Boundedness of fractional integral operators with rough kernels on weighted Morrey spaces

Hua Wang [*]

Department of Mathematics, Zhejiang University, Hangzhou 310027, China

Abstract

Let $M_{\Omega,\alpha}$ and $T_{\Omega,\alpha}$ be the fractional maximal and integral operators with rough kernels, where $0 < \alpha < n$. In this paper, we shall study the continuity properties of $M_{\Omega,\alpha}$ and $T_{\Omega,\alpha}$ on the weighted Morrey spaces $L^{p,\kappa}(w)$. The boundedness of their commutators with BMO functions is also obtained.

MSC(2010): 42B20; 42B25

Keywords: Fractional integral operators; rough kernels; weighted Morrey spaces; commutator

1 Introduction

Let $\Omega \in L^s(S^{n-1})$ be homogeneous of degree zero on $\mathbb{R}^n$, where S^{n-1} denotes the unit sphere of $\mathbb{R}^n (n \geq 2)$ equipped with the normalized Lebesgue measure $d\sigma$ and $s > 1$. For any $0 < \alpha < n$, then the fractional integral operator with rough kernel $T_{\Omega,\alpha}$ is defined by

$$T_{\Omega,\alpha}f(x) = \int_{\mathbb{R}^n} \frac{\Omega(y')}{|y|^{n-\alpha}} f(x-y)\, dy$$

and a related fractional maximal operator $M_{\Omega,\alpha}$ is defined by

$$M_{\Omega,\alpha}f(x) = \sup_{r>0} \frac{1}{r^{n-\alpha}} \int_{|y| \leq r} \left| \Omega(y')f(x-y) \right| dy,$$

where $y' = y/|y|$ for any $y \neq 0$. In 1971, Muckenhoupt and Wheeden [17] studied the weighted norm inequalities for $T_{\Omega,\alpha}$ with the weight $w(x) = |x|^\beta$. The weak type estimates with power weights for $M_{\Omega,\alpha}$ and $T_{\Omega,\alpha}$ was obtained by Ding in [3]. Later, Ding and Lu [4] considered the weighted norm inequalities for $M_{\Omega,\alpha}$ and $T_{\Omega,\alpha}$ with more general weights. More precisely, they proved

Theorem A ([4]). *Let $0 < \alpha < n$, $1 \leq s' < p < n/\alpha$ and $1/q = 1/p - \alpha/n$. If $\Omega \in L^s(S^{n-1})$ and $w^{s'} \in A(p/s', q/s')$, then the operators $M_{\Omega,\alpha}$ and $T_{\Omega,\alpha}$ are all bounded from $L^p(w^p)$ to $L^q(w^q)$.*

---

[*]E-mail address: wanghua@pku.edu.cn.

Let b be a locally integrable function on $\mathbb{R}^n$, then for $0 < \alpha < n$, we shall define the commutators generated by fractional maximal and integral operators with rough kernels and b as follows.

$$[b, M_{\Omega,\alpha}](f)(x) = \sup_{r>0} \frac{1}{r^{n-\alpha}} \int_{|y-x|\leq r} |b(x) - b(y)||\Omega(x-y)f(y)|\, dy,$$

$$[b, T_{\Omega,\alpha}](f)(x) = b(x)T_{\Omega,\alpha}f(x) - T_{\Omega,\alpha}(bf)(x)$$
$$= \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^{n-\alpha}}[b(x) - b(y)]f(y)\, dy.$$

In 1993, by using the Rubio de Francia extrapolation theorem, Segovia and Torrea [21] obtained the weighted boundedness of commutator $[b, T_{\Omega,\alpha}]$, where $b \in BMO(\mathbb{R}^n)$ and Ω satisfies some Dini smoothness condition (see also [20]). In 1999, Ding and Lu [5] improved this result by removing the smoothness condition imposed on Ω. More specifically, they showed (see also [14]).

Theorem B ([5]). *Let $0 < \alpha < n$, $1 \leq s' < p < n/\alpha$ and $1/q = 1/p - \alpha/n$. Assume that $\Omega \in L^s(S^{n-1})$, $w^{s'} \in A(p/s', q/s')$ and $b \in BMO(\mathbb{R}^n)$, then the commutator $[b, T_{\Omega,\alpha}]$ is bounded from $L^p(w^p)$ to $L^q(w^q)$.*

The classical Morrey spaces $\mathcal{L}^{p,\lambda}$ were first introduced by Morrey in [15] to study the local behavior of solutions to second order elliptic partial differential equations. For the boundedness of the Hardy-Littlewood maximal operator, the fractional integral operator and the Calderón-Zygmund singular integral operator on these spaces, we refer the readers to [1, 2, 19]. For the properties and applications of classical Morrey spaces, see [7, 8, 9] and references therein.

In 2009, Komori and Shirai [13] first defined the weighted Morrey spaces $L^{p,\kappa}(w)$ which could be viewed as an extension of weighted Lebesgue spaces, and studied the boundedness of the above classical operators on these weighted spaces. Recently, in [22] and [23], we have established the continuity properties of some other operators on the weighted Morrey spaces $L^{p,\kappa}(w)$.

The purpose of this paper is to discuss the boundedness properties of $M_{\Omega,\alpha}$ and $T_{\Omega,\alpha}$ on the weighted Morrey spaces. Here, and in what follows we shall use the notation $s' = s/(s-1)$ when $1 < s < \infty$ and $s' = 1$ when $s = \infty$. Our main results in the paper are formulated as follows.

Theorem 1.1. *Suppose that $\Omega \in L^s(S^{n-1})$ with $1 < s \leq \infty$. If $0 < \alpha < n$, $1 \leq s' < p < n/\alpha$, $1/q = 1/p - \alpha/n$, $0 < \kappa < p/q$ and $w^{s'} \in A(p/s', q/s')$, then the fractional maximal operator $M_{\Omega,\alpha}$ is bounded from $L^{p,\kappa}(w^p, w^q)$ to $L^{q,\kappa q/p}(w^q)$.*

Theorem 1.2. *Suppose that $\Omega \in L^s(S^{n-1})$ with $1 < s \leq \infty$. If $0 < \alpha < n$, $1 \leq s' < p < n/\alpha$, $1/q = 1/p - \alpha/n$, $0 < \kappa < p/q$ and $w^{s'} \in A(p/s', q/s')$, then the fractional integral operator $T_{\Omega,\alpha}$ is bounded from $L^{p,\kappa}(w^p, w^q)$ to $L^{q,\kappa q/p}(w^q)$.*

Theorem 1.3. *Suppose that $\Omega \in L^s(S^{n-1})$ with $1 < s \leq \infty$ and $b \in BMO(\mathbb{R}^n)$. If $0 < \alpha < n$, $1 \leq s' < p < n/\alpha$, $1/q = 1/p - \alpha/n$, $0 < \kappa < p/q$ and $w^{s'} \in A(p/s', q/s')$, then the commutator $[b, T_{\Omega,\alpha}]$ is bounded from $L^{p,\kappa}(w^p, w^q)$ to $L^{q,\kappa q/p}(w^q)$.*

2 Notations and definitions

Let us first recall some standard definitions and notations. The classical A_p weight theory was first introduced by Muckenhoupt in the study of weighted L^p boundedness of Hardy-Littlewood maximal functions in [16]. A weight w is a nonnegative, locally integrable function on $\mathbb{R}^n$, $B = B(x_0, r_B)$ denotes the ball with the center x_0 and radius r_B. Given a ball B and $\lambda > 0$, λB denotes the ball with the same center as B whose radius is λ times that of B. For a given weight function w, we also denote the Lebesgue measure of B by $|B|$ and the weighted measure of B by $w(B)$, where $w(B) = \int_B w(x)\, dx$. We say that $w \in A_p$, $1 < p < \infty$, if

$$\left(\frac{1}{|B|} \int_B w(x)\, dx\right) \left(\frac{1}{|B|} \int_B w(x)^{-1/(p-1)}\, dx\right)^{p-1} \leq C \quad \text{for every ball } B \subseteq \mathbb{R}^n,$$

where C is a positive constant which is independent of B.

For the case $p = 1$, $w \in A_1$, if

$$\frac{1}{|B|} \int_B w(x)\, dx \leq C \cdot \operatorname*{ess\,inf}_{x \in B} w(x) \quad \text{for every ball } B \subseteq \mathbb{R}^n.$$

For the case $p = \infty$, $w \in A_\infty$ if it satisfies the A_p condition for some $1 < p < \infty$.

We also need another weight class $A(p, q)$ introduced by Muckenhoupt and Wheeden in [18]. A weight function w belongs to $A(p, q)$ for $1 < p < q < \infty$ if there exists a constant $C > 0$ such that

$$\left(\frac{1}{|B|} \int_B w(x)^q\, dx\right)^{1/q} \left(\frac{1}{|B|} \int_B w(x)^{-p'}\, dx\right)^{1/p'} \leq C \quad \text{for every ball } B \subseteq \mathbb{R}^n.$$

A weight function w is said to belong to the reverse Hölder class RH_r if there exist two constants $r > 1$ and $C > 0$ such that the following reverse Hölder inequality holds

$$\left(\frac{1}{|B|} \int_B w(x)^r\, dx\right)^{1/r} \leq C \left(\frac{1}{|B|} \int_B w(x)\, dx\right) \quad \text{for every ball } B \subseteq \mathbb{R}^n.$$

We state the following results that we will use frequently in the sequel.

Lemma 2.1 ([10]). *Let $w \in A_p$ with $p \geq 1$. Then, for any ball B, there exists an absolute constant $C > 0$ such that*

$$w(2B) \leq C\, w(B).$$

In general, for any $\lambda > 1$, we have

$$w(\lambda B) \leq C \cdot \lambda^{np} w(B),$$

where C does not depend on B nor on λ.

Lemma 2.2 ([11]). *Let $w \in RH_r$ with $r > 1$. Then there exists a constant $C > 0$ such that*

$$\frac{w(E)}{w(B)} \leq C \left(\frac{|E|}{|B|} \right)^{(r-1)/r}$$

for any measurable subset E of a ball B.

Next we shall introduce the Hardy-Littlewood maximal operator, its variant and BMO spaces. The Hardy-Littlewood maximal operator M is defined by

$$M(f)(x) = \sup_{x \in B} \frac{1}{|B|} \int_B |f(y)| \, dy,$$

where the supremum is taken over all balls B containing x. For $0 < \alpha < n$, $s \geq 1$, we define the fractional maximal operator $M_{\alpha,s}$ by

$$M_{\alpha,s}(f)(x) = \sup_{x \in B} \left(\frac{1}{|B|^{1-\frac{\alpha s}{n}}} \int_B |f(y)|^s \, dy \right)^{1/s}.$$

Moreover, we denote simply by M_α when $s = 1$.

A locally integrable function b is said to be in $BMO(\mathbb{R}^n)$ if

$$\|b\|_* = \sup_B \frac{1}{|B|} \int_B |b(x) - b_B| \, dx < \infty,$$

where b_B stands for the average of b on B, i.e. $b_B = \frac{1}{|B|} \int_B b(y) \, dy$ and the supremum is taken over all balls B in $\mathbb{R}^n$.

Theorem C ([6, 12]). *Assume that $b \in BMO(\mathbb{R}^n)$. Then for any $1 \leq p < \infty$, we have*

$$\sup_B \left(\frac{1}{|B|} \int_B \left| b(x) - b_B \right|^p \, dx \right)^{1/p} \leq C\|b\|_*.$$

We are going to conclude this section by defining the weighted Morrey space and giving the known result relevant to this paper. For further details, we refer the readers to [13].

Definition 2.3 ([13]). *Let $1 \leq p < \infty$, $0 < \kappa < 1$ and w be a weight function. Then the weighted Morrey space is defined by*

$$L^{p,\kappa}(w) = \left\{ f \in L^p_{loc}(w) : \|f\|_{L^{p,\kappa}(w)} < \infty \right\},$$

where

$$\|f\|_{L^{p,\kappa}(w)} = \sup_B \left(\frac{1}{w(B)^\kappa} \int_B |f(x)|^p w(x) \, dx \right)^{1/p}$$

and the supremum is taken over all balls B in $\mathbb{R}^n$.

In order to deal with the fractional order case, we need to consider the weighted Morrey space with two weights.

Definition 2.4 ([13]). *Let $1 \le p < \infty$ and $0 < \kappa < 1$. Then for two weights u and v, the weighted Morrey space is defined by*

$$L^{p,\kappa}(u,v) = \left\{ f \in L^p_{loc}(u) : \|f\|_{L^{p,\kappa}(u,v)} < \infty \right\},$$

where

$$\|f\|_{L^{p,\kappa}(u,v)} = \sup_B \left(\frac{1}{v(B)^\kappa} \int_B |f(x)|^p u(x)\,dx \right)^{1/p}.$$

Theorem D. *If $0 < \alpha < n$, $1 < p < n/\alpha$, $1/q = 1/p - \alpha/n$, $0 < \kappa < p/q$ and $w \in A(p,q)$, then the fractional maximal operator M_α is bounded from $L^{p,\kappa}(w^p, w^q)$ to $L^{q,\kappa q/p}(w^q)$.*

Throughout this article, we will use C to denote a positive constant, which is independent of the main parameters and not necessarily the same at each occurrence. By $A \sim B$, we mean that there exists a constant $C > 1$ such that $\frac{1}{C} \le \frac{A}{B} \le C$.

3 Proof of Theorem 1.1

Proof of Theorem 1.1. For $\Omega \in L^s(S^{n-1})$, we set

$$\|\Omega\|_{L^s(S^{n-1})} = \left(\int_{S^{n-1}} |\Omega(y')|^s\,d\sigma(y') \right)^{1/s}.$$

From Hölder's inequality, it follows that

$$M_{\Omega,\alpha} f(x) \le \sup_{r>0} \frac{1}{r^{n-\alpha}} \left(\int_{|y| \le r} |\Omega(y')|^s\,dy \right)^{1/s} \left(\int_{|y| \le r} |f(x-y)|^{s'}\,dy \right)^{1/s'}$$

$$\le C\|\Omega\|_{L^s(S^{n-1})} \cdot \sup_{r>0} \left(\frac{1}{r^{n-\alpha s'}} \int_{|y| \le r} |f(x-y)|^{s'}\,dy \right)^{1/s'}$$

$$\le C\|\Omega\|_{L^s(S^{n-1})} \cdot \sup_{r>0} \left(\frac{1}{|B(x,r)|^{1-\frac{\alpha s'}{n}}} \int_{B(x,r)} |f(y)|^{s'}\,dy \right)^{1/s'}$$

$$= C\|\Omega\|_{L^s(S^{n-1})} M_{\alpha,s'}(f)(x).$$

If we let $p_1 = p/s'$, $q_1 = q/s'$ and $\nu = w^{s'}$, then for $0 < \alpha < n$, $1 \le s' < n/\alpha$, we have $1/q_1 = 1/p_1 - (\alpha s')/n$ and $0 < \kappa < p_1/q_1$. Also observe that

$$M_{\alpha,s'}(f) = M_{\alpha s'}(|f|^{s'})^{1/s'}.$$

Hence, by Theorem D, we obtain

$$\left\| M_{\alpha,s'}(f) \right\|_{L^{q,\kappa q/p}(w^q)} = \left\| M_{\alpha s'}(|f|^{s'}) \right\|_{L^{q_1,\kappa q_1/p_1}(\nu^{q_1})}^{1/s'}$$

$$\le C \left\| |f|^{s'} \right\|_{L^{p_1,\kappa}(\nu^{p_1},\nu^{q_1})}^{1/s'}$$

$$\le C\|f\|_{L^{p,\kappa}(w^p,w^q)}.$$

This finishes the proof of Theorem 1.1. $\qquad\square$

4   Proof of Theorem 1.2

Proof of Theorem 1.2. Fix a ball $B = B(x_0, r_B) \subseteq \mathbb{R}^n$ and decompose $f = f_1 + f_2$, where $f_1 = f\chi_{2B}$ and χ_{2B} denotes the characteristic function of $2B$. Since $T_{\Omega,\alpha}$ is a linear operator, then we can write

$$\frac{1}{w^q(B)^{\kappa/p}}\left(\int_B |T_{\Omega,\alpha}f(x)|^q w(x)^q\, dx\right)^{1/q}$$
$$\leq \frac{1}{w^q(B)^{\kappa/p}}\left(\int_B |T_{\Omega,\alpha}f_1(x)|^q w(x)^q\, dx\right)^{1/q} + \frac{1}{w^q(B)^{\kappa/p}}\left(\int_B |T_{\Omega,\alpha}f_2(x)|^q w(x)^q\, dx\right)^{1/q}$$
$$= I_1 + I_2.$$

As in the proof of Theorem 1.1, we also set $p_1 = p/s'$, $q_1 = q/s'$ and $\nu = w^{s'}$. Since $\nu \in A(p_1, q_1)$, then we get $\nu^{q_1} = w^q \in A_{1+q_1/p_1'}$ (see [18]). Hence, by Theorem A and Lemma 2.1, we have

$$I_1 \leq C \cdot \frac{1}{w^q(B)^{\kappa/p}}\left(\int_{2B} |f(x)|^p w(x)^p\, dx\right)^{1/p}$$
$$\leq C\|f\|_{L^{p,\kappa}(w^p, w^q)} \cdot \frac{w^q(2B)^{\kappa/p}}{w^q(B)^{\kappa/p}}$$
$$\leq C\|f\|_{L^{p,\kappa}(w^p, w^q)}.$$

We now turn to deal with the term I_2. An application of Hölder's inequality gives us that

$$\left|T_{\Omega,\alpha}(f_2)(x)\right| \leq \int_{(2B)^c} \frac{|\Omega(x-y)|}{|x-y|^{n-\alpha}}|f(y)|\, dy \tag{1}$$
$$\leq \sum_{k=1}^{\infty}\left(\int_{2^{k+1}B\backslash 2^k B} |\Omega(x-y)|^s\, dy\right)^{1/s}\left(\int_{2^{k+1}B\backslash 2^k B} \frac{|f(y)|^{s'}}{|x-y|^{(n-\alpha)s'}}\, dy\right)^{1/s'}.$$

When $x \in B$ and $y \in 2^{k+1}B\backslash 2^k B$, then we can easily see that $2^{k-1}r_B \leq |y-x| < 2^{k+2}r_B$. Thus, by a simple computation, we deduce

$$\left(\int_{2^{k+1}B\backslash 2^k B} |\Omega(x-y)|^s\, dy\right)^{1/s} \leq C\|\Omega\|_{L^s(S^{n-1})}\left|2^{k+1}B\right|^{1/s}. \tag{2}$$

We also note that if $x \in B$, $y \in (2B)^c$, then $|y-x| \sim |y-x_0|$. Consequently

$$\left(\int_{2^{k+1}B\backslash 2^k B} \frac{|f(y)|^{s'}}{|x-y|^{(n-\alpha)s'}}\, dy\right)^{1/s'} \leq C\cdot\frac{1}{|2^{k+1}B|^{1-\alpha/n}}\left(\int_{2^{k+1}B} |f(y)|^{s'}\, dy\right)^{1/s'}. \tag{3}$$

Substituting the above two inequalities (2) and (3) into (1), we obtain

$$\left|T_{\Omega,\alpha}(f_2)(x)\right| \leq C\|\Omega\|_{L^s(S^{n-1})}\sum_{k=1}^{\infty}\frac{1}{|2^{k+1}B|^{1-\alpha/n-1/s}}\left(\int_{2^{k+1}B} |f(y)|^{s'}\, dy\right)^{1/s'}.$$

6

By using Hölder's inequality and the definition of $\nu \in A(p_1, q_1)$, we can get

$$\left(\int_{2^{k+1}B} |f(y)|^{s'} dy\right)^{1/s'} \leq \left(\int_{2^{k+1}B} |f(y)|^{p_1 s'} \nu(y)^{p_1} dy\right)^{1/(p_1 s')} \left(\int_{2^{k+1}B} \nu(y)^{-p_1'} dy\right)^{1/(p_1' s')}$$

$$\leq C\left(\int_{2^{k+1}B} |f(y)|^p w(y)^p dy\right)^{1/p} \left(\frac{|2^{k+1}B|^{1-1/p_1+1/q_1}}{\nu^{q_1}(2^{k+1}B)^{1/q_1}}\right)^{1/s'}$$

$$\leq C\|f\|_{L^{p,\kappa}(w^p, w^q)} w^q\big(2^{k+1}B\big)^{\kappa/p} \cdot \frac{|2^{k+1}B|^{1/s'-1/p+1/q}}{w^q(2^{k+1}B)^{1/q}}$$

$$= C\|f\|_{L^{p,\kappa}(w^p, w^q)} \big|2^{k+1}B\big|^{1-1/s-\alpha/n} \cdot w^q\big(2^{k+1}B\big)^{\kappa/p-1/q}. \tag{4}$$

So we have

$$\big|T_{\Omega,\alpha}(f_2)(x)\big| \leq C\|f\|_{L^{p,\kappa}(w^p, w^q)} \sum_{k=1}^{\infty} w^q\big(2^{k+1}B\big)^{\kappa/p-1/q},$$

which implies

$$I_2 \leq C\|f\|_{L^{p,\kappa}(w^p, w^q)} \sum_{k=1}^{\infty} \frac{w^q(B)^{1/q-\kappa/p}}{w^q(2^{k+1}B)^{1/q-\kappa/p}}.$$

Observe that $w^q = \nu^{q_1} \in A_{1+q_1/p_1'}$, then we know that there exists $r > 1$ such that $w^q \in RH_r$. Thus, it follows directly from Lemma 2.2 that

$$\frac{w^q(B)}{w^q(2^{k+1}B)} \leq C\left(\frac{|B|}{|2^{k+1}B|}\right)^{(r-1)/r}. \tag{5}$$

Therefore

$$I_2 \leq C\|f\|_{L^{p,\kappa}(w^p, w^q)} \sum_{k=1}^{\infty} \left(\frac{1}{2^{kn}}\right)^{(1-1/r)(1/q-\kappa/p)}$$

$$\leq C\|f\|_{L^{p,\kappa}(w^p, w^q)},$$

where the last series is convergent since $r > 1$ and $0 < \kappa < p/q$. Combining the above estimates for I_1 and I_2 and taking the supremum over all balls $B \subseteq \mathbb{R}^n$, we complete the proof of Theorem 1.2. $\qquad\square$

5   Proof of Theorem 1.3

Proof of Theorem 1.3. Fix a ball $B = B(x_0, r_B) \subseteq \mathbb{R}^n$. Let $f = f_1 + f_2$, where $f_1 = f\chi_{2B}$. Since $[b, T_{\Omega,\alpha}]$ is a linear operator, then we have

$$\frac{1}{w^q(B)^{\kappa/p}} \left(\int_B \big|[b, T_{\Omega,\alpha}]f(x)\big|^q w(x)^q dx\right)^{1/q}$$

$$\leq \frac{1}{w^q(B)^{\kappa/p}} \left(\int_B \big|[b, T_{\Omega,\alpha}]f_1(x)\big|^q w(x)^q dx\right)^{1/q} + \frac{1}{w^q(B)^{\kappa/p}} \left(\int_B \big|[b, T_{\Omega,\alpha}]f_2(x)\big|^q w(x)^q dx\right)^{1/q}$$

$$= J_1 + J_2.$$

7

As before, we set $p_1 = p/s'$, $q_1 = q/s'$ and $\nu = w^{s'}$, then $\nu^{q_1} = w^q \in A_{1+q_1/p_1'}$. Theorem B and Lemma 2.1 imply

$$J_1 \leq C\|b\|_* \cdot \frac{1}{w^q(B)^{\kappa/p}} \left(\int_{2B} |f(x)|^p w(x)^p \, dx \right)^{1/p}$$

$$\leq C\|b\|_*\|f\|_{L^{p,\kappa}(w^p,w^q)} \cdot \frac{w^q(2B)^{\kappa/p}}{w^q(B)^{\kappa/p}}$$

$$\leq C\|b\|_*\|f\|_{L^{p,\kappa}(w^p,w^q)}. \tag{6}$$

In order to estimate the term J_2, for any $x \in B$, we first write

$$\left| [b, T_{\Omega,\alpha}] f_2(x) \right| = \left| \int_{(2B)^c} \frac{\Omega(x-y)}{|x-y|^{n-\alpha}} \big[b(x) - b(y) \big] f(y) \, dy \right|$$

$$\leq \big| b(x) - b_B \big| \cdot \int_{(2B)^c} \frac{|\Omega(x-y)|}{|x-y|^{n-\alpha}} |f(y)| \, dy$$

$$+ \int_{(2B)^c} \frac{|\Omega(x-y)|}{|x-y|^{n-\alpha}} |b(y) - b_B| |f(y)| \, dy$$

$$= I + II.$$

For the term I, it follows from the previous estimates (2) and (4) that

$$I \leq C\|f\|_{L^{p,\kappa}(w^p,w^q)} |b(x) - b_B| \cdot \sum_{k=1}^{\infty} \frac{1}{w^q(2^{k+1}B)^{1/q-\kappa/p}}.$$

Hence

$$\frac{1}{w^q(B)^{\kappa/p}} \left(\int_B I^q \, w(x)^q \, dx \right)^{1/q}$$

$$\leq C\|f\|_{L^{p,\kappa}(w^p,w^q)} \frac{1}{w^q(B)^{\kappa/p}} \cdot \sum_{k=1}^{\infty} \frac{1}{w^q(2^{k+1}B)^{1/q-\kappa/p}} \cdot \left(\int_B |b(x) - b_B|^q w(x)^q \, dx \right)^{1/q}$$

$$= C\|f\|_{L^{p,\kappa}(w^p,w^q)} \sum_{k=1}^{\infty} \frac{w^q(B)^{1/q-\kappa/p}}{w^q(2^{k+1}B)^{1/q-\kappa/p}} \cdot \left(\frac{1}{w^q(B)} \int_B |b(x) - b_B|^q w(x)^q \, dx \right)^{1/q}.$$

We now claim that for any $1 < q < \infty$ and $\mu \in A_\infty$, the following inequality holds

$$\left(\frac{1}{\mu(B)} \int_B |b(x) - b_B|^q \mu(x) \, dx \right)^{1/q} \leq C\|b\|_*. \tag{7}$$

In fact, since $\mu \in A_\infty$, then there must exist $r > 1$ such that $\mu \in RH_r$. Thus, by Hölder's inequality and Theorem C, we obtain

$$\left(\frac{1}{\mu(B)} \int_B |b(x) - b_B|^q \mu(x) \, dx \right)^{1/q} \leq \frac{1}{\mu(B)^{1/q}} \left(\int_B |b(x) - b_B|^{qr'} \, dx \right)^{1/(qr')} \left(\int_B \mu(x)^r \, dx \right)^{1/(qr)}$$

$$\leq C \left(\frac{1}{|B|} \int_B |b(x) - b_B|^{qr'} \, dx \right)^{1/(qr')}$$

$$\leq C\|b\|_*,$$

8

which is our desired result. Note that $w^q \in A_{1+q_1/p_1'} \subset A_\infty$. In addition, we have $w^q \in RH_r$ with $r > 1$. Hence, by the inequalities (5) and (7), we get

$$\frac{1}{w^q(B)^{\kappa/p}} \left(\int_B I^q \, w(x)^q \, dx \right)^{1/q} \le C\|b\|_* \|f\|_{L^{p,\kappa}(w^p, w^q)} \sum_{k=1}^\infty \left(\frac{1}{2^{kn}} \right)^{(1-1/r)(1/q - \kappa/p)}$$

$$\le C\|b\|_* \|f\|_{L^{p,\kappa}(w^p, w^q)}. \tag{8}$$

On the other hand

$$\mathrm{II} \le \sum_{k=1}^\infty \int_{2^{k+1}B \setminus 2^k B} \frac{|\Omega(x-y)|}{|x-y|^{n-\alpha}} |b(y) - b_B||f(y)| \, dy$$

$$\le \sum_{k=1}^\infty \int_{2^{k+1}B \setminus 2^k B} \frac{|\Omega(x-y)|}{|x-y|^{n-\alpha}} |b(y) - b_{2^{k+1}B}||f(y)| \, dy$$

$$+ \sum_{k=1}^\infty \int_{2^{k+1}B \setminus 2^k B} \frac{|\Omega(x-y)|}{|x-y|^{n-\alpha}} |b_{2^{k+1}B} - b_B||f(y)| \, dy$$

$$= \mathrm{III} + \mathrm{IV}.$$

To estimate III and IV, we observe that when $x \in B$, $y \in (2B)^c$, then $|y - x| \sim |y - x_0|$. Thus, it follows from Hölder's inequality and (2) that

$$\mathrm{III} \le C \sum_{k=1}^\infty \frac{1}{|2^{k+1}B|^{1-\alpha/n}} \cdot \int_{2^{k+1}B \setminus 2^k B} |\Omega(x-y)||b(y) - b_{2^{k+1}B}||f(y)| \, dy$$

$$\le C \sum_{k=1}^\infty \frac{1}{|2^{k+1}B|^{1-\alpha/n-1/s}} \cdot \left(\int_{2^{k+1}B} |b(y) - b_{2^{k+1}B}|^{s'} |f(y)|^{s'} \, dy \right)^{1/s'}.$$

An application of Hölder's inequality yields

$$\left(\int_{2^{k+1}B} |b(y) - b_{2^{k+1}B}|^{s'} |f(y)|^{s'} \, dy \right)^{1/s'}$$

$$\le \left(\int_{2^{k+1}B} |f(y)|^{p_1 s'} \nu(y)^{p_1} \, dy \right)^{1/(p_1 s')} \left(\int_{2^{k+1}B} |b(y) - b_{2^{k+1}B}|^{p_1' s'} \nu(y)^{-p_1'} \, dy \right)^{1/(p_1' s')}$$

$$\le \left(\int_{2^{k+1}B} |f(y)|^p w(y)^p \, dy \right)^{1/p} \left(\int_{2^{k+1}B} |b(y) - b_{2^{k+1}B}|^{p_1' s'} \nu(y)^{-p_1'} \, dy \right)^{1/(p_1' s')}.$$

Since $\nu \in A(p_1, q_1)$, then we know that $\nu^{-p_1'} \in A_{1+p_1'/q_1} \subset A_\infty$(see [18]). Hence, by using the inequality (7) and the fact that $\nu \in A(p_1, q_1)$, we obtain

$$\left(\int_{2^{k+1}B} |b(y) - b_{2^{k+1}B}|^{p_1' s'} \nu(y)^{-p_1'} \, dy \right)^{1/(p_1' s')} \le C\|b\|_* \cdot \nu^{-p_1'}(2^{k+1}B)^{1/(p_1' s')}$$

$$\le C\|b\|_* \cdot \left(\frac{|2^{k+1}B|^{1/q_1 + 1/p_1'}}{\nu^{q_1}(2^{k+1}B)^{1/q_1}} \right)^{1/s'}$$

9

$$= C\|b\|_* \cdot \frac{|2^{k+1}B|^{1/s'-1/p+1/q}}{w^q(2^{k+1}B)^{1/q}}. \tag{9}$$

Consequently, by the above inequality (9), we deduce

$$\text{III} \le C\|b\|_*\|f\|_{L^{p,\kappa}(w^p,w^q)} \sum_{k=1}^{\infty} \frac{1}{w^q(2^{k+1}B)^{1/q-\kappa/p}},$$

which implies

$$\frac{1}{w^q(B)^{\kappa/p}} \left(\int_B \text{III}^q\, w(x)^q\, dx \right)^{1/q} \le C\|b\|_*\|f\|_{L^{p,\kappa}(w^p,w^q)} \sum_{k=1}^{\infty} \frac{w^q(B)^{1/q-\kappa/p}}{w^q(2^{k+1}B)^{1/q-\kappa/p}}$$

$$\le C\|b\|_*\|f\|_{L^{p,\kappa}(w^p,w^q)}. \tag{10}$$

Since $b \in BMO(\mathbb{R}^n)$, then a direct calculation shows that

$$\left| b_{2^{k+1}B} - b_B \right| \le C \cdot k\|b\|_*.$$

Moreover, by Hölder's inequality, the estimates (2) and (4), we can get

$$\text{IV} \le C\|b\|_* \sum_{k=1}^{\infty} k \cdot \frac{1}{|2^{k+1}B|^{1-\alpha/n}} \int_{2^{k+1}B \setminus 2^k B} |\Omega(x-y)||f(y)|\, dy$$

$$\le C\|b\|_*\|f\|_{L^{p,\kappa}(w^p,w^q)} \sum_{k=1}^{\infty} k \cdot \frac{1}{w^q(2^{k+1}B)^{1/q-\kappa/p}}.$$

Therefore

$$\frac{1}{w^q(B)^{\kappa/p}} \left(\int_B \text{IV}^q\, w(x)^q\, dx \right)^{1/q} \le C\|b\|_*\|f\|_{L^{p,\kappa}(w^p,w^q)} \sum_{k=1}^{\infty} k \cdot \frac{w^q(B)^{1/q-\kappa/p}}{w^q(2^{k+1}B)^{1/q-\kappa/p}}$$

$$\le C\|b\|_*\|f\|_{L^{p,\kappa}(w^p,w^q)} \sum_{k=1}^{\infty} \frac{k}{2^{kn\delta}}$$

$$\le C\|b\|_*\|f\|_{L^{p,\kappa}(w^p,w^q)}, \tag{11}$$

where $w^q \in RH_r$ and $\delta = (1-1/r)(1/q-\kappa/p)$. Summarizing the estimates (10) and (11) derived above, we thus obtain

$$\frac{1}{w^q(B)^{\kappa/p}} \left(\int_B \text{II}^q\, w(x)^q\, dx \right)^{1/q} \le C\|b\|_*\|f\|_{L^{p,\kappa}(w^p,w^q)}. \tag{12}$$

Combining the inequalities (6) and (8) with the above inequality (12) and taking the supremum over all balls $B \subseteq \mathbb{R}^n$, we conclude the proof of Theorem 1.3. $\quad\square$

It should be pointed out that $[b, M_{\Omega,\alpha}](f)$ can be controlled pointwise by $[b, T_{|\Omega|,\alpha}](|f|)$ for any $f(x)$. In fact, for any $0 < \alpha < n$, $x \in \mathbb{R}^n$ and $r > 0$, we

have

$$[b, T_{|\Omega|,\alpha}](|f|)(x) \geq \int_{|y-x|\leq r} \frac{|\Omega(x-y)|}{|x-y|^{n-\alpha}} |b(x) - b(y)||f(y)| \, dy$$

$$\geq \frac{1}{r^{n-\alpha}} \int_{|y-x|\leq r} |\Omega(x-y)||b(x) - b(y)||f(y)| \, dy.$$

Taking the supremum for all $r > 0$ on both sides of the above inequality, we get

$$[b, M_{\Omega,\alpha}](f)(x) \leq [b, T_{|\Omega|,\alpha}](|f|)(x), \quad \text{for all } x \in \mathbb{R}^n.$$

Hence, as a direct consequence of Theorem 1.3, we finally obtain the following

Corollary 5.1. *Suppose that $\Omega \in L^s(S^{n-1})$ with $1 < s \leq \infty$ and $b \in BMO(\mathbb{R}^n)$. If $0 < \alpha < n$, $1 \leq s' < p < n/\alpha$, $1/q = 1/p - \alpha/n$, $0 < \kappa < p/q$ and $w^{s'} \in A(p/s', q/s')$, then the commutator $[b, M_{\Omega,\alpha}]$ is bounded from $L^{p,\kappa}(w^p, w^q)$ to $L^{q,\kappa q/p}(w^q)$.*

References

[1] D. R. Adams, A note on Riesz potentials, Duke Math. J, **42**(1975), 765–778.

[2] F. Chiarenza and M. Frasca, Morrey spaces and Hardy-Littlewood maximal function, Rend. Math. Appl, **7**(1987), 273–279.

[3] Y. Ding, Weak type bounds for a class of rough operators with power weights, Proc. Amer. Math. Soc, **125**(1997), 2939–2942.

[4] Y. Ding and S. Z. Lu, Weighted norm inequalities for fractional integral operators with rough kernel, Canad. J. Math, **50**(1998), 29–39.

[5] Y. Ding and S. Z. Lu, Higher order commutators for a class of rough operators, Ark. Mat, **37**(1999), 33–44.

[6] J. Duoandikoetxea, Fourier Analysis, American Mathematical Society, Providence, Rhode Island, 2000.

[7] D. S. Fan, S. Z. Lu and D. C. Yang, Regularity in Morrey spaces of strong solutions to nondivergence elliptic equations with VMO coefficients, Georgian Math. J, **5**(1998), 425–440.

[8] G. Di Fazio and M. A. Ragusa, Interior estimates in Morrey spaces for strongly solutions to nondivergence form equations with discontinuous coefficients, J. Funct. Anal, **112**(1993), 241–256.

[9] G. Di Fazio, D. K. Palagachev and M. A. Ragusa, Global Morrey regularity of strong solutions to the Dirichlet problem for elliptic equations with discontinuous coefficients, J. Funct. Anal, **166**(1999), 179–196.

[10] J. Garcia-Cuerva and J. Rubio de Francia, Weighted Norm Inequalities and Related Topics, North-Holland, Amsterdam, 1985.

[11] R. F. Gundy and R. L. Wheeden, Weighted integral inequalities for non-tangential maximal function, Lusin area integral, and Walsh-Paley series, Studia Math, **49**(1974), 107–124.

[12] F. John and L. Nirenberg, On functions of bounded mean oscillation, Comm. Pure Appl. Math, **14**(1961), 415–426.

[13] Y. Komori and S. Shirai, Weighted Morrey spaces and a singular integral operator, Math. Nachr, **282**(2009), 219–231.

[14] S. Z. Lu, Y. Ding and D. Y. Yan, Singular Integrals and Related Topics, World Scientific Publishing, NJ, 2007.

[15] C. B. Morrey, On the solutions of quasi-linear elliptic partial differential equations, Trans. Amer. Math. Soc, **43**(1938), 126–166.

[16] B. Muckenhoupt, Weighted norm inequalities for the Hardy maximal function, Trans. Amer. Math. Soc, **165**(1972), 207–226.

[17] B. Muckenhoupt and R. L. Wheeden, Weighted norm inequalities for singular and fractional integrals, Trans. Amer. Math. Soc, **161**(1971), 249–258.

[18] B. Muckenhoupt and R. L. Wheeden, Weighted norm inequalities for fractional integrals, Trans. Amer. Math. Soc, **192**(1974), 261–274.

[19] J. Peetre, On the theory of $\mathcal{L}_{p,\lambda}$ spaces, J. Funct. Anal, **4**(1969), 71–87.

[20] C. Segovia and J. L. Torrea, Weighted inequalities for commutators of fractional and singular integral, Publ. Mat, **35**(1991), 209–235.

[21] C. Segovia and J. L. Torrea, Higher order commutators for vector-valued Calderón-Zygmund operators, Trans. Amer. Math. Soc, **336**(1993), 537–556.

[22] H. Wang, The boundedness of some operators with rough kernel on the weighted Morrey spaces, Acta Math. Sinica(Chin. Ser), to appear.

[23] H. Wang and H. P. Liu, Some estimates for Bochner-Riesz operators on the weighted Morrey spaces, Acta Math. Sinica(Chin. Ser), to appear.