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BOUNDEDNESS OF GENERALIZED HIGHER COMMUTATORS OF MARCINKIEWICZ INTEGRALS *

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Abstract Let $\vec{b}=(b_1,\cdots,b_m)$ be a finite family of locally integrable functions. Then, we introduce generalized higher commutator of Marcinkiwicz integral as follows:

$$\mu_\Omega^{\vec{b}}(f)(x)=\left(\int_0^\infty |F_{\Omega,t}^{\vec{b}}(f)(x)|^2\frac{\mathrm{d}t}{t}\right)^{1/2},$$

where

$$F_{\Omega,t}^{\vec{b}}(f)(x)=\frac{1}{t}\int_{|x-y|\le t}\frac{\Omega(x-y)}{|x-y|^{n-1}}\prod_{j=1}^m(b_j(x)-b_j(y))f(y)\mathrm{d}y.$$

When $b_j\in\dot{\Lambda}_{\beta_j}, 1\le j\le m$, $0<\beta_j<1$, $\sum\limits_{j=1}^m\beta_j=\beta<n$, and Ω is homogeneous of degree zero and satisfies the cancelation condition, we prove that $\mu_\Omega^{\vec{b}}$ is bounded from $L^p(\mathbb{R}^n)$ to $L^s(\mathbb{R}^n)$, where $1<p<n/\beta$ and $1/s=1/p-\beta/n$. Moreover, if Ω also satisfies some L^q-Dini condition, then $\mu_\Omega^{\vec{b}}$ is bounded from $L^p(\mathbb{R}^n)$ to $\dot{F}_p^{\beta,\infty}(\mathbb{R}^n)$ and on certain Hardy spaces. The article extends some known results.

Key words Generalized higher commutator of Marcinkiewicz integral, Hardy space, Lipschitz space

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1 Introduction

Suppose that S^{n-1} is the unit sphere in $\mathbb{R}^n(n\ge 2)$ equipped with the normalized Lebesgue measure $\mathrm{d}\sigma=\mathrm{d}\sigma(x')$. Let $\Omega\in L^1(S^{n-1})$ be homogeneous of degree zero and satisfy the cancelation condition

$$\int_{S^{n-1}}\Omega(x')\mathrm{d}\sigma(x')=0, \tag{1.1}$$

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where $x' = x/|x|$ for any $x \neq 0$. Then the Marcinkiewicz integral operator of higher dimension is defined by

$$\mu_\Omega(f)(x) = \Big(\int_0^\infty |F_{\Omega,t}(f)(x)|^2 \frac{\mathrm{d}t}{t}\Big)^{1/2},$$

where

$$F_{\Omega,t}(f)(x) = \frac{1}{t}\int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y)\mathrm{d}y.$$

And if H is the Hilbert space $H = \{h : \|h\|_H = (\int_0^\infty |h(t)|^2 \frac{\mathrm{d}t}{t})^{1/2} < \infty\}$, then μ_Ω can be looked as the vector-valued function in H, that is,

$$\mu_\Omega(f)(x) = \Big(\int_0^\infty |F_{\Omega,t}(f)(x)|^2 \frac{\mathrm{d}t}{t}\Big)^{1/2} = \|F_{\Omega,t}(f)(x)\|_H.$$

It is well known that the operator μ_Ω was defined first by Stein in [1]. And he proved that if Ω is continuous and satisfies the Lip_α (S^{n-1}) $(0 < \alpha \le 1)$ condition on S^{n-1}, then μ_Ω is the operator of type (p,p) for $1 < p \le 2$ and of weak (1,1). Benedek et al in [2] proved that if $\Omega \in C^1(S^{n-1})$, then μ_Ω is of type (p,p) for $1 < p < \infty$. Recently, Ding et al in [3] improved the results mentioned above and they gave the $L^p(1 < p < \infty)$ boundedness of μ_Ω for $\Omega \in H^1(S^{n-1})$, where H^1 denotes the Hardy space on S^{n-1}(see [4] etc. for the definition of H^1).

Let $b \in L_{\mathrm{loc}}(\mathbb{R}^n)$, then the commutator of Marcinkiewicz integral is defined by

$$\mu_\Omega^b(f)(x) = \Big(\int_0^\infty |F_{\Omega,b,t}(f)(x)|^2 \frac{\mathrm{d}t}{t}\Big)^{1/2},$$

where

$$F_{\Omega,b,t}(f)(x) = \frac{1}{t}\int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-1}} (b(x)-b(y)) f(y)\mathrm{d}y.$$

In 1990, Torchinsky and Wang [5] proved that if Ω is continuous and satisfies the Lip_α $(0 < \alpha \le 1)$ condition, then for $b \in$BMO, μ_Ω^b is bounded on $L^p(\omega)$, here $\omega \in A_p(1 < p < \infty)$.

For $\beta > 0$, the homogenous Lipschitz space $\dot\Lambda_\beta(\mathbb{R}^n)$ is the space of functions f such that

$$\|f\|_{\dot\Lambda_\beta} = \sup_{x,h\in\mathbb{R}^n, h\neq 0} \frac{|\Delta_h^{[\beta]+1} f(x)|}{|h|^\beta} < \infty,$$

where Δ_h^k denotes the k-th difference operator (see[6]).

When Ω satisfies the Lip_α (S^{n-1}) $(0 < \alpha \le 1)$ condition and $b \in \dot\Lambda_\beta$ $(\mathbb{R}^n)$ $(0 < \beta < \min(1/2, \alpha))$, Liu [7] considered the $(L^p, \dot F_p^{\beta,\infty})$ and (L^p, L^s) boundedness of μ_Ω^b, where $1 < p < n/\beta$ and $1/s = 1/p - \beta/n$. And Xu [8] proved that μ_Ω^b is also bounded on Hardy spaces. Later, we [13] weaken the smoothness condition assumed on Ω and obtained the same conclusion.

Moreover, let $\vec{b} = (b_1, \cdots, b_m)$ be a finite family of locally integrable functions, then generalized higher commutator of Marcinkiwicz integral is defined as follows:

$$\mu_\Omega^{\vec{b}}(f)(x) = \Big(\int_0^\infty |F_{\Omega,t}^{\vec{b}}(f)(x)|^2 \frac{\mathrm{d}t}{t}\Big)^{1/2},$$

where

$$F_{\Omega,t}^{\vec{b}}(f)(x) = \frac{1}{t}\int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-1}} \prod_{j=1}^m (b_j(x)-b_j(y)) f(y)\mathrm{d}y.$$

It is easy to see, when $m=1$, $\mu_\Omega^{\vec b}(f)$ is the commutator of Marcinkiewicz integral and when $b_1=\cdots=b_m$, $\mu_\Omega^{\vec b}(f)$ is the higher commutator of Marcinkiewicz integral.

Let us give some remarks first.

Given any positive integral m, for all $1\le j\le m$, we denote the family of all finite subsets $\sigma=\{\sigma_1,\cdots,\sigma_j\}$ of $\{1,\cdots,m\}$ of different elements by C_j^m and for any $\sigma\in C_j^m$, let $\sigma'=\{1,\cdots,m\}\backslash\sigma$. Let $\vec b=(b_1,\cdots,b_m)$, then for any $\sigma=\{\sigma_1,\cdots,\sigma_j\}\in C_j^m$, we denote $\vec{b_\sigma}=\{b_{\sigma_1},\cdots,b_{\sigma_j}\}$ and $b_\sigma(x)=\prod\limits_{i=1}^{j} b_{\sigma_i}(x)$. With these notations, for any m-tuple $\beta=(\beta_1,\cdots,\beta_m)$ of positive numbers, we denote $\|\vec{b_\sigma}\|_{\dot\Lambda_{\beta_\sigma}}=\prod\limits_{i=1}^{j}\|b_{\sigma_i}\|_{\dot\Lambda_{\beta_{\sigma_i}}}$, where $\beta_{\sigma_1}+\cdots+\beta_{\sigma_j}=\beta_\sigma$. And we simply denote $\|\vec b\|_{\dot\Lambda_\beta}=\prod\limits_{j=1}^{m}\|b_j\|_{\dot\Lambda_{\beta_j}}$.

Now, let us give some definitions and formulate our results.

Definition 1.1 For $\Omega\in L^q(S^{n-1})(q\ge 1)$, the integral modulus $\omega_q(\delta)$ of continuity of order q of Ω is defined by

$$\omega_q(\delta)=\sup_{|\rho|\le\delta}\Big(\int_{S^{n-1}}|\Omega(\rho x')-\Omega(x')|^q\mathrm{d}\delta(x')\Big)^{1/q},$$

where ρ is a rotation on S^{n-1}, $|\rho|=\|\rho-I\|$. When $\omega_q(\delta)$ satisfies

$$\int_0^1\frac{\omega_q(\delta)}{\delta}\mathrm{d}\delta<\infty, \tag{1.2}$$

we say that $\Omega(x')$ satisfies the L^q-Dini condition.

Definition 1.2 Let $\vec b$ be as above such that $b_j\in\dot\Lambda_{\beta_j}(\mathbb{R}^n)$, $1\le j\le m, 0<\beta_j<1$, $\sum\limits_{j=1}^m\beta_j=\beta<n$ and $0<p\le 1$. A function a is said to be a $(p,\infty;\vec b)$ atom, if $a\in L^\infty(\mathbb{R}^n)$ and satisfies the following condition:

(1) supp $a\subset B(x_0,l)=\{x\in\mathbb{R}^n:\ |x-x_0|\le l\}$,

(2) $\|a\|_{L^\infty}\le|B(x_0,l)|^{-1/p}$,

(3) $\int_{\mathbb{R}^n}a(x)b_{\sigma'}(x)\mathrm{d}x=0$, for any $\sigma=(\sigma_1,\cdots,\sigma_j)\in C_m^j,\sigma'=\{1,\cdots,m\}\backslash\sigma$ and $j=1,\cdots,m$.

A temperate distribution f is said to belong to the atomic Hardy space $H_{\vec b}^p(\mathbb{R}^n)$, if it can be written as $f=\sum\limits_j\lambda_ja_j$, in the $\mathcal{S}'$ sense, where a_j is a $(p,\infty;\vec b)$ atom, $\lambda_j\in C$ and $\sum\limits_j|\lambda_j|^p<\infty$.

Moreover, we define the quasinorm on $H_{\vec b}^p(\mathbb{R}^n)$ by

$$\|f\|_{H_{\vec b}^p(\mathbb{R}^n)}=\inf\Big\{\Big(\sum_j|\lambda_j|^p\Big)^{1/p},\quad \text{for all decompostions of } f=\sum_j\lambda_ja_j\Big\}.$$

Obviously, $H_{\vec b}^p(\mathbb{R}^n)(0<p\le 1)$ is a subspace of $H^p(\mathbb{R}^n)$. And when $m=1$ the space $H_{\vec b}^p(\mathbb{R}^n)=H^p(\mathbb{R}^n)$.

Theorem 1 Let $\vec b$ be as above such that $b_j\in\dot\Lambda_{\beta_j}(\mathbb{R}^n)$, $1\le j\le m, 0<\beta_j<1$, $\sum\limits_{j=1}^m\beta_j=\beta<n$, $1<p<n/\beta$ and $1/s=1/p-\beta/n$. If there exists some $q\ge n/(n-\beta)$ such that $\Omega\in L^q(S^{n-1})$ satisfies (1.1), then $\mu_\Omega^{\vec b}(f)$ is bounded from $L^p(\mathbb{R}^n)$ to $L^s(\mathbb{R}^n)$.

Theorem 2 Let $\vec{b}$ be as above such that $b_j \in \dot{\Lambda}_{\beta_j}(\mathbb{R}^n)$, $1 \le j \le m, 0 < \beta_j < 1$, and $\sum_{j=1}^{m} \beta_j = \beta < n$. If $\Omega \in L^q(S^{n-1})(q \ge n/(n-\beta))$ satisfies (1.1), then there exists a constant $C > 0$ such that, for any $\lambda > 0$, $|\{x \in \mathbb{R}^n : |\mu_\Omega^{\vec{b}}(f)(x)| > \lambda\}| \le C(\|b\|_{\dot{\Lambda}_\beta}\|f\|_{L^1}/\lambda)^{n/(n-\beta)}$.

Theorem 3 Let $1 \le q' < p < \infty$, $0 < \varepsilon \le 1$ and $\vec{b}$ be as above such that $b_j \in \dot{\Lambda}_{\beta_j}(\mathbb{R}^n)$, $1 \le j \le m, 0 < \beta_j < 1$, $\sum_{j=1}^{m} \beta_j = \beta < \min\{1/2, \varepsilon\}$. If Ω satisfies (1.1) and

$$\int_0^1 \frac{\omega_q(\delta)}{\delta^{1+\varepsilon}} \mathrm{d}\delta < \infty, \tag{1.3}$$

then

$$\|\mu_\Omega^{\vec{b}}(f)\|_{\dot{F}_p^{\beta,\infty}} \le C\|\vec{b}\|_{\dot{\Lambda}_\beta}\|\Omega\|_{L^q(S^{n-1})}\|f\|_{L^p},$$

where $1/q' + 1/q = 1$.

Theorem 4 Let $0 < \varepsilon \le 1$, $\vec{b}$ be as above such that $b_j \in \dot{\Lambda}_{\beta_j}(\mathbb{R}^n)$, $1 \le j \le m, 0 < \beta_j < 1$, $\sum_{j=1}^{m} \beta_j = \beta \le \min\{1/2, \varepsilon\}$, $n/(n+\beta) < p < 1$, and $1/r = 1/p - \beta/n$. If there exists some $q \ge \max\{r, n/(n-\beta)\}$ such that $\Omega \in L^q(S^{n-1})$ satisfies (1.1) and (1.3), then $\mu_\Omega^{\vec{b}}$ is bounded from $H_{\vec{b}}^p(\mathbb{R}^n)$ to $L^r(\mathbb{R}^n)$.

When $p = 1$, (1.3) can be weakened by (1.2) and we can take $0 < \beta < n$.

Theorem 5 Let $\vec{b}$ be as above such that $b_j \in \dot{\Lambda}_{\beta_j}(\mathbb{R}^n)$, $1 \le j \le m, 0 < \beta_j < 1$, and $\sum_{j=1}^{m} \beta_j = \beta < n$. If there exists some $q \ge n/(n-\beta)$ such that $\Omega \in L^q(S^{n-1})$ satisfies (1.1) and (1.2), then $\mu_\Omega^{\vec{b}}$ is bounded from $H_{\vec{b}}^1(\mathbb{R}^n)$ to $L^{n/(n-\beta)}(\mathbb{R}^n)$.

Remark From [9], it is easy to see that (1.3) is weaker than (1.2) and μ_Ω is bounded from $L^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$, for $1 < p < \infty$ in Theorems 3–5.

2 Some Basic Lemmas

Lemma 2.1 ([6]) (a) For $0 < \beta < 1, 1 \le q < \infty$, we have

$$\|f\|_{\dot{\Lambda}_\beta} \approx \sup_Q \frac{1}{|Q|^{1+\beta/n}} \int_Q |f - f_Q| \approx \sup_Q \frac{1}{|Q|^{\beta/n}} \Big(\frac{1}{|Q|} \int_Q |f - f_Q|^q\Big)^{1/q},$$

for $q = \infty$, the formula should be modified appropriately.

(b) For $0 < \beta < 1, 1 < p < \infty$, we have

$$\|f\|_{\dot{F}_p^{\beta,\infty}} \approx \Big\|\sup_{Q \ni \cdot} \frac{1}{|Q|^{1+\beta/n}} \int_Q |f - f_Q|\Big\|_{L^p},$$

where $f_Q = \frac{1}{|Q|}\int_Q f(x)\mathrm{d}x$.

Lemma 2.2 ([6]) Let $Q^* \subset Q$, then $|f_{Q^*} - f_Q| \le C\|f\|_{\dot{\Lambda}_\beta}|Q|^{\beta/n}$.

Lemma 2.3 ([9]) Suppose that $0 < \lambda < n$ and Ω is homogeneous of degree zero and satisfies the L^q-Dini condition (1.2) for $q > 1$. If there exists a constant $a_0 > 0$ such that $|x| < a_0 R$, then we have

$$\Big(\int_{R<|y|<2R}\Big|\frac{\Omega(y-x)}{|y-x|^{n-\lambda}} - \frac{\Omega(y)}{|y|^{n-\lambda}}\Big|^q \mathrm{d}y\Big)^{1/q} \le CR^{n/q-(n-\lambda)}\Big\{\frac{|x|}{R} + \int_{|x|/2R}^{|x|/R} \frac{\omega_q(\delta)}{\delta}\mathrm{d}\delta\Big\},$$

where the constant $C>0$ is independent of R and x.

Lemma 2.4 ([10]) Let $0<\alpha<n, 1<p<n/\alpha, 1/s=1/p-\alpha/n$, and $q\geq n/(n-\alpha)$. If $\Omega\in L^q(S^{n-1})$, then fractional integral operator $T_{\Omega,\alpha}$ defined by

$$T_{\Omega,\alpha}f(x)=\int_{\mathbb{R}^n}\frac{\Omega(x-y)}{|x-y|^{n-\alpha}}f(y)\mathrm{d}y$$

is bounded from $L^p(\mathbb{R}^n)$ to $L^s(\mathbb{R}^n)$.

Lemma 2.5 ([11]) Let $0<\alpha<n$ and $q\geq n/(n-\alpha)$. If $\Omega\in L^q(S^{n-1})$, then for any $\lambda>0$ and $f\in L^1(\mathbb{R}^n)$, $|\{x\in\mathbb{R}^n:\ |T_{\Omega,\alpha}f(x)|>\lambda\}|\leq C(\|f\|_{L^1}/\lambda)^{n/(n-\alpha)}$.

Lemma 2.6 ([12]) Let $f\in\mathrm{Loc}(\mathbb{R}^n)$, for $x\in Q, f_1=f\chi_{4\sqrt{n}Q}$ and $\vec{b}=(b_1,\cdots,b_m)$ be as above such that $b_j\in\dot{\Lambda}_{\beta_j}(\mathbb{R}^n), 0<\beta_j<1$. Then for any $\sigma\in C_j^m$, $\sigma'=\{1,\cdots,m\}\backslash\sigma, 0\leq j\leq m$,

$$\|(b-\lambda)_{\sigma'}f_1\|_{L^q}\leq C|Q|^{1/r+\beta_{\sigma'}/n}\|\vec{b}_\sigma\|_{\dot{\Lambda}_{\beta_{\sigma'}}}M_r(f)(x)$$

where $M_rf(x)=\sup\limits_{x\in Q}(\frac{1}{|Q|}\int_Q|f(y)|^r\mathrm{d}y)^{1/r}, r>1$ and $\lambda=(\lambda_1,\cdots,\lambda_m), \lambda_j=(b_j)_Q, j=1,\cdots,m$.

3 Proofs of Theorems 1–3

Now, let us first prove Theorem 1 and Theorem 2.

As $b_j\in\dot{\Lambda}_{\beta_j}(\mathbb{R}^n)(0<\beta_j<1)$ for any $j=1,2,\cdots,m$. By the Minkowski inequality,

$$\begin{aligned}\mu_\Omega^{\vec{b}}(f)(x)&=\Big[\int_0^\infty\Big|\frac{1}{t}\int_{|x-y|\leq t}\frac{\Omega(x-y)}{|x-y|^{n-1}}\prod_{j=1}^m(b_j(x)-b_j(y))f(y)\mathrm{d}y\Big|^2\frac{\mathrm{d}t}{t}\Big]^{1/2}\\&\leq\int_{\mathbb{R}^n}\frac{|\Omega(x-y)|}{|x-y|^{n-1}}\prod_{j=1}^m|b_j(x)-b_j(y)||f(y)|\Big[\int_{|x-y|\leq t}^\infty\frac{1}{t^3}dt\Big]^{1/2}\mathrm{d}y\\&\leq C\|\vec{b}\|_{\dot{\Lambda}_\beta}\int_{\mathbb{R}^n}\frac{|\Omega(x-y)|}{|x-y|^{n-\beta}}|f(y)|\mathrm{d}y.\end{aligned}$$

Note that $0<\sum\limits_{j=1}^m\beta_j=\beta<n$, by Lemma 2.4 and Lemma 2.5, we have

$$\|\mu_\Omega^{\vec{b}}(f)\|_{L^s}\leq C\|\vec{b}\|_{\dot{\Lambda}_\beta}\|T_{|\Omega|,\beta}|f|\|_{L^s}\leq C\|\vec{b}\|_{\dot{\Lambda}_\beta}\|f\|_{L^p},$$

and

$$\begin{aligned}|\{x\in\mathbb{R}^n:|\mu_\Omega^{\vec{b}}(f)(x)|>\lambda\}|&\leq|\{x\in\mathbb{R}^n:|T_{|\Omega|,\beta}|f|(x)|>\lambda/C\|\vec{b}\|_{\dot{\Lambda}_\beta}\}\\&\leq C(\|\vec{b}\|_{\dot{\Lambda}_\beta}\|f\|_{L^1}/\lambda)^{n/(n-\beta)}.\end{aligned}$$

So, we complete the proofs of Theorem 1 and Theorem 2. Let us now turn to prove Theorem 3.

For any $x\in\mathbb{R}^n$, fix a cube $Q(x_Q,l)\ni x$ with its center at x_Q and denote the half side-length of Q by l. Let $\vec{\lambda}=(\lambda_1,\cdots,\lambda_m), \lambda_j=(b_j)_Q, 1\leq j\leq m$. For $f\in L^p(\mathbb{R}^n)$, let $f_1=f\chi_{Q^*}, f_2=f-f_1$, where $Q^*=4\sqrt{n}Q$ denotes the $4\sqrt{n}$ times extensions of Q with its center at x_Q. It is obvious that there is an $N\in\mathbb{N}$, such that $2^N\leq4\sqrt{n}<2^{N+1}$.

As

$$
\begin{aligned}
\mu_\Omega^{\vec{b}}(f)(y) &= \Big\{\int_0^\infty \Big|\frac{1}{t}\int_{|y-z|\le t}\frac{\Omega(y-z)}{|y-z|^{n-1}}\prod_{j=1}^m (b_j(y)-b_j(z))f(z)\mathrm{d}z\Big|^2\frac{\mathrm{d}t}{t}\Big\}^{1/2}\\
&= \Big\{\int_0^\infty \Big|\frac{1}{t}\int_{|y-z|\le t}\frac{\Omega(y-z)}{|y-z|^{n-1}}\prod_{j=1}^m [(b_j(y)-\lambda_j)-(b_j(z)-\lambda_j)]f(z)\mathrm{d}z\Big|^2\frac{\mathrm{d}t}{t}\Big\}^{1/2}\\
&= \Big\{\int_0^\infty \Big|\frac{1}{t}\int_{|y-z|\le t}\Big[\prod_{j=1}^m (b_j(y)-\lambda_j)+\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}(-1)^{m-j}(b(y)-\lambda)_\sigma(b(z)-\lambda)_{\sigma'}\\
&\quad +(-1)^m\prod_{j=1}^m (b_j(z)-\lambda_j)\Big]\frac{\Omega(y-z)}{|y-z|^{n-1}}f(z)\mathrm{d}z\Big|^2\frac{\mathrm{d}t}{t}\Big\}^{1/2},
\end{aligned}
$$

and by the definition of $\mu_\Omega(f)$, we have

$$
|\mu_\Omega(f)(y)-\mu_\Omega(g)(x)| = |\|F_{\Omega,t}(f)(y)\|_H-\|F_{\Omega,t}(g)(x)\|_H| \le \|F_{\Omega,t}(f)(y)-F_{\Omega,t}(g)(x)\|_H.
$$

Thus observing $(\mu_\Omega^{\vec{b}}(f))_Q<\infty$ and taking,

$$
\begin{aligned}
a = \Big\{\int_0^\infty \Big|\frac{1}{t}\int_{|x_Q-z|\le t}\Big[\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}(-1)^{m-j}(b(y)-\lambda)_\sigma(b(z)-\lambda)_{\sigma'}\\
+(-1)^m\prod_{j=1}^m (b_j(z)-\lambda_j)\Big]\frac{\Omega(x_Q-z)}{|x_Q-z|^{n-1}}f_2(z)\mathrm{d}z\Big|^2\frac{\mathrm{d}t}{t}\Big\}^{1/2},
\end{aligned}
$$

we have

$$
\begin{aligned}
&\frac{1}{|Q|^{1+\beta/n}}\int_Q |\mu_\Omega^{\vec{b}}(f)(y)-(\mu_\Omega^{\vec{b}}(f))_Q|\mathrm{d}y\\
\le\ &\frac{2}{|Q|^{1+\beta/n}}\int_Q |\mu_\Omega^{\vec{b}}(f)(y)-a|\mathrm{d}y\\
=\ &\frac{2}{|Q|^{1+\beta/n}}\int_Q\Big|\Big\{\int_0^\infty\Big|\frac{1}{t}\int_{|y-z|\le t}\Big[\prod_{j=1}^m (b_j(y)-\lambda_j)\\
&+\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}(-1)^{m-j}(b(y)-\lambda)_\sigma(b(z)-\lambda)_{\sigma'}\\
&+(-1)^m\prod_{j=1}^m (b_j(z)-\lambda_j)\Big]\frac{\Omega(y-z)}{|y-z|^{n-1}}f(z)\mathrm{d}z\Big|^2\frac{\mathrm{d}t}{t}\Big\}^{1/2}\\
&-\Big\{\int_0^\infty\Big|\frac{1}{t}\int_{|x_Q-z|\le t}\Big[\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}(-1)^{m-j}(b(y)-\lambda)_\sigma(b(z)-\lambda)_{\sigma'}\\
&+(-1)^m\prod_{j=1}^m (b_j(z)-\lambda_j)\Big]\frac{\Omega(x_Q-z)}{|x_Q-z|^{n-1}}f_2(z)\mathrm{d}z\Big|^2\frac{\mathrm{d}t}{t}\Big\}^{1/2}\Big|\mathrm{d}y\\
\le\ &\frac{2}{|Q|^{1+\beta/n}}\int_Q\Big\{\int_0^\infty\Big|\int_{|y-z|\le t}\Big[\prod_{j=1}^m (b_j(y)-\lambda_j)+\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}(-1)^{m-j}(b(y)-\lambda)_\sigma\\
&\times(b(z)-\lambda)_{\sigma'}+(-1)^m\prod_{j=1}^m (b_j(z)-\lambda_j)\Big]\frac{\Omega(y-z)}{|y-z|^{n-1}}f(z)\mathrm{d}z
\end{aligned}
$$

$$
\begin{aligned}
&-\int_{|x_Q-z|\le t}\Big[\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}(-1)^{m-j}(b(y)-\lambda)_\sigma(b(z)-\lambda)_{\sigma'}\\
&+(-1)^m\prod_{j=1}^m(b_j(z)-\lambda_j)\Big]\frac{\Omega(x_Q-z)}{|x_Q-z|^{n-1}}f_2(z)\mathrm{d}z\Big|^2\frac{\mathrm{d}t}{t^3}\Big\}^{1/2}\mathrm{d}y\\
\le\ &\frac{2}{|Q|^{1+\beta/n}}\int_Q\Big\{\int_0^\infty\Big|\int_{|y-z|\le t}\prod_{j=1}^m(b_j(y)-\lambda_j)\frac{\Omega(y-z)}{|y-z|^{n-1}}f(z)\mathrm{d}z\Big|^2\frac{\mathrm{d}t}{t^3}\Big\}^{1/2}\mathrm{d}y\\
&+\frac{2}{|Q|^{1+\beta/n}}\int_Q\Big\{\int_0^\infty\Big|\int_{|y-z|\le t}\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}(-1)^{m-j}(b(y)-\lambda)_\sigma(b(z)-\lambda)_{\sigma'}\frac{\Omega(y-z)}{|y-z|^{n-1}}f(z)\mathrm{d}z\\
&-\int_{|x_Q-z|\le t}\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}(-1)^{m-j}(b(y)-\lambda)_\sigma(b(z)-\lambda)_{\sigma'}\frac{\Omega(x_Q-z)}{|x_Q-z|^{n-1}}f_2(z)\mathrm{d}z\Big|^2\frac{\mathrm{d}t}{t^3}\Big\}^{1/2}\mathrm{d}y\\
&+\frac{2}{|Q|^{1+\beta/n}}\int_Q\Big\{\int_0^\infty\Big|\int_{|y-z|\le t}(-1)^m\prod_{j=1}^m(b_j(z)-\lambda_j)\frac{\Omega(y-z)}{|y-z|^{n-1}}f(z)\mathrm{d}z\\
&-\int_{|x_Q-z|\le t}(-1)^m\prod_{j=1}^m(b_j(z)-\lambda_j)\frac{\Omega(x_Q-z)}{|x_Q-z|^{n-1}}f_2(z)\mathrm{d}z\Big|^2\frac{\mathrm{d}t}{t^3}\Big\}^{1/2}\mathrm{d}y\\
:=\ &2(I+II+III).
\end{aligned}
$$

Now, let us estimate I, II, and III, respectively.

From Lemma 2.1 (a), we have

$$
\begin{aligned}
I&=\frac{1}{|Q|^{1+\beta/n}}\int_Q\prod_{j=1}^m|b_j(y)-\lambda_j|\mu_\Omega(f)(y)\mathrm{d}y\\
&\le\prod_{j=1}^m\sup_{y\in Q}\frac{|b_j(y)-\lambda_j|}{|Q|^{\beta/n}}\frac{1}{|Q|}\int_Q|\mu_\Omega(f)(y)|\mathrm{d}y\le C\|\vec{b}\|_{\dot\Lambda_\beta}M(\mu_\Omega(f))(x),
\end{aligned}
$$

where M denotes the Hardy–Littlewood maximal function.

Moreover,

$$
\begin{aligned}
II\le\ &\frac{1}{|Q|^{1+\beta/n}}\int_Q\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}|(b(y)-\lambda)_\sigma|\mu_\Omega((b-\lambda)_{\sigma'}f_1)(y)\mathrm{d}y\\
&+\frac{1}{|Q|^{1+\beta/n}}\int_Q\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}|(b(y)-\lambda)_\sigma|\|F_{\Omega,t}((b-\lambda)_{\sigma'}f_2)(y)-F_{\Omega,t}((b-\lambda)_{\sigma'}f_2)(x_Q)\|_H\mathrm{d}y\\
:=\ &II_1+II_2.
\end{aligned}
$$

Using Lemma 2.1(a), Hölder's inequality, the (L^r,L^r) boundedness of μ_Ω and Lemma 2.6, we have

$$
\begin{aligned}
II_1&\le\frac{1}{|Q|^{1+\beta/n}}\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\prod_{i\in\sigma}\sup_{y\in Q}\frac{|b_i(y)-\lambda_i|}{|Q|^{\beta_\sigma/n}}|Q|^{\beta_\sigma/n}\int_Q\mu_\Omega((b-\lambda)_{\sigma'}f_1)(y)\mathrm{d}y\\
&\le C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\|\vec{b}_\sigma\|_{\dot\Lambda_{\beta_\sigma}}\frac{1}{|Q|^{1+\beta_{\sigma'}/n}}\|\mu_\Omega((b-\lambda)_{\sigma'}f_1)\|_{L^r}|Q|^{1-1/r}
\end{aligned}
$$

$$\begin{aligned}&\leq C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\|\vec{b}_\sigma\|_{\dot{\Lambda}_{\beta_\sigma}}\frac{1}{|Q|^{\beta_{\sigma'}/n+1/r}}\|(b-\lambda)_{\sigma'}f_1\|_{L^r}\\&\leq C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\|\vec{b}_\sigma\|_{\dot{\Lambda}_{\beta_\sigma}}\|\vec{b}_{\sigma'}\|_{\dot{\Lambda}_{\beta_{\sigma'}}}M_r(f)(x)\leq C\|\vec{b}\|_{\dot{\Lambda}_\beta}M_r(f)(x).\end{aligned}$$

For II_2,

$$\begin{aligned}II_2&\leq\frac{1}{|Q|^{\beta/n}}\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\sup_{y\in Q}|(b(y)-\lambda)_\sigma|\|F_{\Omega,t}((b-\lambda)_{\sigma'}f_2)(y)-F_{\Omega,t}((b-\lambda)_{\sigma'}f_2)(x_Q)\|_H\\&\leq\frac{C}{|Q|^{\beta_{\sigma'}/n}}\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\|\vec{b}_\sigma\|_{\dot{\Lambda}_{\beta_\sigma}}\sup_{y\in Q}\|F_{\Omega,t}((b-\lambda)_{\sigma'}f_2)(y)-F_{\Omega,t}((b-\lambda)_{\sigma'}f_2)(x_Q)\|_H\\&:=\frac{C}{|Q|^{\beta_{\sigma'}/n}}\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\|\vec{b}_\sigma\|_{\dot{\Lambda}_{\beta_\sigma}}\sup_{y\in Q}D(y).\end{aligned}$$

Thus,

$$\begin{aligned}D(y)&=\|F_{\Omega,t}((b-\lambda)_{\sigma'}f_2)(y)-F_{\Omega,t}((b-\lambda)_{\sigma'}f_2)(x_Q)\|\\&=\Big\{\int_0^\infty\Big|\frac{1}{t}\Big[\int_{|y-z|\leq t}\frac{\Omega(y-z)}{|y-z|^{n-1}}((b-\lambda)_{\sigma'}f_2)(z)\mathrm{d}z\\&\quad-\int_{|x_Q-z|\leq t}\frac{\Omega(x_Q-z)}{|x_Q-z|^{n-1}}((b-\lambda)_{\sigma'}f_2)(z)\mathrm{d}z\Big]\Big|^2\frac{\mathrm{d}t}{t}\Big\}^{1/2}\\&\leq\Big[\int_0^\infty\Big|\frac{1}{t}\int_{\left\{\substack{|y-z|>t,\\|x_Q-z|\leq t}\right\}}\frac{\Omega(x_Q-z)}{|x_Q-z|^{n-1}}((b-\lambda)_{\sigma'}f_2)(z)\mathrm{d}z\Big|^2\frac{\mathrm{d}t}{t}\Big]^{1/2}\\&\quad+\Big[\int_0^\infty\Big|\frac{1}{t}\int_{\left\{\substack{|y-z|\leq t,\\|x_Q-z|>t}\right\}}\frac{\Omega(y-z)}{|y-z|^{n-1}}((b-\lambda)_{\sigma'}f_2)(z)\mathrm{d}z\Big|^2\frac{\mathrm{d}t}{t}\Big]^{1/2}\\&\quad+\Big[\int_0^\infty\Big|\frac{1}{t}\int_{\left\{\substack{|y-z|\leq t,\\|x_Q-z|\leq t}\right\}}\Big[\frac{\Omega(y-z)}{|y-z|^{n-1}}-\frac{\Omega(x_Q-z)}{|x_Q-z|^{n-1}}\Big]((b-\lambda)_{\sigma'}f_2)(z)\mathrm{d}z\Big|^2\frac{\mathrm{d}t}{t}\Big]^{1/2}\\&:=D_1(y)+D_2(y)+D_3(y).\end{aligned}$$

By Lemma 2.2, it is easy to see,

$$\sup_{z\in Q_{k+1}}|b_j(z)-(b_j)_Q|\leq C|Q_{k+1}|^{\beta_j/n}\|b_j\|_{\dot{\Lambda}_{\beta_j}},\ \text{for any } j=1,2,\cdots,m,\tag{3.1}$$

where $Q_{k+1}=2^{k+1}Q$ and $k=1,2,\cdots$. And when $z\in(Q^*)^c, |z-x_Q|\sim|y-z|$. So, by the Minkowski inequality and Hölder's inequality, we have

$$\begin{aligned}D_1(y)&\leq C\int_{\mathbb{R}^n}\frac{|\Omega(x_Q-z)|}{|x_Q-z|^{n-1}}|((b-\lambda)_{\sigma'}f_2)(z)|\Big(\int_{|x_Q-z|}^{|y-z|}\frac{1}{t^3}\mathrm{d}t\Big)^{1/2}\mathrm{d}z\\&\leq C\int_{Q^*}\frac{|\Omega(x_Q-z)|}{|x_Q-z|^{n-1}}|((b-\lambda)_{\sigma'}f)(z)|\frac{l^{1/2}}{|x_Q-z|^{3/2}}\mathrm{d}z.\\&\leq C\sum_{k=N}^{\infty}2^{-k/2}(2^kl)^{-n}|Q_{k+1}|^{\beta_{\sigma'}/n}\|\vec{b}_{\sigma'}\|_{\dot{\Lambda}_{\beta_{\sigma'}}}\int_{2^kl\leq|x_Q-z|<2^{k+1}l}|\Omega(x_Q-z)||f(z)|\mathrm{d}z\end{aligned}$$

$$\leq C\sum_{k=N}^{\infty}2^{-k/2}(2^kl)^{-n}|Q_{k+1}|^{\beta_{\sigma'}/n}(2^{k+1}l)^{n/q'}\|\vec{b}_{\sigma'}\|_{\dot{\Lambda}_{\beta_{\sigma'}}}M_{q'}(f)(x)$$
$$\times\Big[\int_{|x_Q-z|<2^{k+1}l}|\Omega(x_Q-z)|^q\Big]^{1/q}.$$

As $\Omega\in L^q(S^{n-1})$, we can get

$$\Big[\int_{|x_Q-z|<2^{k+1}l}|\Omega(x_Q-z)|^q\Big]^{1/q}\leq(2^{k+1}l)^{n/q}\|\Omega\|_{L^q(S^{n-1})}. \tag{3.2}$$

Thus,

$$\begin{aligned}D_1(y)&\leq C\sum_{k=N}^{\infty}2^{-k/2}|Q_{k+1}|^{\beta_{\sigma'}/n}\|\vec{b}_{\sigma'}\|_{\dot{\Lambda}_{\beta_{\sigma'}}}\|\Omega\|_{L^q(S^{n-1})}M_{q'}(f)(x)\\&\leq C\sum_{k=N}^{\infty}2^{-k(1/2-\beta_{\sigma'})}|Q|^{\beta_{\sigma'}/n}\|\vec{b}_{\sigma'}\|_{\dot{\Lambda}_{\beta_{\sigma'}}}\|\Omega\|_{L^q(S^{n-1})}M_{q'}(f)(x)\\&\leq C|Q|^{\beta_{\sigma'}/n}\|\vec{b}_{\sigma'}\|_{\dot{\Lambda}_{\beta_{\sigma'}}}\|\Omega\|_{L^q(S^{n-1})}M_{q'}(f)(x).\end{aligned}$$

In the same way,

$$D_2(y)\leq C|Q|^{\beta_{\sigma'}/n}\|\vec{b}_{\sigma'}\|_{\dot{\Lambda}_{\beta_{\sigma'}}}\|\Omega\|_{L^q(S^{n-1})}M_{q'}(f)(x).$$

Let us now estimate $D_3(y)$. By the Minkowski inequality, Hölder's inequality and (3.1),

$$\begin{aligned}D_3(y)\leq&\int_{\mathbb{R}^n}\Big|\frac{\Omega(y-z)}{|y-z|^{n-1}}-\frac{\Omega(x_Q-z)}{|x_Q-z|^{n-1}}\Big||(\vec{b}(z)-\vec{\lambda})_{\sigma'}||f_2(z)|\Big\{\int_{\left\{\substack{|y-z|\leq t,\\|x_Q-z|\leq t}\right\}}\frac{\mathrm{d}t}{t^3}\Big\}^{1/2}\\\leq&\int_{(Q^*)^c}\Big|\frac{\Omega(y-z)}{|y-z|^{n-1}}-\frac{\Omega(x_Q-z)}{|x_Q-z|^{n-1}}\Big|\frac{|(\vec{b}(z)-\vec{\lambda})_{\sigma'}||f(z)|}{|y-z|}\\\leq&\,C\sum_{k=N}^{\infty}(2^kl)^{-1}|Q_{k+1}|^{\beta_{\sigma'}/n}\|\vec{b}_{\sigma'}\|_{\dot{\Lambda}_{\beta_{\sigma'}}}\Big[\int_{2^kl\leq|z-x_Q|<2^{k+1}l}\Big|\frac{\Omega(y-z)}{|y-z|^{n-1}}\\&-\frac{\Omega(x_Q-z)}{|x_Q-z|^{n-1}}\Big|^q\mathrm{d}z\Big]^{1/q}\Big[\int_{|z-x_Q|<2^{j+1}l}|f(z)|^{q'}\mathrm{d}z\Big]^{1/q'}\\\leq&\,C\sum_{k=N}^{\infty}(2^kl)^{-1}(2^{k+1}l)^{n/q'}|Q_{k+1}|^{\beta_{\sigma'}/n}\|\vec{b}_{\sigma'}\|_{\dot{\Lambda}_{\beta_{\sigma'}}}M_{q'}(f)(x)\\&\times\Big[\int_{2^jl\leq|z-x_Q|<2^{j+1}l}\Big|\frac{\Omega(y-z)}{|y-z|^{n-1}}-\frac{\Omega(x_Q-z)}{|x_Q-z|^{n-1}}\Big|^q\mathrm{d}z\Big]^{1/q}.\end{aligned}$$

However, using Lemma 2.3 and condition (1.3), we obtain

$$\begin{aligned}&\Big[\int_{2^kl\leq|z-x_Q|<2^{k+1}l}\Big|\frac{\Omega(y-z)}{|y-z|^{n-1}}-\frac{\Omega(x_Q-z)}{|x_Q-z|^{n-1}}\Big|^q\mathrm{d}z\Big]^{1/q}\\&\leq C(2^kl)^{n/q-(n-1)}\Big\{\frac{|y-x_Q|}{2^kl}+\int_{\frac{|y-x_Q|}{2^{k+1}l}}^{\frac{|y-x_Q|}{2^kl}}\frac{\omega_q(\delta)}{\delta}\mathrm{d}\delta\Big\}\\&\leq C(2^kl)^{n/q-(n-1)}\Big\{2^{-k}+2^{-k\varepsilon}\int_0^1\frac{\omega_q(\delta)}{\delta^{1+\varepsilon}}\mathrm{d}\delta\Big\}\\&\leq C(2^kl)^{n/q-(n-1)}(2^{-k}+2^{-k\varepsilon}).\end{aligned} \tag{3.3}$$

So, by $0<\beta_{\sigma'}<\beta<\min\{1/2,\varepsilon\}$, we have

$$\begin{aligned} D_3(y) &\le C\sum_{k=N}^{\infty}(2^kl)^{-1}(2^{k+1}l)^{n/q'}(2^kl)^{n/q-(n-1)} \\ &\quad\times(2^{-k}+2^{-k\varepsilon})|Q_{k+1}|^{\beta_{\sigma'}/n}\|\vec{b}_{\sigma'}\|_{\dot{\Lambda}_{\beta_{\sigma'}}}M_{q'}(f)(x) \\ &\le C\sum_{k=N}^{\infty}(2^{-k(1-\beta_{\sigma'})}+2^{-k(\varepsilon-\beta_{\sigma'})})|Q|^{\beta_{\sigma'}/n}\|\vec{b}_{\sigma'}\|_{\dot{\Lambda}_{\beta_{\sigma'}}}M_{q'}(f)(x) \\ &\le C|Q|^{\beta_{\sigma'}/n}\|\vec{b}_{\sigma'}\|_{\dot{\Lambda}_{\beta_{\sigma'}}}M_{q'}(f)(x). \end{aligned}$$

Combining the estimates of $D_1(y), D_2(y)$ with $D_3(y)$, we get

$$D(y)\le C|Q|^{\beta_{\sigma'}/n}\|\vec{b}_{\sigma'}\|_{\dot{\Lambda}_{\beta_{\sigma'}}}\|\Omega\|_{L^q(S^{n-1})}M_{q'}(f)(x).$$

So

$$II_2\le C\|\vec{b}\|_{\dot{\Lambda}_\beta}\|\Omega\|_{L^q(S^{n-1})}M_{q'}f(x).$$

Combing II_1 with II_2, we have $II\le C\|\vec{b}\|_{\dot{\Lambda}_\beta}(M_r(f)(x)+M_{q'}f(x))$.

Now, let us estimate III.

$$\begin{aligned} III\ &\le\ \frac{1}{|Q|^{1+\beta/n}}\int_Q\Big|\mu_\Omega(\prod_{j=1}^m(b_j-\lambda_j)f_1)(y)\Big|\mathrm{d}y \\ &\quad+\frac{1}{|Q|^{\beta/n}}\sup_{y\in Q}\Big\|F_{\Omega,t}(\prod_{j=1}^m(b_j-\lambda_j)f_2)(y)-F_{\Omega,t}(\prod_{j=1}^m(b_j-\lambda_j)f_2)(x_Q)\Big\|_H \\ &:=III_1+III_2. \end{aligned}$$

Similar to the estimate of II_1, we have

$$\begin{aligned} III_1&\le\frac{C}{|Q|^{1+\beta/n}}\Big\|\mu_\Omega(\prod_{j=1}^m(b_j-\lambda_j)f_1)\Big\|_{L^r}|Q|^{1-1/r} \\ &\le\frac{C}{|Q|^{1/r+\beta/n}}\Big\|\mu_\Omega(\prod_{j=1}^m(b_j-\lambda_j)f_1)\Big\|_{L^r}\le C\|\vec{b}\|_{\dot{\Lambda}_\beta}M_r(f)(x). \end{aligned}$$

For III_2, set $E(y)=\|F_{\Omega,t}(\prod_{j=1}^m(b_j-\lambda_j)f_2)(y)-F_{\Omega,t}(\prod_{j=1}^m(b_j-\lambda_j)f_2)(x_Q)\|_H$.

Similar to the estimate of $D(y)$, we have

$$E(y)\le C|Q|^{\beta/n}\|\vec{b}\|_{\dot{\Lambda}_\beta}\|\Omega\|_{L^q(S^{n-1})}M_{q'}(f)(x).$$

So

$$III\le C\|\vec{b}\|_{\dot{\Lambda}_\beta}\|\Omega\|_{L^q(S^{n-1})}(M_r(f)(x)+M_{q'}(f)(x)).$$

Combining the estimates of I and II with III, we obtain

$$\begin{aligned} &\frac{1}{|Q|^{1+\beta/n}}\int_Q|\mu_\Omega^{\vec{b}}(f)-(\mu_\Omega^{\vec{b}}(f))_Q|\mathrm{d}y \\ &\le C\|\vec{b}\|_{\dot{\Lambda}_\beta}\|\Omega\|_{L^q(S^{n-1})}\Big[M(\mu_\Omega(f))(x)+M_r(f)(x)+M_{q'}(f)(x)\Big], \end{aligned}$$

here, we take $1<r<p$.

So, by Lemma 2.1(b) and the L^p $(1<p<\infty)$ boundedness of M, M_r, $M_{q'}$, and μ_Ω, we conclude that

$$\begin{aligned}\|\mu_\Omega^{\vec{b}}(f)\|_{\dot{F}_p^{\beta,\infty}} &\approx \Big\|\sup_{Q\ni\cdot}\frac{1}{|Q|^{1+\beta/n}}\int_Q|\mu_\Omega^{\vec{b}}(f)(y)-(\mu_\Omega^{\vec{b}}(f))_Q|\mathrm{d}y\Big\|_{L^p}\\ &\le C\|\vec{b}\|_{\dot{\Lambda}_\beta}\|\Omega\|_{L^q(S^{n-1})}\|f\|_{L^p}.\end{aligned}$$

4 Proofs of Theorems 4 and 5

By a standard argument, it is suffice to show that there exists a constant $C>0$ such that, for each $(p,\infty,\vec{b})$ atom a, $\|\mu_\Omega^{\vec{b}}(a)\|_{L^r}\le C$.

Take a $(p,\infty,\vec{b})$ atom a with supp $a\subset B(x_0,l)$. Then,

$$\begin{aligned}&\|\mu_\Omega^{\vec{b}}(a)\|_{L^r}\le\Big[\int_{2B}|\mu_\Omega^{\vec{b}}(a)(x)|^r\mathrm{d}x\Big]^{1/r}+\Big[\int_{(2B)^c}|\mu_\Omega^{\vec{b}}(a)(x)|^r\mathrm{d}x\Big]^{1/r}\\ \le&\ \Big[\int_{2B}|\mu_\Omega^{\vec{b}}(a)(x)|^r\mathrm{d}x\Big]^{1/r}\\ &+\Big\{\int_{(2B)^c}\Big[\int_0^{|x-x_0|+2l}\Big|\int_{|x-y|\le t}\frac{\Omega(x-y)}{|x-y|^{n-1}}\prod_{j=1}^m(b_j(x)-b_j(y))a(y)\mathrm{d}y\Big|^2\frac{\mathrm{d}t}{t^3}\Big]^{r/2}\mathrm{d}x\Big\}^{1/r}\\ &+\Big\{\int_{(2B)^c}\Big[\int_{|x-x_0|+2l}^\infty\Big|\int_{|x-y|\le t}\frac{\Omega(x-y)}{|x-y|^{n-1}}\prod_{j=1}^m(b_j(x)-b_j(y))a(y)\mathrm{d}y\Big|^2\frac{\mathrm{d}t}{t^3}\Big]^{r/2}\mathrm{d}x\Big\}^{1/r}\\ :=&\ I+II+III.\end{aligned}$$

Choose p_1 and s_1 such that $1<p_1<n/\beta$ and $1/s_1=1/p_1-\beta/n$. It is obvious that $r<s_1$. So, by the Hölder's inequality and the (L^{p_1},L^{s_1}) boundedness of μ_Ω^b (see Theorem 1), we have

$$\begin{aligned}I&\le C\|\mu_\Omega^{\vec{b}}(a)\|_{L^{s_1}}|2B|^{1/r-1/s_1}\le C\|a\|_{L^{p_1}}|2B|^{1/r-1/s_1}\\ &\le C\|a\|_{L^\infty}|B|^{1/p_1}|2B|^{1/r-1/s_1}\le C|B|^{-1/p}|B|^{1/p_1}|2B|^{1/r-1/s_1}\le C.\end{aligned}$$

Because $y\in B, x\in(2B)^c$, we have $|x-y|\sim|x-x_0|\sim|x-x_0|+2l$. By the Minkowski inequality, we have

$$\begin{aligned}II&\le C\Big\{\int_{(2B)^c}\Big[\int_{\mathbb{R}^n}\Big(\int_{|x-y|}^{|x-x_0|+2l}\frac{\mathrm{d}t}{t^3}\Big)^{1/2}\frac{|\Omega(x-y)||a(y)|}{|x-y|^{n-1}}\prod_{j=1}^m|b_j(x)-b_j(y)|\mathrm{d}y\Big]^r\mathrm{d}x\Big\}^{1/r}\\ &\le C\int_B\Big\{\int_{(2B)^c}\Big[\frac{|l|^{1/2}|\Omega(x-y)|}{|x-y|^{n+1/2}}\prod_{j=1}^m|b_j(x)-b_j(y)|\Big]^r\mathrm{d}x\Big\}^{1/r}|a(y)|\mathrm{d}y\\ &\le C\int_B\sum_{k=1}^\infty\Big\{\int_{2^kl\le|x-x_0|<2^{k+1}l}\Big[\frac{|l|^{1/2}|\Omega(x-y)|}{|x-y|^{n+1/2}}\prod_{j=1}^m|b_j(x)-b_j(y)|\Big]^r\mathrm{d}x\Big\}^{1/r}|a(y)|\mathrm{d}y\\ &\le C\int_B\sum_{k=1}^\infty2^{-k/2}(2^kl)^{-n}(2^{k+1}l)^\beta\|\vec{b}\|_{\dot{\Lambda}_\beta}\Big[\int_{|x-x_0|<2^{k+1}l}|\Omega(x-y)|^r\mathrm{d}x\Big]^{1/r}|a(y)|\mathrm{d}y.\end{aligned}$$

Because $|x-x_0|\sim|x-y|$ and $\Omega\in L^q(S^{n-1})$, it is easy to see that, for any $q\ge r\ge1$,

$$\Big[\int_{|x-x_0|<2^{k+1}l}|\Omega(x-y)|^r\mathrm{d}x\Big]^{1/r}\le C(2^{k+1}l)^{n/r}\|\Omega\|_{L^q(S^{n-1})}. \tag{4.1}$$

And, we also have

$$\|a\|_{L^1}\le\|a\|_{L^\infty}|B|\le|B|^{1-1/p}=Cl^{n(1-1/p)}. \tag{4.2}$$

Therefore,

$$\begin{aligned}II&\le C\|\vec{b}\|_{\dot{\Lambda}_\beta}\|\Omega\|_{L^q(S^{n-1})}\int_B\sum_{k=1}^{\infty}2^{-k/2}(2^kl)^{-n}(2^{k+1}l)^{\beta}(2^{k+1}l)^{n/r}|a(y)|\mathrm{d}y\\&\le C\|\vec{b}\|_{\dot{\Lambda}_\beta}\|\Omega\|_{L^q(S^{n-1})}\int_B\sum_{k=1}^{\infty}2^{-k[(1/2-\beta)+n(1-1/r)]}l^{-n(1-1/p)}|a(y)|\mathrm{d}y\\&\le C\|\vec{b}\|_{\dot{\Lambda}_\beta}\|\Omega\|_{L^q(S^{n-1})}l^{-n(1-1/p)}\|a\|_{L^1}\le C\|\vec{b}\|_{\dot{\Lambda}_\beta}\|\Omega\|_{L^q(S^{n-1})}.\end{aligned}$$

Let us now estimate III.

Note that for any $y\in B$, we have $t\ge|x-x_0|+2l\ge|x-x_0|+|y-x_0|\ge|x-y|$. So, by the vanishing condition of a, we have

$$\begin{aligned}&\Big[\int_{|x-x_0|+2l}^{\infty}\Big|\int_{|x-y|\le t}\frac{\Omega(x-y)}{|x-y|^{n-1}}\prod_{j=1}^{m}(b_j(x)-b_j(y))a(y)\mathrm{d}y\Big|^2\frac{\mathrm{d}t}{t^3}\Big]^{1/2}\\&=\Big[\int_{|x-x_0|+2l}^{\infty}\Big|\int_{B}\frac{\Omega(x-y)}{|x-y|^{n-1}}\prod_{j=1}^{m}(b_j(x)-b_j(y))a(y)\mathrm{d}y\Big|^2\frac{\mathrm{d}t}{t^3}\Big]^{1/2}\\&=\Big|\int_{B}\frac{\Omega(x-y)}{|x-y|^{n-1}}\prod_{j=1}^{m}(b_j(x)-b_j(y))a(y)\mathrm{d}y\Big|\Big(\int_{|x-x_0|+2l}^{\infty}\frac{\mathrm{d}t}{t^3}\Big)^{1/2}\\&=\Big|\int_{B}\frac{\Omega(x-y)}{|x-y|^{n-1}}\prod_{j=1}^{m}(b_j(x)-b_j(y))a(y)\mathrm{d}y\Big|\frac{1}{|x-x_0|+2l}\\&\le C\Big\{\Big|\int_B\prod_{j=1}^{m}(b_j(x)-b_j(x_0))\Big[\frac{\Omega(x-y)}{|x-y|^{n-1}}-\frac{\Omega(x-x_0)}{|x-x_0|^{n-1}}\Big]\frac{a(y)}{|x-x_0|+2l}\mathrm{d}y\Big|\\&\quad+\Big|\int_B\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}(-1)^{m-j}(b(x)-b(x_0))_\sigma\Big[\frac{\Omega(x-y)}{|x-y|^{n-1}}-\frac{\Omega(x-x_0)}{|x-x_0|^{n-1}}\Big]\\&\quad\times\frac{(b(y)-b(x_0))_{\sigma'}a(y)}{|x-x_0|+2l}\mathrm{d}y\Big|+\Big|\int_B\frac{\Omega(x-y)}{|x-y|^{n-1}}\frac{\prod_{j=1}^{m}(b_j(y)-b_j(x_0))a(y)}{|x-x_0|+2l}\mathrm{d}y\Big|\Big\}.\end{aligned}$$

So,

$$\begin{aligned}III\le\ &C\Big\{\Big[\int_{(2B)^c}\Big(\int_B\Big|\frac{\Omega(x-y)}{|x-y|^{n-1}}-\frac{\Omega(x-x_0)}{|x-x_0|^{n-1}}\Big|\frac{\prod_{j=1}^{m}|b_j(x)-b_j(x_0)||a(y)|}{|x-x_0|+2l}\mathrm{d}y\Big)^r\mathrm{d}x\Big]^{1/r}\\&+\Big[\int_{(2B)^c}\Big(\int_B\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}|(b(x)-b(x_0))_\sigma|\Big|\frac{\Omega(x-y)}{|x-y|^{n-1}}-\frac{\Omega(x-x_0)}{|x-x_0|^{n-1}}\Big|\\&\times\frac{|(b(y)-b(x_0))_{\sigma'}||a(y)|}{|x-x_0|+2l}\mathrm{d}y\Big)^r\mathrm{d}x\Big]^{1/r}\\&+\Big[\int_{(2B)^c}\Big(\int_B\frac{|\Omega(x-y)|}{|x-y|^{n-1}}\frac{\prod_{j=1}^{m}|b_j(y)-b_j(x_0)||a(y)|}{|x-x_0|+2l}\mathrm{d}y\Big)^r\mathrm{d}x\Big]^{1/r}\Big\}\\&:=C(III_1+III_2+III_3).\end{aligned}$$

For III_1, by the Minkowski inequality, Hölder's inequality, (3.3) and (4.2), we obtain

$$\begin{aligned}
III_1 &\le \int_B \Big\{\int_{(2B)^c}\Big[\Big|\frac{\Omega(x-y)}{|x-y|^{n-1}}-\frac{\Omega(x-x_0)}{|x-x_0|^{n-1}}\Big|\frac{\prod\limits_{j=1}^m|b_j(x)-b_j(x_0)|}{|x-x_0|+2l}\Big]^r\mathrm{d}x\Big\}^{1/r}|a(y)|\mathrm{d}y\\
&\le C\int_B\sum_{k=1}^\infty(2^kl)^{\beta-1}(2^{k+1}l)^{n(1/r-1/q)}\|\vec{b}\|_{\dot{\Lambda}_\beta}\\
&\quad\times\Big[\int_{2^kl\le|x-x_0|<2^{k+1}l}\Big|\frac{\Omega(x-y)}{|x-y|^{n-1}}-\frac{\Omega(x-x_0)}{|x-x_0|^{n-1}}\Big|^q\mathrm{d}x\Big]^{1/q}|a(y)|\mathrm{d}y\\
&\le C\|\vec{b}\|_{\dot{\Lambda}_\beta}\int_B\sum_{k=1}^\infty 2^{-kn(1-1/r)}(2^{-k(1-\beta)}+2^{-k(\varepsilon-\beta)})l^{-n(1-1/p)}|a(y)|\mathrm{d}y\\
&\le C\|\vec{b}\|_{\dot{\Lambda}_\beta}l^{-n(1-1/p)}\|a\|_{L^1}\le C\|\vec{b}\|_{\dot{\Lambda}_\beta}.
\end{aligned}$$

Similar to the estimate of III_1, by the Minkowski inequality and (3.3),

$$\begin{aligned}
III_2 &\le C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\int_B\Big\{\int_{(2B)^c}\Big[|(b(x)-b(x_0))_\sigma|\Big|\frac{\Omega(x-y)}{|x-y|^{n-1}}-\frac{\Omega(x-x_0)}{|x-x_0|^{n-1}}\Big|\\
&\quad\times\frac{|(b(y)-b(x_0))_{\sigma'}|}{|x-x_0|+2l}\Big]^r\mathrm{d}x\Big\}^{1/r}|a(y)|\mathrm{d}y\\
&\le C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\int_B\sum_{k=1}^\infty(2^kl)^{\beta_\sigma-1}l^{\beta_{\sigma'}}\|\vec{b}\|_{\dot{\Lambda}_\beta}\\
&\quad\times\Big[\int_{2^kl\le|x-x_0|<2^{k+1}l}\Big|\frac{\Omega(x-y)}{|x-y|^{n-1}}-\frac{\Omega(x-x_0)}{|x-x_0|^{n-1}}\Big|^r\mathrm{d}x\Big]^{1/r}|a(y)|\mathrm{d}y\\
&\le C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\int_B\sum_{k=1}^\infty(2^kl)^{\beta_\sigma-1}l^{\beta_{\sigma'}}\|\vec{b}\|_{\dot{\Lambda}_\beta}(2^{k+1}l)^{n/r-n+1}(2^{-k}+2^{-k\varepsilon})|a(y)|\mathrm{d}y\\
&\le C\|\vec{b}\|_{\dot{\Lambda}_\beta}\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\int_B\sum_{k=1}^\infty 2^{-kn(1-1/r)}[2^{-k(1-\beta_\sigma)}+2^{-k(\varepsilon-\beta_\sigma)}]l^{-n(1-1/p)}|a(y)|\mathrm{d}y\\
&\le C\|\vec{b}\|_{\dot{\Lambda}_\beta}l^{-n(1-1/p)}\|a\|_{L^1}\le C\|\vec{b}\|_{\dot{\Lambda}_\beta}.
\end{aligned}$$

Let us now estimate III_3.

Note that for any $y\in B, x\in(2B)^c$, we have $|x-y|\sim|x-x_0|+2l$. So, by the Minkowski inequality, (4.1) and (4.2),

$$\begin{aligned}
III_3 &\le C\int_B\Big\{\int_{(2B)^c}\Big[\frac{|\Omega(x-y)|}{|x-y|^{n-1}}\frac{\prod\limits_{j=1}^m|b_j(y)-b_j(x_0)|}{|x-x_0|+2l}\Big]^r\mathrm{d}x\Big\}^{1/r}|a(y)|\mathrm{d}y\\
&\le C\int_B\sum_{k=1}^\infty l^\beta(2^kl)^{-n}\|\vec{b}\|_{\dot{\Lambda}_\beta}\Big[\int_{2^kl\le|x-x_0|<2^{k+1}l}|\Omega(x-y)|^r\mathrm{d}x\Big]^{1/r}|a(y)|\mathrm{d}y\\
&\le C\int_B\sum_{k=1}^\infty 2^{-kn(1-1/r)}l^{-n(1-1/p)}|a(y)|\mathrm{d}y\|\vec{b}\|_{\dot{\Lambda}_\beta}\|\Omega\|_{L^q(S^{n-1})}\\
&\le Cl^{-n(1-1/p)}\|a\|_{L^1}\|\vec{b}\|_{\dot{\Lambda}_\beta}\|\Omega\|_{L^q(S^{n-1})}\le C\|\vec{b}\|_{\dot{\Lambda}_\beta}\|\Omega\|_{L^q(S^{n-1})}.
\end{aligned}$$

So

$$III \le C(III_1 + III_2 + III_3) \le C\|\vec{b}\|_{\dot{\Lambda}_\beta}\|\Omega\|_{L^q(S^{n-1})}.$$

Combining the estimates of I and II with III, we obtain

$$\|\mu_\Omega^{\vec{b}}(a)\|_{L^r} \le C\|\vec{b}\|_{\dot{\Lambda}_\beta}\|\Omega\|_{L^q(S^{n-1})}.$$

Now, let us turn to prove Theorem 5. The main idea is the same as that of proving Theorem 4.

Let $r = n/(n-\beta)$ and a be a $(1, \infty, \vec{b})$ atom with supp $a \subset B(x_0, l)$. Then, we have

$$\|\mu_\Omega^{\vec{b}}(a)\|_{L^r} \le I + II + III,$$

where I, II, and III are the same as in the proof of Theorem 4.

In the same way as proving Theorem 4, we have $I \le C$ and $II \le C\|\vec{b}\|_{\dot{\Lambda}_\beta}\|\Omega\|_{L^q(S^{n-1})}$.

For III, we have

$$\begin{aligned} III \le\ & C\Big\{\Big[\int_{(2B)^c}\Big(\int_B\Big|\frac{\Omega(x-y)}{|x-y|^{n-1}} - \frac{\Omega(x-x_0)}{|x-x_0|^{n-1}}\Big|\frac{\prod\limits_{j=1}^m|b_j(x)-b_j(x_0)||a(y)|}{|x-x_0|+2l}\mathrm{d}y\Big)^r\mathrm{d}x\Big]^{1/r} \\ & + \Big[\int_{(2B)^c}\Big(\int_B\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}|(b(x)-b(x_0))_\sigma|\Big|\frac{\Omega(x-y)}{|x-y|^{n-1}} - \frac{\Omega(x-x_0)}{|x-x_0|^{n-1}}\Big| \\ & \times\frac{|(b(y)-b(x_0))_{\sigma'}||a(y)|}{|x-x_0|+2l}\mathrm{d}y\Big)^r\mathrm{d}x\Big]^{1/r} \\ & + \Big[\int_{(2B)^c}\Big(\int_B\frac{|\Omega(x-y)|}{|x-y|^{n-1}}\frac{\prod\limits_{j=1}^m|b_j(y)-b_j(x_0)||a(y)|}{|x-x_0|+2l}\mathrm{d}y\Big)^r\mathrm{d}x\Big]^{1/r}\Big\} \\ :=\ & C(U+V+W). \end{aligned}$$

In the same way as estimating III_3 in the proof of Theorem 4, we have $W \le C\|\vec{b}\|_{\dot{\Lambda}_\beta}\|\Omega\|_{L^q(S^{n-1})}$.

Let us now estimate U and V. Similar to the estimate of III_1 in the proof of Theorem 4, by Lemma 2.3, $r = n/(n-\beta)$, (1.2) and $\|a\|_\infty \le |B|^{-1}$, we have

$$\begin{aligned} U \le\ & C\int_B\sum_{k=1}^\infty(2^kl)^{\beta-1}(2^{k+1}l)^{n(1/r-1/q)}\|\vec{b}\|_{\dot{\Lambda}_\beta} \\ & \times\Big[\int_{2^kl\le|x-x_0|<2^{k+1}l}\Big|\frac{\Omega(x-y)}{|x-y|^{n-1}} - \frac{\Omega(x-x_0)}{|x-x_0|^{n-1}}\Big|^q\mathrm{d}x\Big]^{1/q}|a(y)|\mathrm{d}y \\ \le\ & C\int_B\sum_{k=1}^\infty(2^kl)^{\beta-1}(2^kl)^{n/r-n+1}\|\vec{b}\|_{\dot{\Lambda}_\beta}\Big[\frac{|y-x_0|}{2^kl} + \int_{|y-x_0|/2^{k+1}l}^{|y-x_0|/2^kl}\frac{\omega_q(\delta)}{\delta}\mathrm{d}\delta\Big]|a(y)|\mathrm{d}y \\ \le\ & C\|\vec{b}\|_{\dot{\Lambda}_\beta}\|a\|_\infty\int_B\Big[\sum_{k=1}^\infty 2^{-k} + \sum_{k=1}^\infty\int_{|y-x_0|/2^{k+1}l}^{|y-x_0|/2^kl}\frac{\omega_q(\delta)}{\delta}\mathrm{d}\delta\Big]\mathrm{d}y \\ \le\ & C\|\vec{b}\|_{\dot{\Lambda}_\beta}\|a\|_\infty|B|\Big[1 + \int_0^1\frac{\omega_q(\delta)}{\delta}\mathrm{d}\delta\Big] \le C\|\vec{b}\|_{\dot{\Lambda}_\beta}. \end{aligned}$$

Using Lemma 2.3, $r = n/(n-\beta)$ and (1.2), similar to the estimate of III_2 in the proof of Theorem 4, we have

$$\begin{aligned}V &\le C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\int_B\sum_{k=1}^{\infty}(2^kl)^{\beta_\sigma-1+n(1/r-1/q)}l^{\beta_{\sigma'}}\|\vec{b}\|_{\dot{\Lambda}_\beta}\\&\quad\times\Big[\int_{2^kl\le|x-x_0|<2^{k+1}l}\Big|\frac{\Omega(x-y)}{|x-y|^{n-1}}-\frac{\Omega(x-x_0)}{|x-x_0|^{n-1}}\Big|^q\mathrm{d}x\Big]^{1/q}|a(y)|\mathrm{d}y\\&\le C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\int_B\sum_{k=1}^{\infty}2^{-k\beta_{\sigma'}}\|\vec{b}\|_{\dot{\Lambda}_\beta}\Big[\frac{|y-x_0|}{2^kl}+\int_{|y-x_0|/2^{k+1}l}^{|y-x_0|/2^kl}\frac{\omega_q(\delta)}{\delta}\mathrm{d}\delta\Big]|a(y)|\mathrm{d}y\\&\le C\|\vec{b}\|_{\dot{\Lambda}_\beta}\int_B\Big[\sum_{k=1}^{\infty}2^{-k}+\sum_{k=1}^{\infty}\int_{|y-x_0|/2^{k+1}l}^{|y-x_0|/2^kl}\frac{\omega_q(\delta)}{\delta}\mathrm{d}\delta\Big]|a(y)|\mathrm{d}y\\&\le C\|a\|_{L^\infty}|B|^{-1}\Big[1+\int_0^1\frac{\omega_q(\delta)}{\delta}\mathrm{d}\delta\Big]\|\vec{b}\|_{\dot{\Lambda}_\beta}\le C\|\vec{b}\|_{\dot{\Lambda}_\beta}.\end{aligned}$$

So

$$III \le C\|\vec{b}\|_{\dot{\Lambda}_\beta}\|\Omega\|_{L^q(S^{n-1})}.$$

Combining the estimates of I and II with III, we have

$$\|\mu_\Omega^{\vec{b}}(a)\|_{L^{n/(n-\beta)}} \le C\|\vec{b}\|_{\dot{\Lambda}_\beta}\|\Omega\|_{L^q(S^{n-1})}.$$

We complete the proof of Theorem 5.

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