=0}^{n-j} \lambda_n \psi_n^j\Big(\frac{u_{i+j} - u_i}{\lambda_n}\Big) & \text{if } u(0) = U_0 \text{ and } u_n = U_L \\ +\infty & \text{otherwise} \end{cases}$$

is

$$\rho_j \int_0^L |u'|^2\, dt + j\alpha_j \#(S(u)) \\ + \alpha_j(1 - \chi_0(u(0+) - U_0)) + \alpha_j(1 - \chi_0(u(L-) - U_L)).$$

Summing up these considerations we obtain that for all K

$$\Gamma\text{-}\liminf_n \tilde{E}_n \geq \rho^K \int_0^L |u'|^2\, dt \\ + \alpha^K \#(S(u)) + \alpha_0^K \#(\{x \in \{0, L\} : u(x+) \neq u(x-)\}),$$

where $\rho^K = \sum_{j=1}^K \rho_j$, $\alpha^K = \sum_{j=1}^K j\alpha_j$ and $\alpha_0^K = \sum_{j=1}^K \alpha_j$. The liminf inequality is obtained by taking the supremum in K.

The upper inequality is obtained by taking $u_n = u$. □

Remark 2.9 Note that we may have $\alpha = +\infty$ but $\alpha_0 < +\infty$, in which case $\tilde{F}$ is finite only on $W^{1,2}(0, L)$ but may be finite also on functions not matching the boundary conditions.

2.1.6 *Homogenization*

We only treat the case of nearest-neighbour interactions. Let $i \mapsto \rho_i$ and $i \mapsto \alpha_i$ define M-periodic functions $\mathbf{Z} \to \mathbf{R}$:

$$\rho_{i+M} = \rho_i, \qquad \alpha_{i+M} = \alpha_i \qquad \text{for all } i.$$

The energies we consider take the form

$$E_n(u) = \sum_{i=0}^{n-1} \min\Big\{\lambda_n \rho_i \Big(\frac{u_{i+1} - u_i}{\lambda_n}\Big)^2, \alpha_i\Big\}. \tag{2.39}$$

Theorem 2.10 *The energies E_n Γ-converge to the energy defined by*

$$F(u) = \begin{cases} \overline{\rho} \displaystyle\int_0^L |u'|^2\, dt + \overline{\alpha}\#(S(u)) & \text{if } u \in P\text{-}W^{1,2}(0, L) \\ +\infty & \text{otherwise,} \end{cases}$$

where

$$\overline{\rho} = M\Big(\sum_{i=1}^{M} \frac{1}{\rho_i}\Big)^{-1}, \qquad \overline{\alpha} = \min_i \alpha_i.$$

Proof By following the proof of Theorem 1.23 we immediately obtain the liminf inequality.

In order to construct a recovery sequence for the Γ-limsup, let $u \in P\text{-}W^{1,2}(0, L)$ and define $v(t) = u(0+) + \int_0^t u'(s)\, ds$. Let v_n be a recovery sequence for the Γ-limit in Theorem 1.23 computed at v. Let $k \in \{0, \dots, M-1\}$ be such that $\overline{\alpha} = \alpha_k$. Then for all $t \in S(u)$ let $j_n(t) \equiv k \bmod M$ be such that $|x^n_{j_n(t)} - t| \le M\lambda_n$. Then the functions

$$u_n(x_i^n) = v_n(x_i^n) + \sum_{t \in S(u):\ j_n(t) \le i} (u(t+) - u(t-))$$

define a recovery sequence for u. □

2.1.7 *Non-local limits*

For all $n \in \mathbf{N}$ let $\rho_n : j\mathbf{Z} \to [0, +\infty)$. We consider the long-range discrete energies of Blake Zisserman type

$$E_n(u) = \sum_{\substack{x,y \in \lambda_n \mathbf{Z} \cap [0,L] \\ x \neq y}} \rho_n(x - y)\, \Psi_n\Big(\frac{u(x) - u(y)}{x - y}\Big) \tag{2.40}$$

defined for $u : \lambda_n \mathbf{Z} \to \mathbf{R}$, where

$$\Psi_n(z) = \min\{\lambda_n z^2, 1\}.$$

We make the same assumptions on (ρ_n) as in Section 1.7.2.

Theorem 2.11 *If conditions* (H1) *and* (H2) *in Section* 1.7.2 *hold, then there exist a subsequence (not relabelled), a Radon measure* μ *on* $\mathbf{R}^2$, *a constant* $c_1 > 0$ *and an even subadditive and lower semicontinuous function* $\varphi : \mathbf{R} \to [0, +\infty]$ *such that the energies* E_n Γ*-converge to the energy* F *defined on* $\mathrm{L}^1(0, L)$ *by*

$$F(u) = \begin{cases} c_1 \displaystyle\int_{(0,L)} |u'|^2\,dt + \sum_{S(u)} \varphi([u]) + \int_{(0,L)^2} \Big(\frac{u(x)-u(y)}{x-y}\Big)^2 d\mu(x,y) \\ \qquad\qquad\qquad\qquad\qquad\qquad \textit{if } u \textit{ is piecewise } W^{1,2} \textit{ on } [0,L] \\ +\infty \qquad\qquad\qquad\qquad\qquad\quad \textit{otherwise,} \end{cases} \tag{2.41}$$

where $S(u)$ *denotes the set of discontinuity points for* u *and* $[u](t) = u(t+) - u(t-)$ *is the jump of* u *at* t. *The measure* μ *and* c_1 *are given by* (1.68) *and* (1.67), *respectively, and the function* φ *is given by the* discrete phase-transition energy density formula

$$\varphi(z) = \liminf_{m\to+\infty} \inf_{|w|<|z|} \lim_n \min\Big\{ \sum_{\substack{j,k\in\mathbf{Z},\ j\neq k \\ -2/m\lambda_n \le j,k \le 2/m\lambda_n}} \rho_n(\lambda_n(j-k))\Psi_n\Big(\frac{u(j)-u(k)}{\lambda_n(j-k)}\Big) \ : $$
$$u : \mathbf{Z} \to \mathbf{R},\ u(j) = 0 \textit{ if } j < -\frac{1}{m\lambda_n},\ u(j) = w \textit{ if } j > \frac{1}{m\lambda_n}\Big\} \tag{2.42}$$

for $z \in \mathbf{R}$.

Remark 2.12 (i) Since φ is subadditive, and it is also non decreasing on $[0, +\infty)$ and even, we have that either it is finite everywhere or $\varphi(z) = +\infty$ for all $z \neq 0$. In the latter case jumps are prohibited and the domain of F is indeed $W^{1,2}(0, L)$.

(ii) We will show below that the function φ may be not constant, in contrast with the case when $\rho_n(z) = \rho(z/\lambda_n)$ for a fixed ρ.

Proof With fixed $m, n \in \mathbf{N}$ the minimum value in (2.42) defines an even function of w which is non-decreasing on $[0, +\infty)$; hence, by Helly's Theorem there exists a sequence (not relabeled) $\{\lambda_n\}$ such that these minimum values converge for all w and for all m. Hence, we can assume, upon passing to this subsequence $\{\lambda_n\}$, that the function φ is well defined. Upon passing to a further subsequence we may also assume that the measures μ_n in Remark 1.21 converge to μ_0. Then, μ and c_1 given by (1.68) and (1.67) are well defined as well. Hence, it suffices to prove the representation for the Γ-limit along this sequence, since the subadditivity and lower semicontinuity of φ are necessary conditions for the lower semicontinuity of F.

We begin by proving the liminf inequality. Let $u_n \to u$ in $\mathrm{L}^1(0, L)$ be such that $\sup_n E_n(u_n) < +\infty$. By hypothesis (H1), if we set

$$S^n = \{x \in \lambda_n\mathbf{Z} :\ |u(x+\lambda_n) - u(x)|^2 > 1/\lambda_n\},$$

then $\#S^n$ is equibounded, and, upon extracting a subsequence, we can suppose that $S^n = \{x_j^n : j = 1, \dots, N\}$ with N independent of n $x_1^n < x_2^n < \dots < x_N^n$ and $x_j^n \to t_j$ for all j. Set $S = \{t_j\} \subset [a, b]$. If $\{x_{M_1}^n\}, \dots, \{x_{M_2}^n\}$ are the sequences converging to $t \in S$ then $u_n(x_{M_1}^n) \to u(t-)$ and $u_n(x_{M_2}^n + \lambda_n) \to u(t+)$. Furthermore, the sequence u_n converges locally weakly in $W^{1,2}((0, L) \setminus S)$.

For all $\eta > 0$ let $S_\eta = \{t \in \mathbf{R} : \operatorname{dist}(t, S) < \eta\}$; set also $\Delta_\eta = \{(x, y) \in \mathbf{R}^2 : |x - y| > \eta\}$. Note that the convergence

$$\frac{u_n(x) - u_n(y)}{x - y} \longrightarrow \frac{u(x) - u(y)}{x - y}$$

as $n \to \infty$ is uniform on $(0, L)^2 \setminus (S_\eta^2 \cup \Delta_\eta)$.

With fixed $m \in \mathbf{N}$, we have the inequality

$$\begin{aligned} E_n(u_n) \geq & \sum_{\substack{x,y \in \lambda_n \mathbf{Z} \cap [0,L] \cap S_{4/m} \\ |x-y| \leq 4/m,\ x \neq y}} \rho_n(x - y) \Psi_n\Big(\frac{u_n(x) - u_n(y)}{x - y}\Big) \\ & + \sum_{\substack{x,y \in \lambda_n \mathbf{Z} \cap [0,L] \setminus S_{4/m} \\ |x-y| \leq 4/m,\, x \neq y}} \rho_n(x - y) \Psi_n\Big(\frac{u_n(x) - u_n(y)}{x - y}\Big) \\ & + \sum_{\substack{x,y \in \lambda_n \mathbf{Z} \cap [0,L] \\ |x-y| > 4/m}} \rho_n(x - y) \Psi_n\Big(\frac{u_n(x) - u_n(y)}{x - y}\Big) \\ =: & \, I_n^1(u_n) + I_n^2(u_n) + I_n^3(u_n). \end{aligned} \tag{2.43}$$

The terms $I_n^2(u_n)$ and $I_n^3(u_n)$ can be dealt with as in Section 1.7.2. We now deal with $I_n^1(u_n)$. We first note that

$$I_n^1(u_n) \geq \sum_{t \in S(u)} \sum_{\substack{x,y \in \lambda_n \mathbf{Z} \cap [t-(2/m), t+(2/m)] \\ x \neq y}} \rho_n(x - y) \Psi_n\Big(\frac{u_n(x) - u_n(y)}{x - y}\Big) \tag{2.44}$$

We use the notation introduced above for the sets S^n and S: let $t_j \in S(u)$ with corresponding sequences $\{x_{M_1}^n\}, \dots, \{x_{M_2}^n\}$ converging to t_j. We can suppose, up to a translation and reflection argument, that $[u](t_j) > 0$, that

$$\max\{u_n(x) : x \in \lambda_n \mathbf{Z},\ t_j - (2/m) \leq x \leq x_{M_1}^n\} = 0$$

and that

$$\min\{u_n(x) : x \in \lambda_n \mathbf{Z},\ x_{M_2}^n + \lambda_n \leq x \leq t_j + (2/m)\} = z_n,$$

with $z_n \to [u](t_j)$. We then have

$$\sum_{\substack{x,y\in\lambda_n\mathbf{Z}\cap[t_j-(2/m),t_j+(2/m)]\\ x\neq y}} \rho_n(x-y)\Psi_n\Big(\frac{u_n(x)-u_n(y)}{x-y}\Big)$$

$$\geq \min\Big\{ \sum_{\substack{x,y\in\lambda_n\mathbf{Z}\cap[t_j-(2/m),t_j+(2/m)]\\ x\neq y}} \rho_n(x-y)\Psi_n\Big(\frac{v(x)-v(y)}{x-y}\Big) :$$

$$v(x^n_{M_1})=u_n(x^n_{M_1}),\ v(x^n_{M_2}+\lambda_n)=u_n(x^n_{M_2}+\lambda_n)\Big\}$$

$$\geq \min\Big\{ \sum_{\substack{x,y\in\lambda_n\mathbf{Z}\cap[t_j-(2/m),t_j+(2/m)]\\ x\neq y}} \rho_n(x-y)\Psi_n\Big(\frac{v(x)-v(y)}{x-y}\Big) :$$

$$v(x)=0 \text{ if } x\leq x^n_{M_1},\ v(x)=z_n \text{ if } x\geq x^n_{M_2}+\lambda_n\Big\}$$

$$\geq \min\Big\{ \sum_{\substack{x,y\in\lambda_n\mathbf{Z}\cap[t_j-(2/m),t_j+(2/m)]\\ x\neq y}} \rho_n(x-y)\Psi_n\Big(\frac{v(x)-v(y)}{x-y}\Big) :$$

$$v(x)=0 \text{ if } t_j-\frac{2}{m}\leq x\leq t_j-\frac{1}{m},\ v(x)=z_n \text{ if } t_j+\frac{1}{m}\leq x\leq t_j+\frac{2}{m}\Big\}$$

$$= \min\Big\{ \sum_{\substack{j,k\in\mathbf{Z}\cap[-2/(m\lambda_n),2/(m\lambda_n)]\\ j\neq k}} \rho_n(\lambda_n(j-k))\Psi_n\Big(\frac{v(j)-v(k)}{\lambda_n(j-k)}\Big) :$$

$$v(j)=0 \text{ if } -\frac{2}{m\lambda_n}\leq j\leq -\frac{1}{m\lambda_n},\ v(j)=z_n \text{ if } \frac{1}{m\lambda_n}\leq j\leq \frac{2}{m\lambda_n}\Big\}. \quad (2.45)$$

Note that we have used the fact that Ψ_n in non decreasing on $(0,+\infty)$ so that our functionals decrease by truncation (namely, when we substitute v by $(v\vee 0)\wedge z_n$). By taking (2.42) into account and summing up for $t_j\in S(u)$, we obtain

$$\liminf_n I^1_n(u_n)\geq \sum_{t\in S(u)}\varphi([u](t))+o(1) \quad (2.46)$$

as $m\to+\infty$.

By summing up this inequality to those obtained in Section 1.7.2 and letting $m\to+\infty$ we eventually get

$$\liminf_n E_n(u_n)\geq c_1\int_{(0,L)}|u'|^2\,dt+\sum_{S(u)}\varphi([u])$$
$$+\int_{(0,L)^2}\Big(\frac{u(x)-u(y)}{x-y}\Big)^2 d\mu(x,y).$$

We now prove the limsup inequality. It suffices to show it for piecewise-affine functions, since this set is strongly dense in the space of piecewise $W^{1,2}$ functions.

We explicitly treat the case when $(0, L)$ is replaced by $(-1, 1)$ and

$$u(t) = \begin{cases} \alpha t & \text{if } t < 0 \\ \beta t + z & \text{if } t > 0 \end{cases}$$

only, as the general case easily follows by repeating the construction we propose locally in the neighbourhood of each point in $S(u)$. It is not restrictive to suppose that $z > 0$, by a reflection argument, and that $\varphi(z) < +\infty$, otherwise there is nothing to prove.

Let $\eta > 0$, let $m \in \mathbf{N}$ with $0 < 1/m < \eta$ and let $z - (1/m) < z_m < z$ be such that

$$\varphi(z) \geq \lim_n \min\Big\{ \sum_{x,y\in\mathbf{Z}, -2/(m\lambda_n)\leq j,k\leq 2/(m\lambda_n)} \rho_n(\lambda_n(j-k))\Psi_n\Big(\frac{u(j)-u(k)}{\lambda_n(j-k)}\Big) \ : \\ u : \mathbf{Z}\to\mathbf{R},\ u(j) = 0 \text{ if } j < -\frac{1}{m\lambda_n},\ u(j) = z_m \text{ if } j > \frac{1}{m\lambda_n}\Big\} - \eta. \tag{2.47}$$

Then there exist functions $v_n^m : \lambda_n\mathbf{Z} \to \mathbf{R}$ such that $v_n^m(x) = 0$ for $x < -1/m$, $v_n^m(x) = z_m$ for $x > T$, $0 \leq v_n^m \leq z_m$ and

$$\lim_n \sum_{\substack{x,y\in\lambda_n\mathbf{Z} \\ -(2/m)\leq x,y\leq(2/m)}} \rho_n(x-y)\,\Psi_n\Big(\frac{v_n^m(x)-v_n^m(y)}{x-y}\Big) \leq \varphi(z) + \eta.$$

We set

$$u_n^m(t) = \begin{cases} u(t+(2/m)) & \text{if } t < -2/m \\ v_n^m(t) & \text{if } -2/m \leq t \leq 2/m \\ u(t-(2/m)) & \text{if } t > 2/m. \end{cases}$$

Note that $u_n^m \to u^m$ in $\mathrm{L}^1((-1,1)\setminus[-1/m, 1/m])$ as $n \to \infty$, where

$$u^m(t) = \begin{cases} u(t+(2/m)) & \text{if } t < -2/m \\ 0 & \text{if } -2/m \leq -1/m \\ z & \text{if } 1/m < t \leq 2/m \\ u(t-(2/m)) & \text{if } t > 2/m. \end{cases}$$

We can then easily estimate

$$\begin{aligned} &\limsup_n E_n(u_n^m) \\ &\leq \limsup_n \sum_{\substack{x,y\in\lambda_n\mathbf{Z}, x\neq y \\ -2/m\leq x,y\leq 2/m}} \rho_n(x-y)\,\Psi_n\Big(\frac{v_n^m(x)-v_n^m(y)}{x-y}\Big) \\ &\quad + \limsup_n \int_{(0,L)^2\setminus\Delta_{2/m}} \rho_n(x-y)\,\frac{1}{\lambda_n}\Psi_n\Big(\frac{u_n^m(x)-u_n^m(y)}{x-y}\Big)\,d\mu_n \end{aligned}$$

$$
\begin{aligned}
&+\limsup_n \sum_{x,y\in\lambda_n\mathbf{Z}\cap[0,L],\ x,y<-1/m,\ |x-y|\le 2/m} \rho_n(x-y)\,\Psi_n\Big(\frac{u_n^m(x)-u_n^m(y)}{x-y}\Big)\\
&+\limsup_n \sum_{x,y\in\lambda_n\mathbf{Z}\cap[0,L],\ x,y>1/m,\ |x-y|\le 2/m} \rho_n(x-y)\,\Psi_n\Big(\frac{u_n^m(x)-u_n^m(y)}{x-y}\Big)\\
&\le \varphi(z)+\eta+\int_{(0,L)^2}\Big(\frac{u^m(x)-u^m(y)}{x-y}\Big)^2 d\mu + c_1\int_{(0,L)}|u'|^2\,dt + o(1)
\end{aligned}
$$

as $m\to+\infty$. Note that we have used the fact that by (1.66) the limit measure μ does not charge $\partial(0,L)^2$. By choosing $m=m(\lambda_n)$ with $m(\lambda_n)\to+\infty$ as $n\to\infty$, and setting $u_n=u_n^{m(\lambda_n)}$ we obtain the desired inequality. □

In the following examples for simplicity we drop the hypothesis that ρ_n is even.

Example 2.13 The function φ is not always constant. As an example, take

$$
\rho_n(z)=\begin{cases}1 & \text{if } z=\lambda_n\\ \sqrt{\lambda_n} & \text{if } z=\lambda_n[1/\sqrt{\lambda_n}]\\ 0 & \text{otherwise.}\end{cases}
$$

Then it can be easily seen that the minimum for the problem defining φ is achieved on the function $v=z\chi_{(0,+\infty)}$, which gives

$$
\varphi(z)=\min\{1+z^2,2\}.
$$

Note that in this case the Γ-limit is

$$
\int_{(0,L)}|u'|^2\,dt+\sum_{S(u)}\varphi([u]),
$$

which is local, but not with φ constant.

Example 2.14 If we take

$$
\rho_n(z)=\begin{cases}1 & \text{if } z=\lambda_n\\ 4\sqrt{\lambda_n} & \text{if } z=\lambda_n[1/\sqrt{\lambda_n}]\\ 0 & \text{otherwise}\end{cases}
$$

then by using the (discretization of) $v=z\chi_{(0,+\infty)}$ as a test function we deduce the estimate

$$
\varphi(z)\le\min\{1+4z^2,5\}.
$$

Since the right hand side is not subadditive, which is a necessary condition for lower semicontinuity, we deduce that the minimum in the definition of φ is obtained by using more than one 'discontinuity'.

Remark 2.15 By the density of the sums of Dirac deltas in the space of Radon measures on the real line, in the limit functional we may obtain any measure μ satisfying the invariance property

$$\mu(A) = \mu(A + t(e_1 + e_2))$$

for all Borel set A and $t \in \mathbf{R}$.

Remark 2.16 In the formula defining φ we cannot substitute the limit of minimum problems on $[-2/(m\lambda_n), 2/(m\lambda_n)]$ by a transition problem on the whole discrete line. In fact, if we take

$$\rho_n(x) = \begin{cases} 1 & \text{if } x = \lambda_n \\ 1 & \text{if } x = \lambda_n[1/\lambda_n] \\ 0 & \text{otherwise,} \end{cases}$$

then the two results are different.

Example 2.17 By again taking ρ_n as in the previous remark, we check that in this case $\mu = (1/\sqrt{2})\mathcal{H}^1 \llcorner (r_1 \cup r_{-1})$, where $r_i = \{x - y = i\}$.

2.2 Lennard Jones potentials

We now consider a function $J : \mathbf{R} \to \mathbf{R} \cup \{+\infty\}$ modeling inter-atomic interactions, with the properties

(i) $J(z) = +\infty$ if $z \leq 0$;
(ii) J is smooth on $(0, +\infty)$;
(iii) $\lim_{z\to 0} J(z) = +\infty$.
(iv) J is strictly convex on $(0, T)$;
(v) J is strictly concave on $(T, +\infty)$;
(vi) $\lim_{z\to+\infty} J(z) = 0$.

Our assumptions are modeled on

$$J(z) = \frac{k_1}{z^{12}} - \frac{k_2}{z^6} \tag{2.48}$$

for $z > 0$ All these conditions can be relaxed, and we refer to the general treatment in the next chapter for weaker assumptions.

Note that hypotheses (ii)–(vi) imply that there exists a unique minimum point, which we denote by $M \in (0, T)$, and that $\min J < 0$.

The energy we will consider are, with fixed $K \geq 1$,

$$E_n(u) = \sum_{j=1}^{K} \sum_{i=0}^{n-j} \lambda_n J\Big(\frac{u_{i+j} - u_i}{\lambda_n}\Big). \tag{2.49}$$

Note the scaling in the argument of J; in terms of the general form considered in the previous chapter, we have $\psi_n^j(z) = J(jz)$.

2.2.1 *Coerciveness conditions*

Note that $E_n(u)$ is finite only if u is strictly increasing; hence, we can use the strong compactness properties of increasing functions. In particular, if (u_n) is a sequence of functions locally equi-bounded on $(0, L)$ then there exists a subsequence converging in $L^1_{\rm loc}(0, L)$, and if all functions are equi-bounded (e.g., if they satisfy some fixed boundary conditions) then there exists a subsequence converging in $L^1(0, L)$ (actually, in $L^p(0, L)$ for all $p < \infty$). Note moreover that, by Helly's Theorem, upon passing to a further subsequence we can obtain convergence *everywhere* on $(0, L)$.

2.2.2 *Nearest-neighbour interactions*

We begin by treating the case $K = 1$; i.e.,

$$E_n(u) = \sum_{i=1}^{n} \lambda_n J\Big(\frac{u_i - u_{i-1}}{\lambda_n}\Big). \tag{2.50}$$

It is easily seen that the Γ-limit is finite on *all* increasing functions. However, deferring the general treatment to the next chapter, we characterize the limit only on $P\text{-}{\rm W}^{1,1}(0, L)$.

Theorem 2.18 *The energies E_n Γ-converge on $P\text{-}{\rm W}^{1,1}(0, L)$ with respect to the $L^1(0, L)$ convergence, to the functional F defined by*

$$F(u) = \begin{cases} \displaystyle\int_{(0,L)} \psi(u')\, dt & \text{if } u(t+) > u(t-) \text{ on } S(u) \\ +\infty & \text{otherwise} \end{cases} \tag{2.51}$$

on $P\text{-}{\rm W}^{1,1}(0, L)$, where

$$\psi(z) = J^{**}(z) = \begin{cases} J(z) & \text{if } z \le M \\ \min J & \text{if } z > M \end{cases}.$$

is the convex envelope of J.

Note that the condition $u(t+) > u(t-)$ on $S(u)$ translates the fact that F must be finite only on increasing functions.

Proof Note preliminarily that F will be finite only on increasing functions so that we need to identify it only on functions u satisfying $u(t+) > u(t-)$ on $S(u)$.

Recall that the functional

$$u \mapsto \int_{(a,b)} \psi(u')\, dt$$

is lower semicontinuous on ${\rm W}^{1,1}(a, b)$ with respect to the $L^1(a, b)$ convergence. Let $u \in P\text{-}{\rm W}^{1,1}(0, L)$ and write

$$(0,L)\setminus S(u)=\bigcup_{k=1}^{N}(y_{k-1},y_k), \tag{2.52}$$

where $0=y_0<\cdots<y_N=L$. Let $u_n\to u$ in $L^1(0,L)$ and $E_n(u_n)<+\infty$ for all n. Then we have

$$\begin{aligned}F(u)&=\sum_{k=1}^{N}\int_{(y_{k-1},y_k)}\psi(u')\,dt\\&\le\sum_{k=1}^{N}\liminf_n\int_{(y_{k-1},y_k)}\psi(u_n')\,dt\\&\le\liminf_n\int_{(0,L)}\psi(u_n')\,dt\\&\le\liminf_n\int_{(0,L)}J(u_n')\,dt=\liminf_n E_n(u_n).\end{aligned}$$

Conversely, let $u\in P\text{-}\mathrm{W}^{1,1}(0,L)$ with $u(t+)>u(t-)$ on $S(u)$, and let $u_n=u$. Then it is easily seen that

$$\lim_n E_n(u_n)=\int_{(0,L)}J(u')\,dt,$$

so that

$$\Gamma\text{-}\limsup_n E_n(u)\le\int_{(0,L)}J(u')\,dt.$$

Now, using the notation (2.52), let $(u_j^k)_j$ converge to u weakly in $\mathrm{W}^{1,1}(y_{k-1},y_k)$ (and hence also uniformly) and satisfy

$$\lim_j\int_{(y_{k-1},y_k)}J((u_j^k)')\,dt=\int_{(y_{k-1},y_k)}\psi(u')\,dt.$$

Note that for j sufficiently large the function u_j defined by

$$u_j=u_j^k\ \text{on}\ (y_{k-1},y_k)$$

satisfies $u_j(t+)>u_j(t-)$ on $S(u_j)=S(u)$ and $u_j\to u$ in $L^1(0,L)$, so that, by the lower semicontinuity of the Γ-limsup we have

$$\begin{aligned}\Gamma\text{-}\limsup_n E_n(u)&\le\liminf_j\Gamma\text{-}\limsup_n E_n(u_j)\\&\le\liminf_j\int_{(0,L)}J(u_j')\,dt\\&=\sum_{k=1}^{N}\lim_j\int_{(y_{k-1},y_k)}J((u_j^k)')\,dt\end{aligned}$$

$$= \sum_{k=1}^{N} \int_{(y_{k-1},y_k)} \psi(u')\,dt = F(u),$$

and the proof is concluded. □

2.2.3 *Higher-order behaviour of nearest-neighbour interactions*

Note that minimum problems involving the limit functional F present a completely different behaviour depending on whether the (trivial) convexification of J is taken into account or not. Consider for example the simple minimum problem

$$m = \min\Big\{ F(u) : u(0) = 0,\ u(L) = h \Big\}, \tag{2.53}$$

with $h > 0$. Then we have:

(*compression*) if $h \leq ML$ then the minimum $m = L\,J(h/L)$ is achieved only by the linear function $u(x) = hx/L$. Note that the minimizer has no jump;

(*tension*) if $h > ML$ then the minimum $m = L \min J$ is achieved by all functions $u \in P\text{-}\mathrm{W}^{1,1}(0,L)$ such that $u' \geq M$ a.e. Note in particular that we can exhibit minimizers with an arbitrary number of jumps.

In this second case, hence, very little information on the behaviour of the minimizers of

$$m_n = \min\Big\{ E_n(u) : u(0) = 0,\ u(L) = h \Big\}, \tag{2.54}$$

can be drawn from the study of the corresponding problem (2.53) for the Γ-limit.

To improve this description, we note now that minimizers of m_n also minimize

$$m_n^{(1)} = \min\Big\{ \frac{E_n(u) - L\,\min J}{\lambda_n} : u(0) = 0,\ u(L) = h \Big\}. \tag{2.55}$$

The choice of the scaling λ_n is suggested by the fact that, choosing $\overline{u}_n = \overline{u}$, where $\overline{u} \in P\text{-}\mathrm{W}^{1,1}(0,L)$ is any function with $u' = M$ a.e. we have $E_n(\overline{u}_n) \leq L\,\min J + c\lambda_n$. We are then lead to studying the Γ-limit of the scaled functions

$$E_n^{(1)}(u) = \frac{E_n(u) - L\,\min J}{\lambda_n}. \tag{2.56}$$

Theorem 2.19 *The functionals $E_n^{(1)}$ Γ-converge with respect to the $L^1(0,L)$ convergence to the functional $F^{(1)}$ given by*

$$F^{(1)}(u) = \begin{cases} -\min J\,\#(S(u)) & \text{if } u \in P\text{-}\mathrm{W}^{1,\infty}(0,L),\ u(t+) > u(t-) \text{ on } S(u) \\ & \text{and } u' = M \text{ a.e.} \\ +\infty & \text{otherwise} \end{cases} \tag{2.57}$$

on $L^1(0,L)$.

Proof Again, note preliminarily that F will be finite only on increasing functions so that we need to identify it only on functions u satisfying $u(t+) > u(t-)$ on $S(u)$.

The liminf inequality will be obtained by comparison. Let $\sup_n E_n(u_n) < +\infty$ and $u_n \to u$ in $L^1(0,L)$. Let

$$v_n(x_i^n) = u_n(x_i^n) - Mx_i^n.$$

Note that $v_n \to v = u - Mx$, and that

$$E_n(u_n) = \tilde{E}_n(v_n) = \sum_{i=1}^n \lambda_n \psi_n\Big(\frac{v_n(x_i^n) - v_n(x_{i-1}^n)}{\lambda_n}\Big),$$

where

$$\psi_n(z) = \frac{1}{\lambda_n}(J(z+M) - \min J).$$

Note that $\psi_n \to +\infty$ if $z \neq 0$, and that

$$\lim_{z\to+\infty} \psi_n(z) = -\frac{\min J}{\lambda_n}.$$

With fixed $k \in \mathbf{N}$ let $\tilde{E}_n^K$ be defined by

$$\tilde{E}_n^K(w) = \sum_{i=1}^n \lambda_n \min\Big\{k\Big(\frac{v_n(x_i^n) - v_n(x_{i-1}^n)}{\lambda_n}\Big)^2, \frac{1}{\lambda_n}(\min J - \frac{1}{k})\Big\}.$$

By the results of the previous chapter $\tilde{E}_n^K$ Γ-converge to F^K defined by

$$F^K(w) = \begin{cases} k\displaystyle\int_{(0,L)} |w'|^2\,dt + (\min J - \frac{1}{k})\#(S(w)) & \text{if } w \in P\text{-}W^{1,2}(0,L) \\ +\infty & \text{otherwise.} \end{cases}$$

Now, note that for n large enough we have

$$E_n \geq \tilde{E}_n^K,$$

so that

$$\begin{aligned} \liminf_n E_n(u_n) &= \liminf_n \tilde{E}_n(v_n) \\ &\geq \liminf_n \tilde{E}_n^K(v_n) \geq F^K(v). \end{aligned}$$

It will then be sufficient to consider the case $v \in P\text{-}W^{1,2}(0,L)$; that is, $u \in P\text{-}W^{1,2}(0,L)$. In this case we get

$$\liminf_n E_n(u_n) \geq k \int_{(0,L)} |u' - M|^2 \, dt + (\min J - \frac{1}{k}) \#(S(u)).$$

By the arbitrariness of k we get the desired inequality.

The limsup inequality is easily obtained. Indeed, if $u \in P\text{-}W^{1,\infty}(0,L)$ is increasing and $u' = M$ a.e. we can take $u_n = u$, in which case $E_n(u_n) = L \min J - \min J \lambda_n + o(\lambda_n)$. □

2.2.4 *Convergence of minimum problems*

From the results of the previous section we can easily derive a description of the limiting behaviour of minimizers of minimum problems (2.54).

Proposition 2.20 *Let $h > ML$; then from every sequence of minimizers of problems (2.54) we can extract a subsequence converging in $L^1(0,L)$ to an increasing function $u \in P\text{-}W^{1,\infty}(0,L)$ such that $u' = M$ a.e. in $(0,L)$ and, after setting $u(0-) = 0$ and $u(L+) = h$, u has only one jump in $[0,L]$. Moreover we have the estimate*

$$m_n = L \min J - \lambda_n \min J + o(\lambda_n)$$

as $n \to +\infty$.

Proof By the coerciveness conditions on E_n, we can suppose that, upon extracting a subsequence, the minimizers of m_n converge in $L^1(0,L)$. We interpret those minimizers also as minimizers of $m_n^{(1)}$. Hence, upon relaxing the boundary conditions, the limit function u solves the problem

$$\begin{aligned} m^{(1)} = -\min J \, \min\{\#(S(u)) : u \in P\text{-}W^{1,\infty}(0,L), \\ u' = M \text{ a.e.}, u(0-) = 0, u(L+) = h\}. \end{aligned}$$

where $S(u)$ is interpreted as a subset of $[0,L]$. The solution of this problem is clearly a function satisfying the thesis of the theorem, and $m^{(1)} = -\min J$. From the convergence of minima

$$\frac{m_n - L \min J}{\lambda_n} = m_n^{(1)} \to m^{(1)} = -\min J$$

we complete the proof. □

2.2.5 *Long-range interactions*

By taking into account the methods of Sections 1.4.2 and 1.4.3 and the proof of Theorem 2.18 we have the following result.

Theorem 2.21 *Let $K \geq 2$ and let E_n be defined by*

$$E_n(u) = \sum_{j=1}^{K} \sum_{i=0}^{n-j} \lambda_n J\Big(\frac{u_{i+j} - u_i}{\lambda_n}\Big) \tag{2.58}$$

The energies E_n Γ*-converge on* $P\text{-}\mathrm{W}^{1,1}(0,L)$ *with respect to the* $L^1(0,L)$ *convergence, to the functional* F *defined by*

$$F(u)=\begin{cases}\displaystyle\int_{(0,L)}\psi(u')\,dt & \textit{if } u(t+)>u(t-) \textit{ on } S(u)\\ +\infty & \textit{otherwise}\end{cases}\tag{2.59}$$

on $P\text{-}\mathrm{W}^{1,1}(0,L)$*, where* $\psi:\mathbf{R}\to\mathbf{R}\cup\{+\infty\}$ *is the convex function given by*

$$\psi(z)=\lim_N\min\Big\{\frac{1}{N}\sum_{j=1}^{K}\sum_{i=0}^{N-j}J(u(i+j)-u(i))$$

$$u:\{0,\dots,N\}\to\mathbf{R},\ u(i)=zi \textit{ for } i\le K \textit{ or } i\ge N-K\Big\}.\tag{2.60}$$

Furthermore, if $K=2$ *then the function* ψ *is also defined as* $\psi=\tilde{J}^{**}$*, where*

$$\tilde{J}(z)=J(2z)+\frac{1}{2}\min\{J(z_1)+J(z_2):\ z_1+z_2=2z\}.\tag{2.61}$$

Proof The proof follows by using the arguments of Theorems 2.18, 1.6 and 1.11, with $\psi_n^j(z)=J(jz)$, after noting that ψ defined above is convex and bounded at $+\infty$. □

Remark 2.22 The function ψ satisfies the same assumptions as J upon replacing (vi) with $\lim_{z\to+\infty}\psi(z)=C<0$, but it can be seen that $\tilde{J}$ in general does not satisfy (iv); i.e., is not of convex/concave form.

3

GENERAL CONVERGENCE RESULTS

In order to state and prove general results for the convergence of discrete schemes we will have to describe the Γ-limits of discrete energies in spaces of functions of bounded variation. We briefly recall some of their properties, referring to [5] for a complete introduction.

3.1 Functions of bounded variation

We recall that the space $BV(a,b)$ of *functions of bounded variation* on (a,b) is defined as the space of functions $u \in \mathrm{L}^1(a,b)$ whose *distributional derivative* Du is a signed Borel measure. For each such u there exists $f \in L^1(a,b)$, a (at most countable) set $S(u) \subset (a,b)$, a sequence of real numbers $(a_t)_{t\in S(u)}$ with $\sum_t |a_t| < +\infty$ and a non-atomic measure $D_c u$ singular with respect to the Lebesgue measure such that the equality of measures $Du = f\,\mathcal{L}_1 + \sum_{t\in S(u)} a_t\delta_t + D_c u$ holds. It can be easily seen that for such functions the left hand-side and right hand-side *approximate limits* $u^-(t)$, $u^+(t)$ exist at every point, and that $S(u) = \{t \in \mathbf{R} : \ u^-(t) \neq u^+(t)\}$ and $a_t = u^+(t) - u^-(t) =: [u](t)$. We will write $\dot u = f$, which is an approximate gradient of u. $D_c u$ is called the *Cantor part* of Du. A sequence u_j converges weakly to u in $BV(a,b)$ if $u_j \to u$ in $\mathrm{L}^1(a,b)$ and $\sup_j |Du_j|(a,b) < +\infty$.

The space $SBV(a,b)$ of *special functions of bounded variation* is defined as the space of functions $u \in BV(a,b)$ such that $D_c u = 0$; i.e., whose distributional derivative Du can be written as $Du = \dot u\,\mathcal{L}_1 + \sum_{t\in S(u)}(u^+(t) - u^-(t))\delta_t$. This notation describes a particular case of a SBV-functions space as introduced by De Giorgi and Ambrosio [15]. We will mainly deal with functionals whose natural domain is that of piecewise-$\mathrm{W}^{1,p}$ functions, which is a particular sub-class of $SBV(a,b)$ corresponding to the conditions $\dot u \in \mathrm{L}^p(a,b)$ and $\#(S(u)) < +\infty$, but we nevertheless use the more general SBV notation for future reference and for further generalization to higher dimensions (see [4]). For an introduction to BV and SBV functions we refer to the book by Ambrosio, Fusco and Pallara [5], while approximation methods for free-discontinuity problems are discussed by Braides [7].

A class of energies on $SBV(a,b)$ are those of the form

$$\int_{(a,b)} f(\dot u)\,dt + \sum_{S(u)} g(u^+(t) - u^-(t)),$$

with $f, g : \mathbf{R} \to [0, +\infty]$. Lower semicontinuity conditions on $\mathcal{E}$ imply that f is lower semicontinuous and convex and g is lower semicontinuous and subadditive;

i.e., $g(x+y) \leq g(x)+g(y)$. The latter can be interpreted as a condition penalizing fracture fragmentation, whereas convexity penalizes oscillations. If φ is not lower semicontinuous and convex (respectively, subadditive) then we may consider its *lower semicontinuous and convex* (respectively, *subadditive*) *envelope*; i.e., the greatest lower semicontinuous and convex (respectively, subadditive) function not greater than φ, that we denote by φ^{**} (respectively, $\mathrm{sub}^{-}\varphi$). For a discussion on the role of these conditions for the lower semicontinuity of $\mathcal{E}$ we refer to [7] Section 2.2 or [8]. Energies in BV must satisfy further compatibility conditions between f and g (see e.g. Theorem 3.1 below and the subsequent remark)

The following theorem is an easy corollary of [2] Theorem 6.3 and will be widely used in the next section.

Theorem 3.1 *For all $n \in \mathbf{N}$ let $f_n, g_n : \mathbf{R} \to [0,+\infty]$ be lower semicontinuous functions. Let $\alpha > 0$ exists such that*

(1) f_n is convex and

$$\alpha(|z|-1) \leq f_n(z) \qquad \text{for every } z \in \mathbf{R},$$

(2) g_n is subadditive and

$$\alpha(|z|-1) \leq g_n(z) \qquad \text{for every } z \in \mathbf{R}.$$

and suppose that $f, g : \mathbf{R} \to [0,+\infty]$ exist such that $\Gamma\text{-}\lim_n f_n = f$ on $\mathbf{R}$ and $\Gamma\text{-}\lim_n g_n = g$ on $\mathbf{R} \setminus \{0\}$. For notation's convenience we set $g(0) = 0$. Let $\mathcal{H}_n : BV(a,b) \to [0,+\infty]$ be defined as

$$\mathcal{H}_n(u) := \begin{cases} \displaystyle\int_a^b f_n(\dot{u})\,dx + \sum_{S(u)} g_n([u]) & \text{if } u \in SBV(a,b) \\ +\infty & \text{otherwise.} \end{cases}$$

Then $\mathcal{H}_n$ Γ-converge with respect to the weak topology of $BV(a,b)$ to the functional $\mathcal{H} : BV(a,b) \to [0,+\infty)$ defined by

$$\mathcal{H}(u) := \int_0^l \overline{f}(\dot{u})\,dx + \sigma^+ D_c u^+(a,b) + \sigma^- D_c u^-(a,b) + \sum_{S(u)} \overline{g}([u])$$

(recall that $D_c u^{\pm}$ denote the positive/negative part of the Cantor measure $D_c u$), where

$$\overline{f}(z) := \inf\{f(z_1)+g^0(z_2) : z = z_1+z_2\},\ \overline{g}(z) := \inf\{f^\infty(z_1)+g(z_2) : z = z_1+z_2\},$$

$$f^\infty(z) = \lim_{t\to+\infty} \frac{f(tz)}{t}, \quad g^0(z) = \lim_{t\to+\infty} tg\Big(\frac{z}{t}\Big), \quad \text{and} \quad \sigma^{\pm} = \lim_{t\to+\infty} \frac{\overline{f}(\pm t)}{t}$$

for all $z \in \mathbf{R}$.

Remark 3.2 Note that if we take $g_n = g$ and $f_n = f$ we recover the well-known compatibility hypothesis $f^\infty = g^0$ for weakly lower semicontinuous functionals on $BV(a,b)$.

If $f(0) = 0$ then it can be easily seen that $\overline{f} = (f \wedge g^0)^{**}$ and $\overline{g} = \mathrm{sub}^-(f^\infty \wedge g)$.

3.2 Nearest-neighbour interactions

For future reference, we state and prove the convergence results allowing for a more general dependence on the underlying lattice than in the previous chapters, at the expense of a slightly more complex notation.

We begin by identifying the functions defined on a lattice with a subset of measurable functions. Consider an open interval (a,b) of $\mathbf{R}$ and two sequences (λ_n), (a_n) of positive real numbers with $a_n \in [a, a+\lambda_n)$ and $\lambda_n \to 0$. For $n \in \mathbf{N}$ let $a \le x_n^1 < \ldots < x_n^{N_n} < b$ be the partition of (a,b) induced by the intersection of (a,b) with the set $a_n + \lambda_n \mathbf{Z}$. We define $\mathcal{A}_n(a,b)$ the set of the restrictions to (a,b) of functions constant on each $[a+k\lambda_n, a+(k+1)\lambda_n)$, $k \in \mathbf{Z}$. A function $u \in \mathcal{A}_n(a,b)$ will be identified by $N_n + 1$ real numbers $c_n^0, \ldots, c_n^{N_n}$ such that

$$u(x) = \begin{cases} c_n^i & \text{if } x \in [x_n^i, x_n^{i+1}),\ i = 1, \ldots, N_n - 1 \\ c_n^0 & \text{if } x \in (a, x_n^1) \\ c_n^{N_n} & \text{if } x \in [x_n^{N_n}, b). \end{cases} \tag{3.1}$$

For $n \in \mathbf{N}$ let $\psi_n : \mathbf{R} \to [0, +\infty]$ be a given Borel function and define $E_n : L^1(a,b) \to [0,+\infty]$ as

$$E_n(u) = \begin{cases} \displaystyle\sum_{i=1}^{N_n - 1} \lambda_n \psi_n \left(\frac{u(x_n^{i+1}) - u(x_n^i)}{\lambda_n} \right) & x \in \mathcal{A}_n(a,b) \\ +\infty & \text{otherwise in } L^1(a,b). \end{cases} \tag{3.2}$$

The following sections contain the description of the asymptotic behaviour of E_n as $n \to +\infty$.

3.2.1 *Potentials with local superlinear growth*

We first treat the case when the potentials ψ satisfy locally a growth condition of order $p > 1$. This is the case of non-convex potentials introduced by Blake and Zisserman and of the scaled Lennard Jones potentials which justify Griffith theory of fracture as a first-order effect.

Theorem 3.3 *For all $n \in \mathbf{N}$ let $T_n^\pm \in \mathbf{R}$ exist with*

$$\lim_n T_n^\pm = \pm\infty, \qquad \lim_n \lambda_n T_n^\pm = 0, \tag{3.3}$$

and such that, if we define $F_n, G_n : \mathbf{R} \to [0,+\infty]$ as

$$F_n(z) = \begin{cases} \psi_n(z) & T_n^- \le z \le T_n^+ \\ +\infty & z \in \mathbf{R} \setminus [T_n^-, T_n^+] \end{cases} \tag{3.4}$$

$$G_n(z) = \begin{cases} \lambda_n \psi_n\left(\dfrac{z}{\lambda_n}\right) & z \in \mathbf{R} \setminus [\lambda_n T_n^-, \lambda_n T_n^+] \\ +\infty & otherwise \end{cases} \tag{3.5}$$

the following conditions are satisfied: there exists $p > 1$ *such that*

$$F_n(z) \geq |z|^p \qquad \forall z \in \mathbf{R} \tag{3.6}$$

$$G_n(z) \geq c > 0 \qquad \forall z \neq 0 \tag{3.7}$$

and, moreover, there exist $F, G : \mathbf{R} \to [0, +\infty]$, *such that*

$$\Gamma\text{-}\lim_n F_n^{**} = F \ \ on \ \mathbf{R}, \tag{3.8}$$

$$\Gamma\text{-}\lim_n \mathrm{sub}^- G_n = G \ \ on \ \mathbf{R}. \tag{3.9}$$

Then, $(E_n)_n$ Γ*-converges to* E *with respect to the convergence in measure on* $L^1(a,b)$*, where*

$$E(u) = \begin{cases} \displaystyle\int_a^b F(\dot{u})\,dt \ + \sum_{t \in S(u)} G([u](t)) & u \in SBV(a,b) \\ +\infty & otherwise \ in \ \ L^1(a,b). \end{cases}$$

Remark 3.4 Note that hypotheses (3.8) and (3.9) are not restrictive upon passing to a subsequence by a compactness argument. This remark also holds for Theorems 3.7 and 3.9. Moreover, if F is finite everywhere then Γ-convergence in (3.8) can be replaced by pointwise convergence.

Proof For simplicity of notation we deal with the case $T_n^+ = -T_n^- =: T_n$, the general case following by simple modifications. Without loss of generality we may assume

$$\sup_n \inf_{z \in \mathbf{R}} F_n(z) < +\infty; \tag{3.10}$$

otherwise we trivially have $F \equiv +\infty$ and consequently $E \equiv +\infty$.

With fixed $u \in L^1(a,b)$ and a sequence $(u_n) \subseteq \mathcal{A}_n(a,b)$ such that $u_n \to u$ in measure and $\sup_n E_n(u_n) < +\infty$. Up to a subsequence, we can suppose in addition that u_n converges to u pointwise a.e. We now construct for each $n \in \mathbf{N}$ a function $v_n \in SBV(a,b)$ and a free-discontinuity energy such that v_n still converges to u and we can use that energy to give a lower estimate for $E_n(u_n)$. Set

$$I_n := \left\{ i \in \{1, \ldots, N_n - 1\} : \left| \frac{u_n(x_n^{i+1}) - u_n(x_n^i)}{\lambda_n} \right| > T_n \right\} \tag{3.11}$$

and

$$v_n(x) := \begin{cases} u_n(x_n^1) & \text{if } x \in (a, x_n^1) \\ c_n^i + \dfrac{(c_n^{i+1} - c_n^i)}{\lambda_n}(x - x_n^i) & x \in [x_n^i, x_n^{i+1}),\ i \notin I_n \\ u_n(x) & x \text{ elsewhere in } (a,b). \end{cases} \tag{3.12}$$

We have that, for $\varepsilon > 0$ fixed,

$$\begin{aligned} &\{x \,:\, |v_n(x) - u_n(x)| > \varepsilon\} \\ &\subseteq \{x \in [x_n^i, x_n^{i+1}),\ i \notin I_n,\ |u_n(x_n^{i+1}) - u_n(x_n^i)| > \varepsilon\} \cup (a, x_n^1). \end{aligned} \tag{3.13}$$

Since, for $i \notin I_n$ we have $|u_n(x_n^{i+1}) - u_n(x_n^i)| \leq \lambda_n T_n$, then $\{x \,:\, |v_n(x) - u_n(x)| > \varepsilon\}$ consists at most of the interval (a, x_n^1) if n is large enough. Hence, the sequence $(v_n)_n$ converges to u in measure and pointwise a.e. Moreover, by (3.7)

$$c\#I_n \leq E_n(u_n) \leq M, \tag{3.14}$$

with $M = \sup_n E_n(u_n)$. By the equiboundedness of $\#I_n$, we can suppose that $S(v_n) = \{x_n^{i+1}\}_{i \in I_n}$ tends to a finite set. For the local nature of the arguments in the following reasoning, we can also assume that S consists of only one point $x_0 \in (a,b)$.

Now, consider the sequence $(w_n)_n$ defined by

$$w_n(x) = \begin{cases} v_n(a) + \displaystyle\int_{(a,x)} \dot v_n(t)\,dt & \text{if } x < x_0 \\ v_n(a) + \displaystyle\int_{(a,x)} \dot v_n(t)\,dt + \sum_{t \in S(v_n)} [v_n](t) & \text{if } x \geq x_0. \end{cases} \tag{3.15}$$

Note that $w_n(a) = v_n(a)$, $\dot w_n = \dot v_n$, $S(w_n) = \{x_0\}$ and $[w_n](x_0) = \sum_{t \in S(v_n)} [v_n](t)$. Such a sequence still converges to u a.e. Indeed, since x_0 is the limit point of the sets $S(v_n)$, for any $\eta > 0$ fixed we can find $n_0(\eta) \in \mathbf{N}$ such that for any $n \geq n_0(\eta)$ and for any $i \in I_n$ $|x_0 - x_n^{i+1}| < \eta$. Hence, by construction, for any $n \geq n_0(\eta)$ and for any $x \in (a,b) \setminus [x_0 - \eta, x_0 + \eta]$, $w_n(x) = v_n(x)$, that is, the two sequences (v_n) and (w_n) have the same pointwise limit. Since $\dot w_n = \dot v_n$ on (a,b), by (3.6) we have that $\|\dot w_n\|_{L^p(a,b)} \leq M$. Then, using Poincaré's inequality on each interval, it can be easily seen that $(w_n)_n$ is equibounded in $W^{1,p}((a,b) \setminus \{x_0\})$. Since it also converges to u pointwise a.e., by using a compactness argument, we get that $u \in W^{1,p}((a,b) \setminus \{x_0\})$ and, up to subsequences,

$$\dot w_n \rightharpoonup \dot u \quad \text{weakly in } L^p(a,b).$$

Moreover, since for any two points $a < x_1 < x_0 < x_2 < b$ we have

$$w_n(x_2) = w_n(x_1) + \int_{x_1}^{x_2} \dot w_n\,dt + [w_n](x_0)$$

$$u(x_2) = u(x_1) + \int_{x_1}^{x_2} \dot{u}\, dt + [u](x_0),$$

taking points x_1, x_2 in which w_n converges to u and passing to the limit as $n \to +\infty$, we have

$$[w_n](x_0) \to [u](x_0). \tag{3.16}$$

We can now rewrite our functionals in terms of v_n:

$$\begin{aligned} E_n(u_n) &= \sum_{i \notin I_n} \lambda_n \psi_n(\dot{v}_n) + \sum_{i \in I_n} G_n([v_n](x_n^{i+1})) \\ &= \int_a^b F_n(\dot{v}_n)\, dt + \sum_{t \in S(v_n)} G_n([v_n](t)). \end{aligned}$$

From (3.14) we also have

$$\begin{aligned} E_n(u_n) &\geq \int_a^b F_n(\dot{v}_n)\, dt + \mathrm{sub}^- G_n\Big(\sum_{t \in S(v_n)} [v_n](t)\Big) \\ &\geq \int_a^b F_n^{**}(\dot{w}_n)\, dt + \mathrm{sub}^- G_n([w_n](x_0)) \end{aligned}$$

as $n \to +\infty$. Passing to the liminf as $n \to +\infty$, using (3.16) we have

$$\begin{aligned} \liminf_n E_n(u_n) &\geq \liminf_n \int_a^b F_n^{**}(\dot{w}_n)\, dt + \liminf_n \mathrm{sub}^- G_n([w_n](x_0)) \\ &\geq \int_a^b F(\dot{u})\, dt + G([u](x_0)) \end{aligned}$$

as desired.

We now turn our attention to the construction of recovery sequences for the Γ-limsup. We may assume in what follows that $\inf_{z \in \mathbf{R}} F_n(z) = F_n(0)$.

Step 1 We first prove the limsup inequality for u affine on (a, b). Set $\xi = \dot{u}$; we can assume, upon a slight translation argument, that $F(\xi) = \lim_n F_n^{**}(\xi)$. Then, for each n in $\mathbf{N}$ we can find $\xi_n^1, \xi_n^2 \in \mathbf{R}$, $t_n \in [0,1]$ such that

$$\begin{aligned} |t_n \xi_n^1 + (1-t_n)\xi_n^2 - \xi| &\leq \frac{\sqrt{\lambda_n}}{2(b-a)} \\ t_n F_n(\xi_n^1) + (1-t_n) F_n(\xi_n^2) &\leq F_n^{**}(\xi) + o(1) \\ |\xi_n^i| &\leq c = c(\xi). \end{aligned} \tag{3.17}$$

Note that in the last inequality the choice of the constant c can be chosen independent of n thanks to (3.6) and (3.10). It can be easily seen that it is not restrictive to make the following assumptions on ξ_n^i:

$$\xi_n^1 > \xi, \quad F_n(\xi_n^1) \le F_n(\xi_n^2), \quad (|\xi_n^1| + |\xi_n^2|)\sqrt{\lambda_n} \le 1. \tag{3.18}$$

We define a piecewise-affine function $v_n \in L^1(a,b)$ with the following properties:

$$v_n(x) = u(x) \text{ on } (a, x_n^1],$$
$$\dot{v}_n{}_{\,|\,[x_n^i, x_n^{i+1})} := v_n^i \in \{\xi_n^1, \xi_n^2\},$$

and v_n^i is defined recursively by

$$\begin{cases} v_n^1 = \xi_n^1 \\ v_n^{i+1} = \begin{cases} v_n^i & \text{if } \frac{\sqrt{\lambda_n}}{2} \le v_n(a_n) + \sum\limits_{j=1}^{i} v_n^j \lambda_n + v_n^i \lambda_n - u(x_n^{i+1}) \le \sqrt{\lambda_n} \\ \xi_n^1 + \xi_n^2 - v_n^i & \text{otherwise.} \end{cases} \end{cases} \tag{3.19}$$

Since $0 \le v_n - u \le \sqrt{\lambda_n}$ by definition, $(v_n)_n$ converges to u uniformly, and hence in measure and, moreover,

$$\beta_n^1 := \#\left\{i \in \{0, \ldots, N_n\} : \ v_n^i = \xi_n^1\right\} \ge t_n N_n. \tag{3.20}$$

Indeed, from (3.17), (3.18) and (3.19) we deduce

$$\begin{aligned} \lambda_n N_n \left(t_n(\xi_n^1 - \xi) + (1-t_n)(\xi_n^2 - \xi)\right) \le \frac{\sqrt{\lambda_n}}{2} &\le v_n(x_n^{N_n}) - u(x_n^{N_n}) \\ &= \beta_n^1(\xi_n^1 - \xi)\lambda_n + (N_n - \beta_n^1)(\xi_n^2 - \xi)\lambda_n, \end{aligned}$$

so that

$$(\beta_n^1 - t_n N_n)(\xi_n^1 - \xi_n^2) \ge 0.$$

Now, consider the sequence $(u_n) \subseteq A_n(a,b)$ defined by

$$\begin{aligned} u_n(x_n^i) &= v_n(x_n^i) \qquad \text{for } i = 1, \ldots, N_n, \\ u_n(a) &= v_n(a) \ \text{ and } \ u_n(b) = v_n(b). \end{aligned}$$

Since (3.13) still holds with u_n, v_n as above, it can be easily checked that $(u_n)_n$ converges to u in measure. Hence, recalling (3.17), (3.18) and (3.20),

$$\begin{aligned} E_n(u_n) &= \lambda_n F_n(\xi_n^1)\beta_n^1 + (N_n - \beta_n^1)\lambda_n F_n(\xi_n^2) \\ &\le t_n \lambda_n N_n F_n(\xi_n^1) + (1-t_n) N_n \lambda_n F_n(\xi_n^2) \\ &\le N_n \lambda_n \left(F_n^{**}(\xi) + o(1)\right) \le (b-a) F_n^{**}(\xi) + o(1). \end{aligned}$$

Taking the limsup as $n \to +\infty$ we get

$$\limsup_n E_n(u_n) \le F(\xi)(b-a) = E(u).$$

The same construction as above works also in the case of a piecewise-affine function: let $[a,b] = \bigcup [a_j, b_j]$ with $a_1 = a$, $b_j = a_{j+1}$ and $\dot{u}$ constant on each

(a_j, b_j), then it suffices to repeat the procedure above on each (a_j, b_j) to provide functions v_n^j in $A_n(a_j, b_j)$ such that

$$v_n^j \to u_{|(a_j,b_j)} \qquad \text{in measure}$$

$$\limsup_n \sum_{\{i:\ x_n^i \in (a_j,b_j)\}} \lambda_n \psi_n \left(\frac{v_n^j(x_n^{i+1}) - v_n^j(x_n^i)}{\lambda_n} \right) \leq \int_{a_j}^{b_j} F(\dot{u})\, dx.$$

With j fixed define $y_n^j := \max\{x_n^i \in (a_j, b_j)\}$. Then, the recovery sequence u_n is defined in (a_j, b_j) as

$$u_n(x) = v_n^j(x) - \sum_{\ell < j} (v_n^{\ell+1}(y_n^\ell + \lambda_n) - v_n^\ell(y_n^\ell)).$$

Since $u(x) + \frac{\sqrt{\lambda_n}}{2} \leq v_n^j(x) \leq u(x) + \sqrt{\lambda_n}$ by construction, and $|u(y_n^\ell + \lambda_n) - u(y_n^\ell)| \leq c\lambda_n$, we have that $u_n \to u$ in measure and

$$E_n(u_n) = \sum_j \sum_{\{i:\ x_n^i \in (a_j,b_j)\}} \lambda_n \psi_n \left(\frac{v_n^j(x_n^{i+1}) - v_n^j(x_n^i)}{\lambda_n} \right) + cF_n(0)\lambda_n.$$

By a density argument we can extend the result to functions in $\mathrm{W}^{1,p}(a, b)$.

Step 2 Let u be of the form $z\chi_{(x_0,b)}$ with $G(z) < +\infty$ and let z_n be a recovery sequence for $G(z) = \Gamma\text{-}\lim_n \mathrm{sub}^- G_n(u)$. The sequence $\mathrm{sub}^- G_n(z_n)$ is bounded, hence, by (3.7), upon possibly considering a suitable subsequence, there exists an integer N not depending on n such that

$$\mathrm{sub}^- G_n(z_n) = \sup_\varepsilon \inf \left\{ \sum_{i=1}^N G_n(z^i) : \left| \sum_{i=1}^N z^i - z_n \right| < \varepsilon \right\}.$$

Hence, for all n we can find N points $\{z_n^1, \ldots, z^N\}$ such that

$$\lim_n \sum_{i=1}^N z_n^i = z \quad \text{and} \quad \lim_n \sum_{i=1}^N G_n(z_n^i) = G(z). \tag{3.21}$$

Let $i_n \in \{1, \ldots, N_n\}$ be the index such that $x_0 \in [x_n^{i_n}, x_n^{i_n+1})$ and, for n large, define w_n as in (3.1) with

$$c_n^i = \begin{cases} 0 & \text{if } i \leq i_n \\ \sum\limits_{j \leq (i - i_n)} (z_n^j) & \text{if } i_n < i \leq i_n + N \\ \sum\limits_{j=1}^N (z_n^j) & \text{if } i > i_n + N. \end{cases} \tag{3.22}$$

Clearly $(w_n)_n \to u$ in measure and

$$E_n(w_n) = \sum_{i=1}^{N} \lambda_n \psi_n \left(\frac{z_n^i}{\lambda_n} \right) = \sum_{i=1}^{N} G_n(z_n^i) + (b-a)F_n(0)\,;$$

the estimate follows from (3.21) by passing to the limit as $n \to +\infty$.

Step 3 Let $u \in SBV(a,b)$ be such that $E(u) < +\infty$, then

$$u = v + w \text{ with } v(x) = \int_a^x \dot{u}\, dt + c \text{ and } w(x) = \sum_{j=1}^{m} z_j \chi_{[x_j,b)}.$$

For $j = 1, \dots m$ let w_n^j be defined as in Step 2 with jumps in $\bigcup_j \{x_n^{i+i_{n,j}}\}_{i=1}^{N_j}$ and let v_n be a recovery sequence for v such that it is constant on each $[x_n^{i_{n,j}}, x_n^{i_{n,j}} + \lambda_n N_j)$. The sequence $u_n = v_n + \sum\limits_{j=1}^{m} w_n^j$ converges in measure to u and

$$\limsup_n E_n(u_n) = \limsup_n \Big(E_n(v_n) + \sum_{j+1}^{m} E_n(w_n^j) \Big) \le E(v) + E(w) = E(u),$$

as desired. □

Corollary 3.5 *Let $\psi_n : \mathbf{R} \to [0,+\infty]$ satisfy the hypotheses of Theorem* 3.3. *Assume that, in addition, for all $n \in \mathbf{N}$, $F_n = F_n^{**}$ on $[T_n^-, T_n^+]$ and $G_n = \mathrm{sub}^- G_n$ on $\mathbf{R} \setminus [\lambda_n T_n^-, \lambda_n T_n^+]$. Then, for any $u \in L^1(a,b)$, $E_n(u)$ Γ-converges to $E(u)$ with respect to the strong topology of $L^1(a,b)$.*

Proof It suffices to produce a recovery sequence converging strongly in $L^1(a,b)$. Note that in Step 1, by the convexity of F_n, we can choose $\xi_n^1 = \xi_n^2 = \xi_n$ in (3.17). Then $v_n = u$ and u_n turns out to be the piecewise-constant interpolation of u at points $\{x_n^i\}$. It is easy to check that $u_n \to u$ strongly in $L^1(a,b)$. It remains to show that also for functions of the form $z\chi_{[x_0,b)}$ it is possible to exhibit a sequence that converges strongly in $L^1(a,b)$. To this end it suffices to note that in Step 2, since $G_n = \mathrm{sub}^- G_n$ locally on $\mathbf{R} \setminus \{0\}$, we can find a sequence (z_n) such that (3.21) is replaced by $\lim_n z_n = z$ and $\lim_n G_n(z_n) = G(z)$. Hence, the sequence w_n defined by (3.22) converges to u strongly in $L^1(a,b)$ and it is a recovery sequence. □

Remark 3.6 Note that the hypotheses of the previous corollary are satisfied if ψ_n is convex and lower semicontinuous on $[T_n^-, T_n^+]$ and concave and lower semicontinuous on $(-\infty, T_n^-]$ and $[T_n^+, +\infty)$

3.2.2 *Potentials with linear growth*

In this section we will consider energy potentials ψ_n such that

$$\psi_n(z) \ge \alpha(|z| - 1) \qquad \forall z \in \mathbf{R} \tag{3.23}$$

for some $\alpha > 0$. For this kind of energies we can still prove a convergence result to a free-discontinuity energy, whose volume and surface densities are obtained by a suitable interaction of the limit functions F, G of the two 'regularized' scalings of ψ_n. Note that in the following statement the sequences $T_n^{\pm}$ are arbitrary.

Theorem 3.7 *Let $\psi_n : \mathbf{R} \to [0, +\infty]$ satisfy (3.23). For all $n \in \mathbf{N}$ let $T_n^{\pm} \in \mathbf{R}$ satisfy properties (3.3) and let $F_n, G_n : \mathbf{R} \to [0, +\infty]$ be defined as in (3.4). Assume that $F, G : \mathbf{R} \to [0, +\infty]$ exist such that*

$$\Gamma\text{-}\lim_n F_n^{**} = F \ \textit{on}\ \mathbf{R}, \tag{3.24}$$

$$\Gamma\text{-}\lim_n \mathrm{sub}^- G_n = G \ \textit{on}\ \mathbf{R} \setminus \{0\}. \tag{3.25}$$

For notation's convenience we set $G(0) = 0$. Then, $(E_n)_n$ Γ-converges to E with respect to the convergence in $L^1(a, b)$ and the convergence in measure, where

$$E(u) = \begin{cases} \displaystyle\int_a^b \overline{F}(\dot{u})\, dx + \sum_{S(u)} \overline{G}([u]) + c_1 Du_c^+(a, b) + c_{-1} Du_c^-(a, b) & \textit{if } u \in BV(a, b) \\ +\infty & \textit{otherwise,} \end{cases}$$

$$\overline{F}(z) := \inf\{F(z_1) + G^0(z_2) : z_1 + z_2 = z\},$$

$$\overline{G}(z) := \inf\{F^\infty(z_1) + G(z_2) : z_1 + z_2 = z\},$$

$c_1 := \overline{F}^\infty(1)$ and $c_{-1} := \overline{F}^\infty(-1)$.

Remark 3.8 Thanks to (3.23) the theorem can be restated also with respect to the weak convergence in $BV(a, b)$. Indeed, sequences converging in measure along which the functionals E_n are equibounded are weakly compact in BV.

Proof Again we deal with the case $T_n^+ = -T_n^- =: T_n$, the general case being achieved by slight modifications. Let $u_n, u \in L^1(a, b)$ be such that $u_n \to u$ in measure and $E_n(u_n) \leq c$. Analogously to the proof of Theorem 3.3, we will estimate $E_n(u_n)$ by a free-discontinuity energy computed on a sequence v_n converging to u weakly in $BV(a, b)$. Let I_n and v_n be defined as in (3.11) and (3.12), respectively. Note that $v_n \to u$ in measure and that v_n has equibounded total variation on (a, b). Indeed, by hypothesis (3.23) we have

$$|Dv_n|(a, b) = \sum_{i=1}^{N_n - 1} |u_n(x_n^{i+1}) - u_n(x_n^i)| \leq \frac{1}{\alpha} E_n(u_n) + c.$$

From this inequality we easily get that $u \in BV(a, b)$, in particular the Γ-liminf is finite only on $BV(a, b)$.

Up to passing to a subsequence we may assume that v_n converges to u weakly in $BV(a,b)$; moreover, by construction we have

$$E_n(u_n) \geq \int_a^b F_n^{**}(\dot{v}_n)\,dt + \sum_{S(v_n)} \mathrm{sub}^- G_n([v_n]).$$

Hence, it suffices to apply Theorem 3.1 to the functionals on the left hand side to get the Γ-liminf inequality.

To obtain the converse inequality it suffices to provide a sequence v_n converging to u in $L^1(a,b)$ such that

$$\limsup_n E_n(v_n) \leq E_1(u) := \int_a^b F(\dot{u})\,dt + \sum_{S(u)} G([u])$$

when $u \in SBV(a,b)$. The general estimate will be then obtained by relaxation (i.e. by taking $f_n = F$ and $g_n = G$ in Theorem 3.1). By a standard approximation argument it is sufficient to prove this inequality in the simpler cases of u linear and of u with a single jump. Let $u(t) = \xi t$; we may assume that $F(\xi) < +\infty$. Moreover, we may assume in what follows that $\inf_{z\in\mathbf{R}} F_n(z) = F_n(0)$. Then we can find ξ_n^1, ξ_n^2 such that the analog of (3.17) holds. In this case, $|\xi_n^1|, |\xi_n^2|$ are not necessarily equibounded; nevertheless we have by definition $|\xi_n^1|, |\xi_n^2| \leq T_n$ since $F(\xi) < +\infty$. Thus we can construct the functions v_n as in the proof of the Γ-limsup-inequality of Theorem 3.3, up to a slight modification. Indeed, if we replace $\sqrt{\lambda_n}$ with $\lambda_n T_n$ in (3.17) and (3.19), all those inequalities still hold. In particular we have that $|u(x) - v_n(x)| \leq \lambda_n T_n$ in (a,b). Thus v_n is a recovery sequence converging to u in $L^\infty(a,b)$.

As for the case of $u = z\chi_{(x_0,b)}$ with $G(z) < +\infty$, let $z_n = \sum_{i=1}^{M_n} z_n^i$ be such that $G(z) = \lim_n \sum_{i=1}^{M_n} G_n(z_n^i)$. Note that since $\lim_n \sum_{i=1}^{M_n} z_n^i = z$, by taking (3.23) into account, we may assume that $\sum_{i=1}^{M_n} (z_n^i)^+ \leq c|z|$ and the same for the negative terms. We may assume also that $|z_n^i| \geq \lambda_n T_n$ for any i. Hence, by arguing separately on the positive and negative part of (z_n^i), we easily get that

$$M_n \leq \frac{c|z|}{\lambda_n T_n}. \tag{3.26}$$

Finally we can construct a sequence of functions w_n defined as in (3.22) where we replace N with M_n. By taking (3.26) into account we easily get

$$|\{x : w_n(x) \neq u(x)\}| \leq \lambda_n M_n \leq \frac{c|z|}{T_n}.$$

Then $w_n \to u$ in $L^\infty(a,b)$ and, by construction,

$$\limsup_n E_n(w_n) = G(z) = E_1(u).$$

The desired upper estimate follows then by standard arguments. □

3.2.3 *Potentials of Lennard Jones type*

We now treat the case of potentials with non-symmetric growth conditions, which still ensure weak-BV compactness of sequences with equibounded energies. These conditions are satisfied for example by Lennard Jones potentials.

Theorem 3.9 *Let $\psi_n : \mathbf{R} \to [0, +\infty]$ satisfy*

$$\psi_n(z) \geq (|z|^p - 1) \qquad \text{for all } z < 0. \tag{3.27}$$

for some $p > 1$. For all $n \in \mathbf{N}$ let $T_n \in \mathbf{R}$ satisfy

$$\lim_n T_n = +\infty, \qquad \lim_n \lambda_n T_n = 0, \tag{3.28}$$

and let $F_n, G_n : \mathbf{R} \to [0, +\infty]$ be defined by

$$F_n(z) = \begin{cases} \psi_n(z) & z \leq T_n \\ \\ +\infty & z > T_n^+ \end{cases} \tag{3.29}$$

$$G_n(z) = \begin{cases} \lambda_n \psi_n \left(\dfrac{z}{\lambda_n} \right) & \text{if } z > \lambda_n T_n \\ \\ +\infty & \text{otherwise.} \end{cases} \tag{3.30}$$

Assume that there exist $F, G : \mathbf{R} \to [0, +\infty]$ such that

$$\Gamma\text{-}\lim_n F_n^{**} = F \text{ on } \mathbf{R}, \tag{3.31}$$

$$\Gamma\text{-}\lim_n \mathrm{sub}^- G_n = G \text{ on } \mathbf{R} \setminus \{0\}. \tag{3.32}$$

For notation's convenience we set $G(0) = 0$. Then, $(E_n)_n$ Γ-converges to E with respect to the convergence in $L^1_{\rm loc}(a, b)$ and the convergence in measure, where

$$E(u) = \begin{cases} \displaystyle\int_a^b \overline{F}(\dot{u})\, dx + \sum_{S(u)} \overline{G}([u]) + \sigma D u_c^+(a, b) & \text{if } u \in BV_{\rm loc}(a, b)\ D_c u^- = 0\ , \\ & \text{and } [u] > 0 \text{ on } S(u), \\ \\ +\infty & \text{otherwise in } L^1(a, b). \end{cases}$$

where $\overline{F}$ and $\overline{G}$ are defined as in Theorem 3.7 and $\sigma := \overline{F}^\infty(1)$.

Proof Let $u_n \to u$ in measure and be such that $E_n(u_n) \leq c$, and assume that $u_n \to u$ also pointwise. Set

$$I_n = \{ i \in \{1, \ldots, N_n\} :\ u_n(x_n^{i+1}) - u_n(x_n^i) > \lambda_n T_n \},$$

and let v_n be the sequence of functions defined as in (3.12) with this choice of I_n. Note that $v_n \to u$ in measure. By taking hypothesis (3.27) into account we have the following estimate on the negative part of the (classical) derivative of v_n

$$\int_a^b |(\dot{v}_n)^-|^p\,dt \le \sum_{i \notin I_n} \lambda_n \Big(\frac{(u_n(x_n^{i+1}) - u_n(x_n^i))^-}{\lambda_n}\Big)^p \le \frac{E_n(u_n)}{\alpha} + c.$$

Hence, with fixed $\delta > 0$ and with fixed x_1, x_2 points in $(a, a+\delta)$, $(b-\delta, b)$, respectively, in which v_n converges pointwise to u, we get

$$\Big|\int_{a+\delta}^{b-\delta} (\dot{v}_n)^+\,dt + \sum_{S(v_n)\cap(a+\delta,b-\delta)} [v_n]\Big| \le |v_n(x_2) - v_n(x_1)| + \int_a^b (\dot{v}_n)^-\,dt.$$

It follows that v_n is bounded in $BV_{\rm loc}(a,b)$. Since $v_n \to u$ in measure we get that v_n converges in $BV_{\rm loc}(a,b)$ to u and hence $u \in BV_{\rm loc}(a,b)$. With fixed $\eta > 0$, consider

$$F_n^\eta(z) = (F_n)^{**}(z) + \eta|z|, \qquad G_n^\eta(z) = {\rm sub}^- G_n(z) + \eta|z|.$$

For every $\delta \in (0, (b-a)/2)$ we have

$$E_n(u_n) + \eta c(\delta) \ge \int_{a+\delta}^{b-\delta} F_n^\eta(\dot{v}_n)\,dt + \sum_{S(v_n)\cap(a+\delta,b-\delta)} G_n^\eta([v_n]),$$

where $c(\delta) = \sup_n |Dv_n|(a+\delta, b-\delta)$. We can apply Theorem 3.1 and obtain for every η and δ

$$\begin{aligned}&\liminf_n E_n(u_n) + \eta c(\delta)\\ &\ge \int_{a+\delta}^{b-\delta} \overline{F}(\dot{u})\,dt + \overline{F}^\infty(1)|D_c u^+|(a+\delta, b-\delta) + \sum_{S(u)\cap(a+\delta,b-\delta)} \overline{G}([u]).\end{aligned}$$

By letting $\eta \to 0$ and subsequently $\delta \to 0$ we obtain the desired inequality.

The construction of a recovery sequence for the Γ-limsup follows the same procedure as in the proof of Theorem 3.7. □

3.2.4 *Examples*

Example 3.10 (i) The typical example of a sequence of functions which satisfy the hypotheses of Theorem 3.3 (and indeed of Corollary 3.5) is given (fixed (λ_n) converging to 0 and $C > 0$) by

$$\psi_n(z) = \frac{1}{\lambda_n}\big((\lambda_n z^2) \wedge C\big),$$

with $p = 2$, $T_n = \sqrt{C/\lambda_n}$,

$$F_n(z) = \begin{cases} z^2 & |z| \le \sqrt{C/\lambda_n} \\ +\infty & \text{otherwise}, \end{cases} \qquad G_n(z) = \begin{cases} C & |z| > \sqrt{C/\lambda_n} \\ +\infty & |z| \le \sqrt{C/\lambda_n}, \end{cases}$$

so that

$$E(u) = \int_a^b |\dot{u}|^2\,dt + C\#(S(u))$$

on $SBV(a,b)$. Discrete energies of this form have been proposed by Blake and Zisserman.

(ii) Theorem 3.3 allows also to treat asymmetric cases. As an example, let

$$\psi_n(z) = \begin{cases} \frac{1}{\lambda_n}\big((\lambda_n z^2) \wedge C\big) & \text{if } z > 0 \\ \\ z^2 & \text{if } z \le 0. \end{cases}$$

In this case the Γ-limit (with respect to both the convergence in measure and L^1 convergence) is given by

$$E(u) = \begin{cases} \displaystyle\int_a^b |\dot{u}|^2\,dt + C\#(S(u)) & \text{if } u \in SBV(a,b) \text{ and } u^+ > u^- \text{ on } S(u) \\ \\ +\infty & \text{otherwise.} \end{cases}$$

Note that

$$G(z) = \begin{cases} C & \text{if } z > 0 \\ 0 & \text{if } z = 0 \\ +\infty & \text{if } z < 0 \end{cases}$$

forbids negative jumps.

(iii) (Lennard Jones type potentials) Let $\psi : \mathbf{R} \to [0,+\infty]$ be a lower semicontinuous function and satisfy $\psi(z) = 0$ if and only if $z = 0$, $\psi(z) \ge \alpha(|z|^p - 1)$ for $z < 0$ and $\lim_{z\to+\infty} \psi(z) = C$. Let $\psi_n = \psi$ for all n. Then we can apply Theorem 3.9 and obtain $F = \psi^{**}$ and

$$G(z) = \begin{cases} 0 & \text{if } z \ge 0 \\ +\infty & \text{if } z < 0. \end{cases}$$

Note that $F(z) = 0$ if $z \ge 0$.

(iv) (scaled Lennard Jones type potentials) Let ψ be as in the previous example, and choose

$$\psi_n(z) = \frac{1}{\lambda_n}\psi(z).$$

Then we can apply Theorem 3.3 with

$$F(z) = \begin{cases} 0 & \text{if } z = 0 \\ +\infty & \text{otherwise}, \end{cases}$$

and G as in Example (ii) above. In this case the limit energy E is finite only on piecewise-constant functions with a finite number of positive jumps. On such functions $E(u) = C\#(S(u))$.

We now give an example which illustrates the effect of the operation of the subadditive envelope.

Example 3.11 If we take

$$\psi_n(z) = z^2 \wedge \Big(\frac{1}{\lambda_n} + (|z|\sqrt{\lambda}_n - 1)^2\Big)$$

with λ_n converging to 0, then we obtain $F(z) = z^2$ and

$$G(z) = \mathrm{sub}^-(1+z^2) = \min\Big\{k + \frac{z^2}{k} : \ k = 1, 2, \dots\Big\}.$$

3.2.5 *A remark on second-neighbour interactions*

Consider functionals of the form

$$E_n(u) = \sum_i \lambda_n \psi_n^1\Big(\frac{u(x_n^{i+1}) - u(x_n^i)}{\lambda_n}\Big) + \sum_i 2\lambda_n \psi_n^2\Big(\frac{u(x_n^{i+2}) - u(x_n^i)}{2\lambda_n}\Big). \tag{3.33}$$

If both sequences of functions $(\psi_n^i)_n$ satisfy conditions of Corollary 3.5 and some additional growth conditions from above, then it can be seen that the conclusions of Theorem 3.3 hold with

$$F(z) = \lim_n \Big(\psi_n^1(z) + 2\psi_n^2(z)\Big),$$

and

$$G(z) = \lim_n \lambda_n \Big(\psi_n^1\Big(\frac{z}{\lambda_n}\Big) + 4\psi_n^2\Big(\frac{z}{2\lambda_n}\Big)\Big).$$

This means that E_n can be decomposed as the sum of three 'nearest-neighbour type' functionals, with underlying lattices $\lambda_n\mathbf{Z}$, $2\lambda_n\mathbf{Z}$ and $\lambda_n(2\mathbf{Z}+1)$, respectively, whose Γ-convergence can be studied separately. We now show that a similar conclusion does not hold if we remove the convexity/concavity hypothesis on ψ_n^i.

Example 3.12 Let (λ_n) be a sequence of positive numbers converging to 0, and let $M > 2$ be fixed. Let E_n be given by (3.33) with

$$\psi_n^k(z) = \begin{cases} z^2 & \text{if } |z| \le 1/\sqrt{k\lambda_n} \\ \frac{1}{k\lambda_n} G^k\Big(k\lambda_n z\Big) & \text{if } |z| > 1/\sqrt{k\lambda_n} \end{cases}$$

$(k = 1, 2)$, where

$$G^1(z) = \begin{cases} M & \text{if } |z| < 8 \\ 1 & \text{if } |z| \ge 8 \end{cases} \qquad G^2(z) = \begin{cases} 1 & \text{if } |z| \le 1 \\ M & \text{if } |z| > 1. \end{cases}$$

Neither G^i is subadditive and we have

$$\text{sub}^{-} G^1(z) = \begin{cases} 2 & \text{if } |z| < 8 \\ 1 & \text{if } |z| \geq 8 \end{cases} \qquad \text{sub}^{-} G^2(z) = \begin{cases} 1 & \text{if } |z| \leq 1 \\ 2 & \text{if } |z| > 1. \end{cases}$$

We can view E_n as the sum of a first-neighbour interaction functional and two second-neighbour interaction functionals, to whom we can apply separately Theorem 3.3, obtaining the limit functionals

$$E^1(u) = \int_a^b |\dot{u}|^2 \, dt + \sum_{S(u)} \text{sub}^{-} G^1([u])$$

for the first, and

$$E^2(u) = \int_a^b |\dot{u}|^2 \, dt + \sum_{S(u)} \text{sub}^{-} G^2([u])$$

for each of the second ones. We will show that the Γ-limit of E_n is strictly greater than $E^1(u) + 2E^2(u)$ at some $u \in SBV(a,b)$.

Let u be given simply by $u = \chi_{(t_0,b)}$ with $t_0 \in (a,b)$. In this case $E^1(u) + 2E^2(u) = 4$. Suppose that there exist $u_n \in \mathcal{A}_n(a,b)$ converging to u and such that $\limsup_n E_n(u_n) \leq 4$. In this case it can be easily seen that for n large enough there must exist i_n such that

$$u_n(x^{i_n}) - u_n(x^{i_n - 1}) > 4, \qquad u_n(x^{i_n+1}) - u_n(x^{i_n}) < -4,$$

but

$$|u_n(x^{i_n-1}) - u_n(x^{i_n-2})| < 1, \qquad |u_n(x^{i_n+2}) - u_n(x^{i_n+1})| < 1.$$

This implies that

$$u_n(x^{i_n}) - u_n(x^{i_n-2}) > 3, \qquad u_n(x^{i_n+2}) - u_n(x^{i_n}) < -3,$$

so that $\limsup_n E_n(u_n) \geq 2M$, which gives a contradiction.

3.3 Long-range interactions

We conclude this chapter with a general statement whose proof can be obtained by carefully using the arguments of Section 1.4.3 and of the previous sections in this chapter. We use the notation of Chapter 1.

Let $K \in \mathbf{N}$ be fixed. For all $n \in \mathbf{N}$ and $j \in \{1, \ldots, K\}$ let $\psi_n^j : \mathbf{R} \to (-\infty, +\infty]$ be given Borel functions bounded below. Define $E_n : L^1(0,L) \to [0,+\infty]$ as

$$E_n(u) = \begin{cases} \displaystyle\sum_{j=1}^{K} \sum_{i=0}^{n-j} \lambda_n \psi_n^j \left(\frac{u(x_n^{i+j}) - u(x_n^i)}{j\lambda_n} \right) & x \in \mathcal{A}_n(0,L) \\ +\infty & \text{otherwise in } L^1(0,L). \end{cases} \tag{3.34}$$

We will describe the asymptotic behaviour of E_n as $n \to +\infty$ when the energy densities are potentials of Lennard Jones type. More precisely, we will make the following assumptions.

(H1) (*growth conditions*) There exists a convex function $\Psi : \mathbf{R} \to [0, +\infty]$ and $p > 1$ such that

$$\lim_{z \to -\infty} \frac{\Psi(z)}{|z|} = +\infty$$

and there exist constants c_j^1, $c_j^2 > 0$ such that

$$c_j^1(\Psi(z) - 1) \leq \psi_n^j(z) \leq c_j^2 \max\{\Psi(z), |z|\}$$

for all $z \in \mathbf{R}$.

Remark 3.13 Hypothesis (H1) is designed to cover the case of Lennard Jones potentials (and potential of the same shape. Another case included in hypotheses (H1) is when all functions satisfy a uniform growth condition of order $p > 1$; i.e.,

$$(|z|^p - 1) \leq \psi_n^j(z) \leq c(|z|^p + 1)$$

for all j and n.

Before stating our main result, we have to introduce the counterpart of the energy densities F_n and G_n in the previous section for the case $K > 1$. The idea is roughly speaking to consider clusters of N subsequent points (N large) and define an average discrete energy for each of those clusters, so that the energy E_n may be approximately regarded as a 'nearest neighbour interaction energy' acting between such clusters, to which the above description applies.

We fix a sequence (N_n) of natural numbers with the property

$$\lim_n N_n = +\infty, \qquad \lim_n \frac{N_n}{n} = 0. \tag{3.35}$$

We define

$$\psi_n(z) = \min\Big\{ \frac{1}{N_n} \sum_{j=1}^{K} \sum_{i=0}^{N_n - j} \psi_n^j\Big(\frac{u(i+j) - u(i)}{j}\Big) : \ u : \{0, \dots, N_n\} \to \mathbf{R},$$
$$u(x) = zx \text{ if } x = 0, \dots, K, N_n - K, \dots, N_n \Big\}. \tag{3.36}$$

By using the energies ψ_n we will regard a system of N_n neighbouring points as a single interaction between the two extremal ones, up to a little error which is negligible as $N_n \to +\infty$. We can now state our convergence result, whose thesis is exactly the same as that of Theorem 3.9 upon replacing λ_n by $\varepsilon_n := N_n \lambda_n$.

Theorem 3.14 *Let ψ_n^j satisfy* (H1) *and let (E_n) be given by* (3.34)*. Let ψ_n be given by* (3.36) *and let $\varepsilon_n = N_n\lambda_n$. For all $n \in \mathbf{N}$ let $T_n \in \mathbf{R}$ be defined as in* (3.28)*, and let $F_n, G_n : \mathbf{R} \to [0, +\infty]$ be defined by*

$$F_n(z) = \begin{cases} \psi_n(z) & z \leq T_n \\ +\infty & z > T_n \end{cases} \tag{3.37}$$

$$G_n(z) = \begin{cases} \varepsilon_n \psi_n\left(\dfrac{z}{\varepsilon_n}\right) & \textit{if } z > \varepsilon_n T_n \\ +\infty & \textit{otherwise.} \end{cases} \tag{3.38}$$

Assume that there exist $F, G : \mathbf{R} \to [0, +\infty]$ such that

$$\Gamma\text{-}\lim_n F_n^{**} = F \ \textit{ on } \mathbf{R}, \tag{3.39}$$

$$\Gamma\text{-}\lim_n \mathrm{sub}^- G_n = G \ \textit{on } \mathbf{R} \setminus \{0\}. \tag{3.40}$$

Note that this assumption is always satisfied, upon extracting a subsequence. Then, $(E_n)_n$ Γ-converges to E with respect to the convergence in $L^1_{\mathrm{loc}}(0, L)$ and the convergence in measure, where

$$E(u) = \begin{cases} \displaystyle\int_0^L \overline{F}(\dot{u})\,dx + \sum_{S(u)} \overline{G}([u]) + \sigma D u_c^+(0, L) & \textit{if } u \in BV_{\mathrm{loc}}(0, L) \ D_c u^- = 0\ , \\ & \textit{and } [u] > 0 \ \textit{on } S(u), \\ +\infty & \textit{otherwise in } L^1(0, L). \end{cases}$$

where $\overline{F}$ and $\overline{G}$ are defined by (for notational convenience we set $G(0) = 0$)

$$\overline{F}(z) := \inf\{F(z_1) + G^0(z_2) : z_1 + z_2 = z\},$$

$$\overline{G}(z) := \inf\{F^\infty(z_1) + G(z_2) : z_1 + z_2 = z\},$$

and $\sigma := \overline{F}^\infty(1)$.

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