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G-continued fractions for basic hypergeometric functions II

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Abstract

In this paper we apply a modification of a generalized Pringsheim's theorem to obtain a G-continued fraction expansion for the quotient of two contiguous basic hypergeometric functions in arbitrarily many variables. As an application we obtain a G-continued fraction extension of the Rogers–Ramanujan continued fraction.

1. Introduction

In [1,2,5] the problem of expanding basic hypergeometric functions into continued fractions was studied. The goal of [1] was to maximize the number of free parameters in a continued fraction which extended the famous Rogers–Ramanujan continued fraction as well as the Rogers–Ramanujan identities. One problem is that as one adds free parameters to basic hypergeometric functions, the order of the difference equations they satisfy increases and soon it is not possible to find the second-order recurrences necessary for continued fractions.

In the previous paper [4] we applied a generalization of the continued fraction process which arises from higher-order recurrences. This generalization is known as a G-continued

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fraction [10]. In this paper using q-difference equations we find a G-continued fraction expansion for the quotient of two basic hypergeometric functions in arbitrarily many variables. In [4] we gave a similar expansion, but here the functions involved are different as is the method to prove convergence. For convergence in [4] we used the theorem of Zahar [14] and the fact that for the basic hypergeometric functions considered, the full vector space of solutions of their q-difference equations is easy to write down. The theorem of Zahar [14] extends the theorem of Pincherle [11] for ordinary continued fractions. However, when the full vector space of solutions of the q-difference equation is not known, or is difficult to describe, the theorem of Zahar [14] can be hard to apply. An example of this for ordinary continued fractions is the q-difference equation which generates the famous Rogers–Ramanujan continued fraction. In this paper we consider the general case using a different method to obtain convergence results. To approach the general case we use a modification of a generalization of Pringsheim's theorem due to Levrie [7]. In [7] the theory on infinite systems of equations, developed by Kantorovich and Krylov [6], was used to prove a generalization of Pringsheim's theorem. Using a variant of this method, we obtain convergence for an explicit G-continued fraction which represents the quotient of two contiguous basic hypergeometric series. As an application of this result, we obtain an infinite family of G-continued fractions generalizing the famous Rogers–Ramanujan continued fraction. See Theorem 7 and its corollaries for this result.

First we review basic results and prove the variant of Levrie's theorem which we use to derive our G-continued fraction expansion.

2. Definition and notation

We begin with the definition of a G-continued fraction. Consider the mth-order linear homogeneous recurrence relation of the form

$$\sum_{i=0}^{m} a_i(n) y_{n+m-i} = 0, \quad a_0(n)a_m(n) \neq 0, \quad n = 0, 1, \ldots. \tag{2.1}$$

Here the $a_i(n)$, $0 \leqslant i \leqslant m$, are given sequences of complex constants. Define the following transformations s_n and S_n from $\mathbb{C}^{m-1}$ into $\mathbb{C} \cup \{\infty\}$ iteratively in terms of the sequences $a_i(n)$ by the following equations:

$$\begin{aligned} &s_n(w_1, \ldots, w_{m-1}) \\ &\quad = \frac{-a_m(n)}{a_{m-1}(n) + a_{m-2}(n)w_1 + a_{m-3}(n)w_1w_2 + \cdots + a_0(n)w_1w_2\ldots w_{m-1}}, \\ &S_1(w_1, \ldots, w_{m-1}) = s_1(w_1, \ldots, w_{m-1}), \\ &S_n(w_1, \ldots, w_{m-1}) = S_{n-1}\big(s_n(w_1, \ldots, w_{m-1}), w_1, \ldots, w_{m-2}\big), \end{aligned}$$

for $n \geqslant 1$, and $f_n = S_n(0, 0, \ldots, 0)$. Then f_n is called the $(n+1)$st approximant of the G-continued fraction

$$\mathbf{K}_{n=0}^{\infty}\left[\frac{-a_m(n)}{a_{m-1}(n); \ldots; a_0(n)}\right].$$

If $f = \lim_{n\to\infty} f_n$ exists, then the G-continued fraction is said to converge to this limiting value and we write

$$\mathbf{K}_{n=0}^{\infty}\left[\frac{-a_m(n)}{a_{m-1}(n);\ldots;a_0(n)}\right] = f.$$

3. Infinite systems

In this section we review the basics of infinite systems from [6] to give a context for the next section. Recall the following definitions on infinite system of equations. The system of equations

$$\begin{aligned} &a_{0,0}x_0 + a_{0,1}x_1 + \cdots = b_0,\\ &a_{1,0}x_0 + a_{1,1}x_1 + \cdots = b_1,\\ &\quad\vdots \end{aligned}$$

is called an infinite system of linear equations in an infinite set of unknowns. Here $a_{i,k}$ and b_i are complex numbers and the x_i are unknowns. Notice that by singling out x_i in the ith equation and transposing it to the left side, the given system can be written in the following form:

$$x_i = \sum_{k=0,\ k\neq i}^{\infty} c_{i,k}x_k + b_i, \quad i = 0, 1, \ldots. \tag{3.1}$$

Definition 1. The infinite system of equations (3.1) is said to have a solution x_i ($i = 0, 1, \ldots$) if on substituting these values in the equations we obtain convergent series and all the equations are satisfied.

Definition 2. The system

$$X_i = \sum_{k=0}^{\infty} C_{i,k}X_k + B_i, \quad i = 0, 1, \ldots, \tag{3.2}$$

is called *majorant* for the system (3.1) if the following inequalities hold:

$$|c_{i,k}| < C_{i,k} \quad \text{and} \quad |b_i| < B_i, \quad i = 0, 1, \ldots.$$

An infinite system (3.1) is called a *regular system* if

$$\sum_{k=0}^{\infty} |c_{i,k}| < 1, \quad i = 0, 1, \ldots,$$

and is said to be (3.1) *fully regular* if the sum of the moduli of the coefficients of each row does not exceed a constant number less than unity:

$$\sum_{k=0}^{\infty} |c_{i,k}| \leqslant 1 - \epsilon < 1, \quad i = 0, 1, \ldots,\ \epsilon > 0.$$

The solutions x_i^* and X_i^* of systems (3.1) and obtained by the successive approximations with zero initial values are called *principal solutions* of these systems. The successive approximation method is defined by the following equations:

$$x_i^{(0)} = 0, \quad i = 0, 1, \ldots,$$

$$x_i^{(n+1)} = \sum_{k=0}^{\infty} c_{i,k} x_k^{(n)} + b_i, \quad i, n = 0, 1, \ldots,$$

$$\lim_{n\to\infty} x_i^{(n)} = x_i^*, \quad i = 0, 1, \ldots.$$

In [6] the following theorems were proved which are used in [7] to derive a generalization of Pringsheim's theorem.

Let

$$\rho_i = 1 - \sum_{k=0}^{\infty} |c_{i,k}| \; (> \epsilon) \quad (i = 0, 1, \ldots).$$

Suppose that a system (3.1) is fully regular and satisfies

$$|b_i| < M\rho_i, \quad M > 0. \tag{3.3}$$

Then the system

$$X_i = \sum_{k=0}^{\infty} |c_{i,k}| X_k + M\rho_i \tag{3.4}$$

is majorant system for system (3.1). It is clear that the system (3.4) has a positive solution $X_i = M > 0$.

Theorem 1 [6]. *A fully regular system* (3.1) *with the condition* (3.3) *has a unique bounded solution* x_i $(i = 0, 1, \ldots)$. *Moreover, the principal solution of the system of equations* (3.4) X_i^* *is* $X_i^* = M$.

Theorem 2 [6]. *Let a system* (3.1) *be regular and satisfy* (3.3). *Then the principal solution* x_i^* *of an infinite system of equations can be found by the method of reduction, i.e., if* $x_{i,N}$ *is the principal solution of the finite system*

$$x_i = \sum_{k=0}^{N} c_{i,k} x_k + b_i, \quad i = 0, 1, \ldots, N, \tag{3.5}$$

then $x_i^* = \lim_{N\to\infty} x_{i,N}$.

Observe that the principal solution x_i^* which may be found by the method of reduction is bounded by M.

4. *G*-continued fractions in degenerate cases

Let $a_i(n)$ be functions $g_i(zq^n)$. Here we consider following recurrence relation:

$$\sum_{i=0}^{m} g_i(zq^n) f(zq^{n+m-i}) = 0, \quad n = 0, 1, \ldots, \tag{4.1}$$

where $g_0 g_m$ is not identically zero. We apply Theorems 1 and 2 to obtain a G-continued expansion of a quotient of q-series. As usual throughout we take $|q| < 1$.

Theorem 3. *Fix $\epsilon > 0$ and let $D = \{z \in \mathbb{C}\colon |z| < r\}$. Assume for $z \in D$ and for $n = 0, 1, \ldots,$*

$$\frac{\sum_{i=0,\ i\neq m-1}^{m} |g_i(zq^n)|}{|g_{m-1}(zq^n)|} \leqslant 1 - \epsilon. \tag{4.2}$$

Suppose that there is a bounded function $f(z)$ in D so that $f(zq^i)$ $(i = 0, 1, \ldots)$ satisfies (4.1)*. Then $f(zq^i)$ $(i = 0, 1, \ldots)$ is the only bounded solution for each $z \in D$ which satisfies* (4.1) *and the G-continued fraction*

$$\mathbf{K}_{n=0}^{\infty}\left[\frac{-g_m(zq^n)}{g_{m-1}(zq^n); \ldots; g_0(zq^n)}\right]$$

converges to $f(zq)/f(z)$ for $z \in D$.

Proof. Fix $z \in D$ and $|q| < 1$. Let $w_i = f(zq^i)$ and write the recurrence relation (4.1) as

$$w_{i+1} = -\sum_{k=0,\ k\neq 1}^{m} \frac{g_{m-k}(zq^i)}{g_{m-1}(zq^i)} w_{i+k} \quad (i = 0, 1, \ldots). \tag{4.3}$$

By changing variables we can easily see that (4.3) can be written as

$$w_i = -\sum_{k=i-1,\ k\neq i}^{m+i-1} \frac{g_{m-1+i-k}(zq^{i-1})}{g_{m-1}(zq^{i-1})} w_k \quad (i = 1, 2, \ldots).$$

Let $g_i(zq^n) = 0$ for all $n > 0$ if either $i < 0$ or $i > m$ and put

$$c_{i,k} = -\frac{g_{m-1+i-k}(zq^{i-1})}{g_{m-1}(zq^{i-1})}.$$

It is clear that $c_{i,k} = 0$ if $k < i - 1$ or $k > m + i - 1$. So we can write the recurrence relation (4.1) as an infinite system of equations

$$w_i = \sum_{k=0,\ k\neq i}^{\infty} c_{i,k} w_k \quad (i = 1, 2, \ldots). \tag{4.4}$$

Using the inequality (4.2), one can easily see that the above system is fully regular. Since $b_i = 0$ for all i, $|b_i| < M\rho_i$ for some $M > 0$. Since the infinite system (4.4) satisfies all the requirements in Theorem 1, there is a unique bounded solution w_i $(i = 0, 1, \ldots)$ for each $z \in D$. Since $f(z)$ is bounded in D and satisfies (4.1), it is evident that $f(zq^i)$ $(i = 0, 1, \ldots)$

is a bounded solution of (4.4) for each $z \in D$. Hence $f(zq^i)$ $(i = 0, 1, \ldots)$ is the unique bounded solution for each $z \in D$ which fulfills (4.1).

Let N be a fixed nonnegative integer. Define

$$y^N_{N+i} = 0, \quad i = 2, \ldots, m, \tag{4.5}$$

$$y^N_{N+1} = 1, \tag{4.6}$$

$$y^N_n = -\sum_{i=0}^{m-1} \frac{g_i(zq^n)}{g_m(zq^n)} y^N_{n+m-i}, \quad n = 0, \ldots, N, \tag{4.7}$$

and

$$T^N(n) = \begin{cases} \frac{y^N_{n+1}}{y^N_n}, & \text{if } n = 0, \ldots, N, \\ 0, & \text{if } n = N+1, \ldots, N+m-1. \end{cases} \tag{4.8}$$

Here, (4.7) is defined backward, that is, (4.7) is defined from $n = N$ to 0. As an immediate consequence of the definition we have

$$S_N(0, \ldots, 0) = T^N(0).$$

Since y^N_n $(n = 0, 1, \ldots, N+2)$ satisfies the recurrence relation (4.1), y^N_n can be considered as a solution of the finite system

$$y^N_n = \sum_{k=0,\ k\neq n}^{N} c_{n,k} y^N_k \quad (n = 0, 1, \ldots, N).$$

Thus $\lim_{N\to\infty} y^N_n$ is bounded for all $n = 0, 1, \ldots$ and is a solution of infinite system (4.4) by Theorem 2. Since the bounded solution of (4.4) is unique, $\lim_{N\to\infty} y^N_n = f(zq^n)$. Hence

$$\lim_{N\to\infty} T^N(n) = \lim_{N\to\infty} \frac{y^N_{n+1}}{y^N_n} = \frac{f(zq^{n+1})}{f(zq^n)}.$$

From (4.5)–(4.8), we have for N sufficiently large and $n = 0, 1, \ldots,$

$$T^N(n) = \frac{-g_m(n)}{g_{m-1}(n)+g_{m-2}(n)T^N(n+1)+\cdots+g_{m-i}(n)\prod_{j=1}^{n+i-1}T^N(n+j)+\cdots+g_0(n)\prod_{j=1}^{n+m-1}T^N(n+j)}. \tag{4.9}$$

By repeated application of (4.9), $T^N(0)$ converges to $f(zq)/f(z)$. Therefore the G-continued fraction converges to $f(zq)/f(z)$. □

Theorem 3 will be applied to the basic hypergeometric series by making use of the theorem of Thomae [13], which states that the analytic solutions at the origin (analytic up to a multiple of z^γ, $\gamma \in \mathbb{R}$) of the q-difference equation

$$(\beta_0 - \alpha_0 zq^n) f(zq^n) + (\beta_1 - \alpha_1 zq^n) f(zq^{n+1}) + \cdots + (\beta_m - \alpha_m zq^n) f(zq^{n+m}) = 0 \tag{4.10}$$

are given for $i = 1, 2, \ldots, k - l$ by

$$f_i(zq^n) = (zq^n)^{\gamma_i} \sum_{j \geqslant 0} \frac{(c_1 d_i^{-1})_j \ldots (c_{k'-l'} d_i^{-1})_j}{(d_1 q d_i^{-1})_j \ldots (d_{k-l} q d_i^{-1})_j} q^{\binom{j}{2}(l'-l)} \left(\frac{\alpha_{l'} d_i^l}{\beta_l q^l d_i^{l'}} \right)^j (zq^n)^j, \tag{4.11}$$

where $|q| < 1$, $q^{-\gamma_i} = d_i$, $(a)_0 = 1$,

$$(a)_k = (1-a)(1-aq)\ldots(1-aq^{k-1}),$$

$$\sum_{i=0}^{m} \alpha_i z^i = \alpha_{l'} z^{l'} (1-c_1 z)\ldots(1-c_{k'-l'} z), \tag{4.12}$$

and

$$\sum_{i=0}^{m} \beta_i z^i = \beta_l z^l (1-d_1 z)\ldots(1-d_{k-l} z). \tag{4.13}$$

Equation (4.10) is called the canonical q-difference equation for the functions defined by (4.11). Here we consider the special case ($\beta_0 = 1$, $\alpha_0 = 0$) of (4.11):

$$f_i(zq^n) = (zq^n)^{\gamma_i} \sum_{j \geqslant 0} \frac{(c_1 d_i^{-1})_j \ldots (c_{k'-l'} d_i^{-1})_j}{(d_1 q d_i^{-1})_j \ldots (d_k q d_i^{-1})_j} q^{\binom{j}{2} l'} \left(\frac{\alpha_{l'}}{d_i^{l'}} \right)^j (zq^n)^j \quad \text{for all } z \in \mathbb{C}, \tag{4.14}$$

where $|q| < 1$, $q^{-\gamma_i} = d_i$ and c_i, d_i are zeroes of Eqs. (4.12) and (4.13). From (4.10), $f_i(z)$ $(i = 1, \ldots, k)$ satisfies

$$f(zq^n) + (\beta_1 - \alpha_1 zq^n) f(zq^{n+1}) + \cdots + (\beta_m - \alpha_m zq^n) f(zq^{n+m}) = 0. \tag{4.15}$$

Here, unlike in [4], the f_i are not necessarily distinct functions. Let $\epsilon > 0$ be fixed. Assume

$$(1-\epsilon)|\beta_1| > 1 + \sum_{i=2}^{m} |\beta_i|. \tag{4.16}$$

Let $g(z) = \beta_1 z$ and $h(z) = 1 + \beta_2 z^2 + \cdots + \beta_m z^m$. Then by (4.16), $|h(z)| < |g(z)|$ on $|z| = 1$. Applying Rouche's theorem to $g(z)$ and $g(z) + h(z)$, we can say that $g(z) + h(z)$ has the same number of zeros as $g(z)$ inside $|z| = 1$. Since $g(z)$ has only one zero inside the unit disk, so does $g(z) + h(z)$. It is easy to see that $|g(z) + h(z)| > 0$ on $|z| = 1$. Hence there is only one d_i such that $|d_i| > 1$ and the others are inside the unit disk. Since $|q^{-\gamma_i}| = |d_i|$ and $|q| < 1$, z^{γ_i} is bounded in a domain containing 0 if and only if $|d_i| \geqslant 1$. Let d be the one with $|d| > 1$ and $f(zq^n)$ be the solution corresponding to d.

In this situation we have the following theorem.

Theorem 4. *Assume* (4.16). *Then the G-continued fraction*

$$\mathbf{K}_{n=0}^{\infty} \left[\frac{-1}{(\beta_1 - \alpha_1 zq^n); \ldots; (\beta_{m-1} - \alpha_{m-1} zq^n); (\beta_m - \alpha_m zq^n)} \right] \tag{4.17}$$

converges to $f(zq)/f(z)$ *for each* $z \in \mathbb{C}$.

Proof. Note that $f(zq^n) = (zq^n)^\gamma f_1(zq^n)$ with $|q^{-\gamma}| = |d| > 1$ and f_1 analytic. So $f(zq^n)$ $(n = 0, 1, \ldots)$ is bounded for each $z \in D$, where

$$D = \{z \in \mathbb{C}\colon\ |z| < R\}, \quad R > 1.$$

First we observe there is a integer $N \geqslant 0$ such that for all $n \geqslant N$ and for each fixed $z \in D$,

$$zq^n\left((1-\epsilon)|\alpha_1| + \sum_{i=2}^{m} |\alpha_i|\right) \leqslant (1-\epsilon)|\beta_1| - \left(1 + \sum_{i=2}^{m} |\beta_i|\right).$$

Hence for each $z \in D$ and for all $n \geqslant N$,

$$(1-\epsilon)|\beta_1 - \alpha_1 zq^n| \geqslant 1 + \sum_{i=2}^{m} |\beta_i - \alpha_i zq^n|, \quad n = 0, 1, 2, \ldots. \tag{4.18}$$

By Theorem 3, for each $z \in D$,

$$\mathbf{K}_{n=N}^{\infty}\left[\frac{-1}{(\beta_1 - \alpha_1 zq^n); \ldots; (\beta_{m-1} - \alpha_{m-1} zq^n); (\beta_m - \alpha_m zq^n)}\right] = \frac{f(zq^{N+1})}{f(zq^N)}. \tag{4.19}$$

Since (4.19) holds for arbitrary R, (4.19) is true for $z \in \mathbb{C}$. Hence (4.17) converges to $f(zq)/f(z)$ for $z \in \mathbb{C}$. □

5. Application to a general Rogers–Ramanujan continued fraction

First, recall some results concerning the equivalent G-continued fractions given in [8]. We define for arbitrary solutions $x_n^{(1)}, x_n^{(2)}, \ldots, x_n^{(p)}$ of (2.1):

$$E_N\left(x_n^{(1)}, \ldots, x_n^{(p)}\right) := \begin{vmatrix} x_{N+p-m+1}^{(1)} & \cdots & x_{N+p-m+1}^{(m)} \\ \vdots & & \vdots \\ x_{N+p}^{(1)} & \cdots & x_{N+p}^{(m)} \end{vmatrix}.$$

Theorem 5 [8]. *The* $(N+1)$*st approximant of the* G*-continued fraction*

$$\mathbf{K}\left[\frac{-a_m(n)}{a_{m-1}(n); \ldots; a_0(n)}\right]$$

is given by

$$-\frac{E_N(A_n^{(1)}, A_n^{(3)}, \ldots, A_n^{(m)})}{E_N(A_n^{(2)}, A_n^{(3)}, \ldots, A_n^{(m)})},$$

where $A_n^{(i)}$ $(i = 1, \ldots, m)$ *are the solutions of* (2.1) *with initial values*

$$A_n^{(i)} = \delta_{n,i-1}, \quad n = 0, 1, \ldots, m-1. \tag{5.1}$$

Definition 3. The G-continued fractions

$$\mathbf{K}\left[\frac{-a_m(n)}{a_{m-1}(n);\ldots;a_0(n)}\right]$$

and

$$\mathbf{K}\left[\frac{-b_m(n)}{b_{m-1}(n);\ldots;b_0(n)}\right]$$

are said to be equivalent if they have the same approximants.

Thus equivalent G-continued fractions have the same limit if it exists.

Theorem 6 [8]. *If there exists a sequence of constants $\{r_n\}$ with $r_{-m+1}=1$, $r_n\neq 0$ $(n=-m+1,\ldots,\infty)$ such that*

$$b_i(n)=r_{n+1}r_n\ldots r_{n-i+1}a_i(n),\quad i=0,\ldots,m,$$

for all $n=0,1,\ldots,$ then the G-continued fractions

$$\mathbf{K}_{n=0}^{\infty}\left[\frac{-a_m(n)}{a_{m-1}(n);\ldots;a_0(n)}\right]$$

and

$$\mathbf{K}_{n=0}^{\infty}\left[\frac{-b_m(n)}{b_{m-1}(n);\ldots;b_0(n)}\right]$$

are equivalent.

Now consider the recurrence relation

$$f(zq^n)-f(zq^{n+1})-zq^{n+1}f(zq^{n+m})=0.$$

The G-continued fraction arising from the above recurrence relation is

$$\mathbf{K}_{n=0}^{\infty}\left[\frac{1}{1;0;\ldots;0;zq^{n+1}}\right].$$

Unfortunately, Theorem 4 does not immediately apply as the only d_i is $d_1=1$. This is remedied by applying Theorem 6 as follows. Let $b_m(0)=2^{-m}$ and for $n\geqslant 1$ put $b_m(n)=2^{-m-1}$. Also for $n\geqslant 0$ let $b_{m-1}(n)=2^{-m}$ and put $b_0(n)=2^{-1}zq^{n+1}$. Then the two G-continued fractions

$$\mathbf{K}_{n=0}^{\infty}\left[\frac{1}{1;0;\ldots;0;zq^{n+1}}\right]$$

and

$$\mathbf{K}_{n=0}^{\infty}\left[\frac{b_m(n)}{b_{m-1}(n);0;\ldots;0;b_0(n)}\right]$$

are equivalent by applying Theorem 10 with $r_n=1/2$ $(n\neq -m+1)$ and $r_{-m+1}=1$. Also note that

$$\mathbf{K}_{n=0}^{\infty}\left[\frac{b_m(n)}{b_{m-1}(n); 0; \ldots; 0; b_0(n)}\right] = \frac{2^{-m}}{2^{-m} + 2^{-1}zq \prod_{i=1}^{m-1} \mathbf{K}_{n=i}^{\infty}\left[\frac{2^{-m-1}}{2^{-m}; 0; \ldots; 2^{-1}(zq^{n+1})}\right]}. \tag{5.2}$$

The G-continued fraction

$$\mathbf{K}_{n=1}^{\infty}\left[\frac{2^{-m-1}}{2^{-m}; 0; \ldots; 2^{-1}(zq^{n+1})}\right]$$

is generated from the recurrence relation

$$-2^{-m-1} f(zq^n) + 2^{-m} f(zq^{n+1}) + 2^{-1} zq^{n+1} f(zq^{n+m}) = 0, \quad n = 1, 2, \ldots,$$

which simplifies to

$$f(zq^n) - 2f(zq^{n+1}) - 2^m zq^{n+1} f(zq^{n+m}) = 0, \quad n = 1, 2, \ldots, \tag{5.3}$$

and we find using (4.11) that $f(zq^n)$ is a solution of the recurrence relation (5.3), where

$$f(zq^n) = (zq^n)^{\gamma} \sum_{k \geqslant 0} \frac{q^{k(mk-m+2n+2)/2}}{(q)_k} z^k,$$

with $q^{\gamma} = 2^{-1}$. Clearly the coefficients of the recurrence relation (5.3) satisfies the inequality (4.16). Hence by Theorem 4 we can conclude that for any $l \geqslant 1$ and for $z \in \mathbb{C}$,

$$\mathbf{K}_{n=l}^{\infty}\left[\frac{2^{-m-1}}{2^{-m}; 0; \ldots; 0; 2^{-1}zq^{n+1}}\right] = \frac{f(zq^{l+1})}{f(zq^l)}. \tag{5.4}$$

Thus we have the following theorem.

Theorem 7. *For $z \in \mathbb{C}$,*

$$\mathbf{K}_{n=0}^{\infty}\left[\frac{1}{1; 0; \ldots; 0; zq^{n+1}}\right] = \frac{\sum_{k \geqslant 0} \frac{q^{k(mk-m+4)/2}}{(q)_k} z^k}{\sum_{k \geqslant 0} \frac{q^{k(mk-m+2)/2}}{(q)_k} z^k}. \tag{5.5}$$

Proof. By using (5.2) and (5.4), we have for all $z \in \mathbb{C}$,

$$\mathbf{K}_{n=0}^{\infty}\left[\frac{1}{1; 0; \ldots; 0; zq^{n+1}}\right] = \frac{2^{-m}}{2^{-m} + 2^{-1}zq \prod_{i=1}^{m-1} \frac{f(zq^{i+1})}{f(zq^i)}} = \frac{2^{-m}}{2^{-m} + 2^{-1}zq \frac{f(zq^m)}{f(zq)}}$$
$$= \frac{f(zq)}{f(zq) + 2^{m-1} zq f(zq^m)} = \frac{2f(zq)}{f(z)} = \frac{\sum_{k \geqslant 0} \frac{q^{k(mk-m+4)/2}}{(q)_k} z^k}{\sum_{k \geqslant 0} \frac{q^{k(mk-m+2)/2}}{(q)_k} z^k}. \quad \square$$

The formula (5.5) is a G-continued fraction generalizing the Rogers–Ramanujan continued fraction to which it reduces when $m = 2$. Corollary 1 can be found in [9,12] and the more general case is treated in [3].

Corollary 1 ([9,12] and Corollary to Entry 15 in [3]).

$$\frac{1}{1+}\,\frac{zq}{1+}\,\frac{zq^2}{1+}\,\frac{zq^3}{1+\cdots}=\frac{\sum_{k\geqslant 0}\frac{q^{k(k+1)}}{(q)_k}z^k}{\sum_{k\geqslant 0}\frac{q^{k^2}}{(q)_k}z^k}.$$

Proof. In Theorem 7, let $m=2$. □

The next corollary displays the form of a G-continued fraction when it is expanded out as an infinite process.

Corollary 2. *The G-continued fraction*

$$\frac{1}{1+zq\left(\frac{1}{1+zq^2\left(\frac{1}{1+zq^3+\cdots}\right)\left(\frac{1}{1+zq^4+\cdots}\right)}\right)\left(\frac{1}{1+zq^3\left(\frac{1}{1+zq^4+\cdots}\right)\left(\frac{1}{1+zq^5+\cdots}\right)}\right)}$$

converges to

$$\frac{\sum_{k\geqslant 0}\frac{q^{k(3k+1)/2}}{(q)_k}z^k}{\sum_{k\geqslant 0}\frac{q^{k(3k-1)/2}}{(q)_k}z^k}.$$

Proof. In Theorem 7, let $m=3$. The expansion given above is obtained by iterating the G-continued fraction algorithm. □

6. Conclusion

In this paper we have given the G-continued fraction for the quotients of contiguous basic hypergeometric functions obtained from the canonical q-difference equation which characterizes the functions. There are also noncanonical q-difference equations satisfied by these functions with certain parameters specialized. In a future paper we will consider the G-continued fractions resulting from these q-difference equations. In general, these equations are rather complicated, so we will consider those special cases which result in particularly nice G-continued fraction expansions. We also plan to consider number-theoretic applications.

References

[1] G. Andrews, D. Bowman, A full extension of the Rogers–Ramanujan continued fraction, Proc. Amer. Math. Soc. 123 (1995) 3343–3350.

[2] D. Bowman, Modified convergence for q-continued fraction defined by functional relations, Contemp. Math. 166 (1994) 155–165.

[3] B.C. Berndt, Ramanujan's Notebooks Part III, Springer-Verlag, New York, 1991.

[4] D. Bowman, G. Choi, G-continued fractions for basic hypergeometric functions, J. Math. Anal. Appl. 243 (2000) 338–343.

[5] D. Bowman, J. Sohn, Partial q-difference equations for basic hypergeometric functions and their q-continued fractions, J. Reine Angew. Math., under revision.
[6] L.V. Kantorovich, V.I. Krylov, Approximate Methods of Higher Analysis, Interscience, New York, 1958.
[7] P. Levrie, Pringsheim's theorem revisited, J. Comput. Appl. Math. 25 (1989) 93–104.
[8] P. Levrie, R. Piessens, Convergence accelerations for Miller's algorithm, in: A. Cuyt (Ed.), Nonlinear Numerical Methods and Rational Approximation, 1988, pp. 349–370.
[9] L.J. Rogers, Second expansion on the expansion of some infinite products, Proc. London Math. Soc. 25 (1894) 318–343.
[10] L. Lorentzen, H. Waadeland, Continued Fractions with Applications, North-Holland, Amsterdam, 1992.
[11] S. Pincherle, Delle funzioni ipergeometriche e di varie questioni ad esse attinenti, Giorn. Mat. Battaglini 32 (1894) 209–291.
[12] S. Ramanujan, Collected Papers, Chelsea, New York, 1962.
[13] J. Thomae, Les séries Heinéennes supérieures, ou les éries de la forme $1+\sum_{n=0}^{\infty} x^n \frac{1-q^a}{1-q}\frac{1-q^{a+1}}{1-q^2}\cdots\frac{1-q^{a+n-1}}{1-q^n}\frac{1-q^{a'}}{1-q^{b'}}\frac{1-q^{a'+1}}{1-q^{b'+1}}\cdots\frac{1-q^{a'+n-1}}{1-q^{b'+n-1}}\cdots\frac{1-q^{a(h)}}{1-q^{b(h)}}\frac{1-q^{a(h)+1}}{1-q^{b(h)+1}}\cdots\frac{1-q^{a(h)+n-1}}{1-q^{b(h)+n-1}}$, Ann. Mat. Pura Appl. 4 (1870) 105–138.
[14] R.V.M. Zahar, Computational algorithms for linear difference equations, Thesis, Purdue University (1968).