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On the zeros of polynomials: An extension of the Eneström–Kakeya theorem

Vanessa Botta [a], Messias Meneguette [a], José A. Cuminato [b,*], Sean McKee [c]

[a] Faculdade de Ciências e Tecnologia, UNESP – Univ Estadual Paulista, Departamento de Matemática, Estatística e Computação, P.O. Box 467, 19060-900 Presidente Prudente, SP, Brazil

[b] Departamento de Matemática Aplicada e Estatística, Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, P.O. Box 668, 13560-970 São Carlos, SP, Brazil

[c] Department of Mathematics and Statistics, University of Strathclyde, 26 Richmond Street, Glasgow G1 1XH, Scotland, UK

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ABSTRACT

This paper presents an extension of the Eneström–Kakeya theorem concerning the roots of a polynomial that arises from the analysis of the stability of Brown (K, L) methods. The generalization relates to relaxing one of the inequalities on the coefficients of the polynomial. Two results concerning the zeros of polynomials will be proved, one of them providing a partial answer to a conjecture by Meneguette (1994) [6].

1. Introduction

One of the classical results concerning the roots of a polynomial is known in the literature as the Eneström–Kakeya theorem [4]. This theorem is particularly important in the study of stability of numerical methods for differential equations.

Theorem 1 *(Eneström–Kakeya, real coefficients case). Let $P(z) = \sum_{i=0}^{n} a_i z^i$ be a polynomial such that $0 < a_0 \leqslant a_1 \leqslant \cdots \leqslant a_n$. Then, $P(z)$ has all its zeros inside or on the unit circle.*

In 1994, one of the authors published the following result in the *problems section* of SIAM Review as a conjecture, see [6].

Conjecture 2. *Let the polynomial $P(z) = \sum_{i=0}^{n} a_i z^i$, $a_i \in \mathbb{R}$, be such that*

$$0 < a_0 \leqslant a_1 \leqslant \cdots \leqslant a_{n-1} > a_n, \qquad a_n > 0 \quad \text{and} \quad n a_n > (n-1) a_{n-1},$$

and all its zeros lie in the unit disk. Then, the zeros of the perturbed polynomial

$$S(z) = (a_n + \gamma) z^n + a_{n-1} z^{n-1} + \cdots + a_0$$

lie in the unit disk, for all $\gamma > 0$.

* Corresponding author.
E-mail addresses: botta@fct.unesp.br (V. Botta), messias@fct.unesp.br (M. Meneguette), jacumina@icmc.usp.br (J.A. Cuminato), s.mckee@strath.ac.uk (S. McKee).

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The motivation for this conjecture arose from a study of Brown methods [1], but it is an interesting conjecture in its own right and may be regarded as an extension of Eneström–Kakeya theorem.

Substantial computational experimentation has indicated that Conjecture 2 is true. Nonetheless a proof has not been found up to now and in this work we prove a result that will provide partial validation. In addition it will be seen to have application to the analysis of stability of multistep multiderivative methods. For the sake of completeness, in the next section we present some well-known results that will be needed later.

2. Classical results

Lemma 3. *If the polynomial $P(z)=\sum_{i=0}^{n}a_i z^i$, $a_i\in\mathbb{R}$, has all its zeros in $|z|\leqslant 1$, then $|a_0|\leqslant|a_n|$. If at least one of the roots is inside the unit circle then $|a_0|<|a_n|$.*

Proof. Let $\alpha_1,\alpha_2,\dots,\alpha_n$ be the zeros of $P(z)$ in $|z|\leqslant 1$. Using Vieta's formulae [7], we have

$$(-1)^n\frac{a_0}{a_n}=\alpha_1\alpha_2\cdots\alpha_n.$$

Then

$$\left|\frac{a_0}{a_n}\right|=|\alpha_1\alpha_2\cdots\alpha_n|=|\alpha_1||\alpha_2|\cdots|\alpha_n|\leqslant 1,$$

with strict inequality if at least one of the roots is inside the unit circle. □

Definition 4. Let the polynomial $P(z)=\sum_{i=0}^{n}a_i z^i$, $a_i\in\mathbb{R}$. We define the associated polynomial

$$P^*(z)=z^nP\left(\frac{1}{z}\right)=a_0z^n+a_1z^{n-1}+\cdots+a_n=a_0\prod_{j=1}^{n}\left(z-z_j^*\right),$$

whose zeros z_k^* are, relative to the unit circle, the inverses of the zeros z_k of $P(z)$, that is, $z_k^*=\frac{1}{z_k}$.

Definition 5. Given $P(z)$, the sequence of polynomials $P_j(z)$ is defined by:

$$P_j(z)=\sum_{k=0}^{n-j}a_k^{(j)}z^k,\quad \text{where } P_0(z)=P(z)\quad\text{and}$$

$$P_{j+1}(z)=\Delta P_j(z)\doteq a_0^{(j)}P_j(z)-a_{n-j}^{(j)}P_j^*(z),\quad j=0,1,\dots,n-1,\tag{1}$$

with $P_0^*(z)=P^*(z)$ and $P_j^*(z)=(P_j(z))^*$.

It is clear from (1) that the coefficients of $P_{j+1}(z)$ satisfy the recurrence relation

$$a_k^{(j+1)}=a_0^{(j)}a_k^{(j)}-a_{n-j}^{(j)}a_{n-j-k}^{(j)},\quad k=0,1,\dots,n-j \text{ and } j=0,1,\dots,n.\tag{2}$$

Lemma 6. *The polynomial $P_j(z)$ satisfies for all j, $\Delta(P_j(z))^*=\Delta P_j^*(z)=-P_{j+1}(z)$.*

Proof. From Definition 5 we have that $P_{j+1}(z)=\sum_{k=0}^{n-j-1}a_k^{(j+1)}z^k$ where the coefficients $a_k^{(j+1)}$ are given by (2). Now let $P_j(z)=a_0^{(j)}+a_1^{(j)}z+\cdots+a_{n-j}^{(j)}z^{n-j}$ be so that, from Definition 4, we have $P_j^*(z)=a_0^{(j)}z^{n-j}+a_1^{(j)}z^{n-j-1}+\cdots+a_{n-j}^{(j)}$. Let us now write the polynomial $P_j^*(z)$ in the form $P_j^*(z)=Q(z)=b_0+b_1z+\cdots+b_{n-j}z^{n-j}$ where obviously $b_k=a_{n-j-k}^{(j)}$. From Definition 5 we can calculate the polynomial $\Delta Q(z)=\Delta P_j^*(z)=b_0^{(1)}+b_1^{(1)}z+\cdots+b_{n-j-1}^{(1)}z^{n-j-1}$ where the coefficients $b_k^{(1)}$ are calculated from (2) to give $b_k^{(1)}=b_0b_k-b_{n-j}b_{n-j-k}$. But $b_k=a_{n-j-k}^{(j)}$ and hence $b_k^{(1)}=a_{n-j}^{(j)}a_{n-j-k}^{(j)}-a_0^{(j)}a_k^{(j)}$ and from (2) we see that $b_k^{(1)}=-a_k^{(j+1)}$, and this proves the lemma. □

Definition 7. For each polynomial $P_j(z)$ we shall denote the constant term $a_0^{(j)}$ by δ_j. Thus we see that

$$\delta_{j+1}=a_0^{(j+1)}=\left|a_0^{(j)}\right|^2-\left|a_{n-j}^{(j)}\right|^2,\quad j=0,1,\dots,n-1.$$

Lemma 8. *If P_j has p_j zeros in $|z|<1$ and if $\delta_{j+1}\neq 0$, then P_{j+1} has*

$$p_{j+1}=\begin{cases} p_j, & \text{if } \delta_{j+1}>0,\\ n-j-p_j, & \text{if } \delta_{j+1}<0\end{cases}$$

zeros in $|z|<1$. Furthermore, P_{j+1} has the same zeros on $|z|=1$ as P_j.

Proof. The proof of this lemma may be found in Marden [4, p. 195]. □

Regarding the last statement of Lemma 8 we remark that it has, as a consequence, the following result that will be important in the proof of the main theorem in the next section.

Lemma 9. *Let $P(z)$ be a polynomial with real coefficients. If $P(z)$ has q zeros on the unit circle then $P_{q+1}(z)\equiv 0$. In particular if $P(z)$ has all its roots on the unit circle then $\Delta P(z)\equiv 0$.*

Proof. From the last statement of Lemma 8, $P_{q+1}(z)$ has the same roots as $P_q(z)$ on the unit circle. By a recursive argument this polynomial has the same zeros as $P_{q-1}(z)$ on the unit circle and so on, leading to the conclusion that it has the same zeros as $P(z)$ on the unit circle. In conclusion $P_{q+1}(z)$ has q roots on the unit circle, but $P_{q+1}(z)$ is a polynomial of degree $q-1$ by construction and hence it must vanish. □

The next result is due to Schur [8,9]. The proof of this result is included below as the argument provides some insight into the main result of this paper.

Lemma 10. *If $0<|a_0|<|a_n|$, then $P(z)$ has all its zeros on or inside the unit circle if and only if $\Delta P^*(z)$ has all its zeros on or inside the unit circle.*

Proof. First we prove the lemma for the case where $|a_0|<|a_n|$ and both $P(z)$ and $\Delta P^*(z)$ have all their zeros strictly inside the unit circle. Observe that $\delta_1=a_0^2-a_n^2<0$; hence from Lemma 8 we have that $p_{j+1}=n-j-p_j$. As the zeros of $P(z)$ lie inside the unit circle we have $p_0=n$ in Lemma 8, giving $p_1=0$, that is all the zeros of $P_1(z)$ lie outside the unit circle, consequently those of $\Delta P^*(z)$ lie inside the unit circle. Conversely, if $\Delta P^*(z)$ has all its zeros inside the unit circle, then $p_1=0$. So, from Lemma 8, $p_0=n$, that is $P(z)$ has all the zeros inside the unit circle, and this proves the lemma for this simpler case.

Let us consider now the case where $P(z)$ has m zeros in $|z|<1$ and q zeros on the unit circle ($m+q=n$). Then we have

$$P(z)=a_n(z-\alpha_1)(z-\alpha_2)\cdots(z-\alpha_q)R(z),$$

where α_i ($i=1,2,\dots,q$) are the zeros of $P(z)$ such that $|\alpha_i|=1$ and $R(z)=c_0+c_1z+\cdots+c_mz^m$ is a polynomial that has all its zeros inside the unit circle (consequently, $R^*(z)$ has all its zeros outside the unit circle). Observe that $c_m=1$,

$$c_0=\frac{a_0}{(-1)^q a_n\alpha_1\alpha_2\cdots\alpha_q} \tag{3}$$

and $0<|c_0|<|c_m|$. As $P(z)$ and $P^*(z)$ have the same zeros on the unit circle, we can write

$$P^*(z)=z^nP\left(\frac{1}{z}\right)=\frac{a_0}{c_0}(z-\alpha_1)(z-\alpha_2)\cdots(z-\alpha_q)R^*(z)$$

and $P^*(z)$ has all the zeros in $|z|\geqslant 1$. From Lemma 8, $P(z)$ has the same zeros on $|z|=1$ as $\Delta P(z)$, and from (1) we may write

$$\begin{aligned}\Delta P(z)&=a_0a_n(z-\alpha_1)(z-\alpha_2)\cdots(z-\alpha_q)R(z)-\frac{a_na_0}{c_0}(z-\alpha_1)(z-\alpha_2)\cdots(z-\alpha_q)R^*(z)\\ &=\frac{a_0a_n}{c_0}(z-\alpha_1)(z-\alpha_2)\cdots(z-\alpha_q)\big(c_0R(z)-c_mR^*(z)\big)\\ &=\frac{a_0a_n}{c_0}(z-\alpha_1)(z-\alpha_2)\cdots(z-\alpha_q)\Delta R(z)\end{aligned}$$

and recalling (3) we get:

$$\Delta P^*(z)=\left(\frac{a_0}{c_0}\right)^2(z-\alpha_1)(z-\alpha_2)\cdots(z-\alpha_q)\Delta R^*(z).$$

To conclude the proof we need only apply Lemma 8 to the polynomial $R(z)$. Observe that, in this case, $\delta_1=c_0^2-c_m^2<0$. As $R(z)$, by construction, has all its zeros inside the unit circle, that is, $r_0=m$, then, by Lemma 8, for $j=0$ we have $r_1=0$.

Then, the zeros of $\Delta R(z)$ lie outside the unit circle. Consequently, the zeros of $\Delta R^*(z)$ lie inside the unit circle and $\Delta P^*(z)$ has all its zeros inside or on the unit circle. Conversely, if $\Delta R^*(z)$ has all its zeros inside the unit circle, then $r_1 = 0$. So, by Lemma 8, $r_0 = m$. Thus, $P(z)$ has all its zeros inside or on the unit circle. □

We note that if $|a_0| > |a_n|$ and $P(z)$ has all zeros outside the unit circle, from Lemma 8, $\Delta P(z)$ has all zeros inside the unit disk and, consequently, $\Delta P^*(z)$ has all zeros outside the unit circle.

3. Main result

We first require the following lemma.

Lemma 11. *Let the sequence* $\{b_k^{(j)}:\ j = 1, 2, \ldots, n-1;\ k = 0, 1, \ldots, n-j\}$ *be defined by*

$$b_k^{(j+1)} = b_0^{(j)} b_k^{(j)} - b_{n-j}^{(j)} b_{n-j-k}^{(j)}, \qquad b_k^{(0)} = b_k \tag{4}$$

where b_k, $k = 0, 1, \ldots, n$, *is a given sequence of real numbers. If*

$$b_1^{(2)} > b_2^{(2)} > \cdots > b_{n-2}^{(2)} > 0 \quad \text{and} \quad b_k^{(2)} < r b_{k-1}^{(2)}, \quad k = 2, 3, \ldots, n-2,$$

and $b_0^{(2)} > r b_1^{(2)}$, *for some real* $r \in (0, 1)$, *then for* $j = 2, 3, \ldots, n-1$ *we have*:

$$b_0^{(j)} > r b_1^{(j)} > 0, \tag{5}$$

$$b_1^{(j)} > b_2^{(j)} > \cdots > b_{n-j}^{(j)} > 0, \tag{6}$$

$$b_k^{(j)} < r b_{k-1}^{(j)}, \quad k = 2, 3, \ldots, n-j. \tag{7}$$

Proof. The proof is by induction on j. Firstly, we note that the inductive hypothesis is true for $\nu = 2$ from the assumptions of the lemma. Let us suppose that the statement is true for all $j = 2, 3, \ldots, \nu$. We shall prove that it is true for $j = \nu + 1$. For $i = 1, 2, \ldots, n - \nu - 2$, from (4) and the inductive hypothesis,

$$\begin{aligned} b_i^{(\nu+1)} - b_{i+1}^{(\nu+1)} &= b_0^{(\nu)} b_i^{(\nu)} - b_{n-\nu}^{(\nu)} b_{n-\nu-i}^{(\nu)} - b_0^{(\nu)} b_{i+1}^{(\nu)} + b_{n-\nu}^{(\nu)} b_{n-\nu-i-1}^{(\nu)} \\ &= b_0^{(\nu)} \big(b_i^{(\nu)} - b_{i+1}^{(\nu)}\big) + b_{n-\nu}^{(\nu)} \big(b_{n-\nu-i-1}^{(\nu)} - b_{n-\nu-i}^{(\nu)}\big) > 0. \end{aligned}$$

Thus

$$b_i^{(\nu+1)} > b_{i+1}^{(\nu+1)}, \quad i = 1, 2, \ldots, n - \nu - 2. \tag{8}$$

Now

$$\begin{aligned} b_{n-(\nu+1)}^{(\nu+1)} &= b_0^{(\nu)} b_{n-(\nu+1)}^{(\nu)} - b_{n-\nu}^{(\nu)} b_1^{(\nu)} \\ &> r b_1^{(\nu)} b_{n-(\nu+1)}^{(\nu)} - b_{n-\nu}^{(\nu)} b_1^{(\nu)}, \quad \text{using (5)} \\ &> b_1^{(\nu)} \big(r b_{n-(\nu+1)}^{(\nu)} - b_{n-\nu}^{(\nu)}\big) > 0, \quad \text{using (7)}. \end{aligned} \tag{9}$$

Then from (8) and (9),

$$b_1^{(\nu+1)} > b_2^{(\nu+1)} > \cdots > b_{n-(\nu+1)}^{(\nu+1)} > 0.$$

Now

$$\begin{aligned} b_0^{(\nu+1)} - r b_1^{(\nu+1)} &= \big(b_0^{(\nu)}\big)^2 - \big(b_{n-\nu}^{(\nu)}\big)^2 - r\big(b_0^{(\nu)} b_1^{(\nu)} - b_{n-\nu}^{(\nu)} b_{n-\nu-1}^{(\nu)}\big) \\ &= b_0^{(\nu)} \big(b_0^{(\nu)} - r b_1^{(\nu)}\big) + b_{n-\nu}^{(\nu)} \big(r b_{n-\nu-1}^{(\nu)} - b_{n-\nu}^{(\nu)}\big) > 0, \end{aligned} \tag{10}$$

using (4) and the inductive hypothesis. Thus $b_0^{(\nu+1)} > r b_1^{(\nu+1)}$ and as $b_1^{(\nu+1)} > 0$ we have

$$b_0^{(\nu+1)} > 0.$$

However $b_0^{(\nu+1)} = (b_0^{(\nu)})^2 - (b_{n-\nu}^{(\nu+1)})^2$ from (4) and so we have

$$\big(b_0^{(\nu)}\big)^2 - \big(b_{n-\nu}^{(\nu)}\big)^2 > 0.$$

Finally, for $i = 2, 3, \ldots, n - (\nu + 1)$,

$$\begin{aligned}
b_i^{(\nu+1)} - rb_{i-1}^{(\nu+1)} &= b_0^{(\nu)} b_i^{(\nu)} - b_{n-\nu}^{(\nu)} b_{n-i-\nu}^{(\nu)} - r\big(b_0^{(\nu)} b_{i-1}^{(\nu)} - b_{n-\nu}^{(\nu)} b_{n-i-\nu+1}^{(\nu)}\big) \\
&= b_0^{(\nu)}\big(b_i^{(\nu)} - rb_{i-1}^{(\nu)}\big) + b_{n-\nu}^{(\nu)}\big(rb_{n-i-\nu+1}^{(\nu)} - b_{n-i-\nu}^{(\nu)}\big) \\
&< b_0^{(\nu)}\big(b_i^{(\nu)} - rb_{i-1}^{(\nu)}\big) + b_{n-\nu}^{(\nu)}\big(rb_{n-i-\nu}^{(\nu)} - b_{n-i-\nu}^{(\nu)}\big) \\
&= b_0^{(\nu)}\big(b_i^{(\nu)} - rb_{i-1}^{(\nu)}\big) + (r-1) b_{n-\nu}^{(\nu)} b_{n-i-\nu}^{(\nu)} < 0,
\end{aligned}$$

using (4) and the inductive hypothesis. Furthermore, (10) implies that $b_0^{(\nu+1)} > rb_1^{(\nu+1)}$ for $i = 1, 2, \ldots, n-(\nu+1)$ and the proof is complete. □

The principal theorem of this article is the following:

Theorem 12. *Let the polynomial $P(z) = \sum_{i=0}^n a_i z^i$, $a_i \in \mathbb{R}$, be such that all its zeros lie on or inside the unit circle and the coefficients satisfy*

$$0 < a_0 < a_1 < \cdots < a_{n-1} > a_n, \qquad a_n > 0, \tag{11}$$

$$a_i < ra_{i+1}, \quad i = 0, 1, \ldots, n-2, \quad \text{and} \quad a_n > ra_{n-1}, \tag{12}$$

for some real $0 < r < 1$. Then, the polynomial $S(z) = P(z) + \gamma z^n$, $\gamma > 0$, has all its zeros inside the unit circle.

Proof. We define the polynomials

$$S(z) = b_n z^n + b_{n-1} z^{n-1} + \cdots + b_0,$$

where $b_n = a_n + \gamma$ and $b_i = a_i$, $i = 0, 1, \ldots, n-1$, and

$$S_j(z) = b_{n-j}^{(j)} z^{n-j} + b_{n-j-1}^{(j)} z^{n-j-1} + \cdots + b_0^{(j)}, \quad j = 0, 1, \ldots, n,$$

where the coefficients $b_k^{(j)}$, $k = 0, 1, \ldots, n-j$, are defined by

$$b_k^{(j+1)} = b_0^{(j)} b_k^{(j)} - b_{n-j}^{(j)} b_{n-j-k}^{(j)} \tag{13}$$

and

$$S_0(z) = S(z).$$

Note that, since the roots of $P(z)$ are on or inside the unit circle, from Lemma 3 $a_0 \leqslant a_n$; consequently, as $0 < a_0$ and $a_0 = b_0$ we have $0 < b_0 \leqslant a_n < a_n + \gamma = b_n$ and then Lemma 10 can be applied to conclude that the zeros of $S(z)$ lie inside the unit circle if and only if the zeros of $S_1^*(z)$ do.

The idea of the proof is to apply Lemma 10 recursively to show that the zeros of $\Delta S_(z) \doteq S_1^*(z)$ lie inside the unit circle if and only if the zeros of $S_2^*(z)$ do and so on until we reach the polynomial $S_{n-1}^*(z) = b_1^{(n-1)} + b_0^{(n-1)} z$ for which we can calculate the zero and show that it lies inside the unit circle. With this purpose in mind we divide the proof into three parts. First we have to show, at each stage, that the polynomial that we are dealing with is indeed a member of the sequence defined by (1), since in Lemma 10 we have that the zeros of $S(z)$ lie inside the unit circle if and only if those of $Q(z) = S_1^*(z)$ do. So to apply this lemma again we have to show that the polynomial $\Delta Q^*(z)$ is a member of the sequence (1). But from Lemma 6, $\Delta Q^*(z) = -S_2(z)$, and so it has the same roots as $S_2(z)$. This brings us back to the sequence (1), and the argument can be carried on. The next step is to prove that at each step of the recursive argument the coefficients of the polynomials $S_j(z)$ satisfy the hypothesis of Lemma 10, that is to show that $|b_{n-j}^{(j)}| < |b_0^{(j)}|$. Finally, in the third step, we show that the recursive application of Lemma 10, will always yield a stage where a non-vanishing polynomial of first degree, $S_{n-1}^*(z) = b_1^{(n-1)} + b_0^{(n-1)} z$ is obtained. It will be shown that the root of this polynomial lies inside the unit circle. We remark that the fact that the recursive argument can be carried out all the way down to first degree is due to the fact that in every stage we get a non-vanishing polynomial $S_j(z)$, as will be shown. This, on the other hand, implies, by Lemma 9, that the roots of $S(z)$ lie strictly inside the unit circle as stated in the theorem.

Steps 2 and 3 of the above approach will be proved in what follows. However, we start by presenting the proof that $|b_{n-j}^{(j)}| < |b_0^{(j)}|$. For $j = 1$, from the assumptions (11) and (13) it can be easily verified that

$$b_1^{(1)} < b_2^{(1)} < \cdots < b_{n-1}^{(1)}. \tag{14}$$

Now

$$\begin{aligned}
b_{n-1}^{(1)} &= b_0 b_{n-1} - (a_n + \gamma) b_1 < rb_1 b_{n-1} - (a_n + \gamma) b_1 \\
&= b_1(rb_{n-1} - a_n) - \gamma b_1 < 0
\end{aligned}$$

using (13), the definition of b_k given in (12) and (13). Then, from (14) we have $b_k^{(1)} < 0$ for $k = 1, 2, \dots, n-1$, and

$$\left|b_1^{(1)}\right| > \left|b_2^{(1)}\right| > \cdots > \left|b_{n-1}^{(1)}\right|.$$

We now prove that the coefficients of $S_1(z)$ satisfy

$$\left|b_k^{(1)}\right| < r\left|b_{k-1}^{(1)}\right|, \quad \text{for } k = 2, 3, \dots, n-1, \quad \text{and} \quad \left|b_0^{(1)}\right| > r\left|b_1^{(1)}\right|.$$

In fact, from (12) and (13),

$$\begin{aligned} b_0^{(1)} - rb_1^{(1)} &= b_0^2 - b_n^2 - r(b_0 b_1 - b_n b_{n-1}) \\ &= b_0(b_0 - rb_1) + b_n(rb_{n-1} - b_n) \\ &= a_0(a_0 - ra_1) + (a_n + \gamma)(ra_{n-1} - a_n) - \gamma(a_n + \gamma) < 0. \end{aligned}$$

Then $b_0^{(1)} < rb_1^{(1)}$. As $b_1^{(1)} < 0$ and $r > 0$ we get $b_0^{(1)} < 0$ and

$$\left|b_0^{(1)}\right| > r\left|b_1^{(1)}\right|. \tag{15}$$

Now, for $k = 2, 3, \dots, n-1$ and $0 < r < 1$,

$$\begin{aligned} b_k^{(1)} - rb_{k-1}^{(1)} &= b_0 b_k - b_n b_{n-k} - r(b_0 b_{k-1} - b_n b_{n-k+1}) \\ &= b_0(b_k - rb_{k-1}) + b_n(rb_{n-k+1} - b_{n-k}). \end{aligned}$$

Observe that, for $0 < r < 1$,

$$b_k - rb_{k-1} > \frac{1}{r} b_{k-1} - rb_{k-1} = \frac{1 - r^2}{r} b_{k-1} > 0.$$

Then, $b_k^{(1)} - rb_{k-1}^{(1)} > 0$, for $k = 2, 3, \dots, n-1$ and $0 < r < 1$. Therefore, since $b_1^{(1)} < b_2^{(1)} < \cdots < b_{n-1}^{(1)} < 0$ we have, for $k = 2, 3, \dots, n-1$,

$$\left|b_k^{(1)}\right| < r\left|b_{k-1}^{(1)}\right|. \tag{16}$$

From (13) and (14), we can easily deduce that

$$b_1^{(2)} > b_2^{(2)} > \cdots > b_{n-2}^{(2)}. \tag{17}$$

Now, from (13), the fact that $b_k^{(1)} < 0$, $k = 1, 2, \dots, n-1$ and (15) it follows that

$$\begin{aligned} b_{n-2}^{(2)} &= b_0^{(1)} b_{n-2}^{(1)} - b_{n-1}^{(1)} b_1^{(1)} = \left|b_0^{(1)}\right|\left|b_{n-2}^{(1)}\right| - \left|b_{n-1}^{(1)}\right|\left|b_1^{(1)}\right| \\ &> r\left|b_1^{(1)}\right|\left|b_{n-2}^{(1)}\right| - \left|b_{n-1}^{(1)}\right|\left|b_1^{(1)}\right| = \left|b_1^{(1)}\right|\left(r\left|b_{n-2}^{(1)}\right| - \left|b_{n-1}^{(1)}\right|\right) > 0, \quad \text{using (16)}. \end{aligned}$$

Hence $b_{n-2}^{(2)} > 0$ and from (17), $b_k^{(2)} > 0$ for $k = 1, 2, \dots, n-2$. That is,

$$b_1^{(2)} > b_2^{(2)} > \cdots > b_{n-2}^{(2)} > 0.$$

We shall now prove that the coefficients of $S_2(z)$ satisfy

$$\begin{aligned} &b_k^{(2)} < rb_{k-1}^{(2)}, \quad \text{for } k = 2, 3, \dots, n-2, \\ &b_0^{(2)} > rb_1^{(2)}. \end{aligned}$$

In fact, from (13), (15), (16) and $b_i^{(1)} < 0$, $i = 1, 2, \dots, n-1$, we have

$$\begin{aligned} b_0^{(2)} - rb_1^{(2)} &= \left(b_0^{(1)}\right)^2 - \left(b_{n-1}^{(1)}\right)^2 - r\left(b_0^{(1)} b_1^{(1)} - b_{n-2}^{(1)} b_{n-1}^{(1)}\right) \\ &= b_0^{(1)}\left(b_0^{(1)} - rb_1^{(1)}\right) + b_{n-1}^{(1)}\left(rb_{n-2}^{(1)} - b_{n-1}^{(1)}\right) > 0. \end{aligned}$$

Thus,

$$b_0^{(2)} > rb_1^{(2)} \quad \text{and as} \quad b_1^{(2)} > 0 \quad \text{we have } b_0^{(2)} > 0.$$

Since

$$b_0^{(2)} = \left(b_0^{(1)}\right)^2 - \left(b_{n-1}^{(1)}\right)^2 > 0, \quad \text{we have } \left|b_0^{(1)}\right| > \left|b_{n-1}^{(1)}\right|.$$

Finally for $k = 2, 3, \dots, n-2$ we have

$$b_k^{(2)} - rb_{k-1}^{(2)} = b_0^{(1)} b_k^{(1)} - b_{n-1}^{(1)} b_{n-k-1}^{(1)} - r\big(b_0^{(1)} b_{k-1}^{(1)} - b_{n-1}^{(1)} b_{n-k}^{(1)}\big)$$
$$= b_0^{(1)}\big(b_k^{(1)} - rb_{k-1}^{(1)}\big) + b_{n-1}^{(1)}\big(rb_{n-k}^{(1)} - b_{n-k-1}^{(1)}\big) < 0.$$

In summary, we proved that

$$b_1^{(2)} > b_2^{(2)} > \cdots > b_{n-2}^{(2)} > 0,$$
$$b_k^{(2)} < rb_{k-1}^{(2)}, \quad k = 2, 3, \dots, n-2 \quad \text{and}$$
$$b_0^{(2)} > rb_1^{(2)}.$$

We may now invoke Lemma 11 to argue that

$$b_1^{(j)} > b_2^{(j)} > \cdots > b_{n-j}^{(j)} > 0, \quad j = 2, 3, \dots, n-1,$$
$$b_k^{(j)} < rb_{k-1}^{(j)}, \quad k = 2, 3, \dots, n-j, \ j = 2, 3, \dots, n-1, \quad \text{and}$$
$$b_0^{(j)} > rb_1^{(j)}, \quad j = 2, 3, \dots, n-1,$$

and this completes the second part of the proof of Theorem 12.

To complete the proof we proceed as follows. We observe first that in the proof of part two it has been demonstrated that the coefficients $b_k^{(j)}$, $k = 0, 1, \dots, n-j$, are non-zero for all j. In fact they are either positive or negative numbers, so that $S_j(z)$ is non-vanishing for all j and from Lemma 9 we deduce that the roots of $S(z)$ are inside the unit circle if and only if the root of the first degree polynomial $S_{n-1}^*(z) = b_1^{(n-1)} + b_0^{(n-1)} z$ is in the unit circle.

This argument will now be given.

From (5) and (6), $b_1^{(n-1)} > 0$ and $b_0^{(n-1)} > 0$, respectively.

For $j = 2, 3, \dots, n-1$, from (13), we have

$$0 < b_0^{(j)} = \big(b_0^{(j-1)}\big)^2 - \big(b_{n-(j-1)}^{(j-1)}\big)^2 = \big(b_0^{(j-1)} - b_{n-(j-1)}^{(j-1)}\big)\big(b_0^{(j-1)} + b_{n-(j-1)}^{(j-1)}\big),$$

where

$$b_0^{(j-1)} - b_{n-(j-1)}^{(j-1)} > 0 \quad \text{and} \quad b_0^{(j-1)} + b_{n-(j-1)}^{(j-1)} > 0. \tag{18}$$

We first note that

$$b_0^{(n-1)} = \big(b_0^{(n-2)}\big)^2 - \big(b_2^{(n-2)}\big)^2 = \big(b_0^{(n-2)} - b_2^{(n-2)}\big)\big(b_0^{(n-2)} + b_2^{(n-2)}\big)$$

and

$$b_1^{(n-1)} = b_0^{(n-2)} b_1^{(n-2)} - b_1^{(n-2)} b_2^{(n-2)} = b_1^{(n-2)}\big(b_0^{(n-2)} - b_2^{(n-2)}\big)$$

so that

$$b_0^{(n-1)} - b_1^{(n-1)} = \big(b_0^{(n-2)} - b_2^{(n-2)}\big)\big(b_0^{(n-2)} - b_1^{(n-2)} + b_2^{(n-2)}\big). \tag{19}$$

We shall now prove by induction on k that:

$$b_0^{(n-1)} - b_1^{(n-1)} = \big(b_0^{(n-2)} - b_2^{(n-2)}\big)\big(b_0^{(n-3)} + b_3^{(n-3)}\big)\big(b_0^{(n-4)} - b_4^{(n-4)}\big)$$
$$\cdots \big(b_0^{(n-k)} + (-1)^{k+1} b_k^{(n-k)}\big)\left(\sum_{i=0}^{k} (-1)^i b_i^{(n-k)}\right). \tag{20}$$

First note that (19) represents the induction hypothesis for $k = 2$. We first assume that it is true for $k = m$ and then show that it is true for $k = m+1$.

Expanding the last bracket on the right-hand side of (20) for $k = m$ we have

$$\sum_{i=0}^{m} (-1)^i b_i^{(n-m)} = \sum_{i=0}^{m} (-1)^i \big[b_0^{(n-m-1)} b_i^{(n-m-1)} - b_{m+1}^{(n-m-1)} b_{m+1-i}^{(n-m-1)}\big]$$
$$= \sum_{i=0}^{m} (-1)^i b_0^{(n-m-1)} b_i^{(n-m-1)} + \sum_{i=0}^{m} (-1)^{i+1} b_{m+1}^{(n-m-1)} b_{m+1-i}^{(n-m-1)}$$
$$= b_0^{(n-m-1)} \sum_{i=0}^{m} (-1)^i b_i^{(n-m-1)} + b_{m+1}^{(n-m-1)} \sum_{i=0}^{m} (-1)^{m-i+1} b_{i+1}^{(n-m-1)}$$

where the order of summation has been reversed in the second term. Now

$$\begin{aligned}\sum_{i=0}^{m}(-1)^i b_i^{(n-m)} &= \big(b_0^{(n-m-1)}\big)^2 + b_0^{(n-m-1)}\sum_{i=1}^{m}(-1)^i b_i^{(n-m-1)} \\ &\quad + b_{m+1}^{(n-m-1)}\sum_{i=0}^{m-1}(-1)^{m-i+1} b_{i+1}^{(n-m-1)} - \big(b_{m+1}^{(n-m-1)}\big)^2 \\ &= \big(b_0^{(n-m-1)} - b_{m+1}^{(n-m-1)}\big)\big(b_0^{(n-m-1)} + b_{m+1}^{(n-m-1)}\big) \\ &\quad + \sum_{i=1}^{m}(-1)^i b_i^{(n-m-1)}\big(b_0^{(n-m-1)} + (-1)^m b_{m+1}^{(n-m-1)}\big) \\ &= \begin{cases} (b_0^{(n-m-1)} + b_{m+1}^{(n-m-1)})(\sum_{i=0}^{m+1}(-1)^i b_i^{(n-m-1)}), & m \text{ even}, \\ (b_0^{(n-m-1)} - b_{m+1}^{(n-m-1)})(\sum_{i=0}^{m+1}(-1)^i b_i^{(n-m-1)}), & m \text{ odd}, \end{cases}\end{aligned}$$

which completes the induction.

Select $k = n$ in (20) to obtain

$$b_0^{(n-1)} - b_1^{(n-1)} = \big(b_0^{(n-2)} - b_2^{(n-2)}\big)\big(b_0^{(n-3)} + b_3^{(n-3)}\big)\cdots\big(b_0^{(1)} + (-1)^n b_{n-1}^{(1)}\big)\big(b_0 + (-1)^{n+1} b_n\big)\big((-1)^n\gamma + P(-1)\big).$$

From (18), the sign of $b_0^{(n-1)} - b_1^{(n-1)}$ depends on the sign of the term

$$\big(b_0^{(1)} + (-1)^n b_{n-1}^{(1)}\big)\big(b_0 + (-1)^{n+1} b_n\big)\big((-1)^n\gamma + P(-1)\big),$$

since all other terms are positive.

Now since $P(z)$ has all its zeros inside or on the unit circle we have that $P(-1) \geqslant 0$ for n even and $P(-1) \leqslant 0$ for n odd. So, for n even,

$$b_0^{(1)} + (-1)^n b_{n-1}^{(1)} < 0, \qquad b_0 + (-1)^{n+1} b_n < 0 \quad \text{and} \quad (-1)^n\gamma + P(-1) > 0,$$

and, for n odd,

$$b_0^{(1)} + (-1)^n b_{n-1}^{(1)} < 0, \qquad b_0 + (-1)^{n+1} b_n > 0 \quad \text{and} \quad (-1)^n\gamma + P(-1) < 0.$$

Consequently,

$$b_0^{(n-1)} - b_1^{(n-1)} > 0$$

and the root of $S_{n-1}^*(z)$ is

$$z_1 = -\frac{b_1^{(n-1)}}{b_0^{(n-1)}}$$

which is clearly inside the unit circle.

This completes the proof that $S(z)$ has all its zeros inside the unit circle. □

4. Numerical examples

The following are examples of polynomials that satisfy the conditions of Theorem 12.

Example 13. Let us consider the polynomial $S(z) = 0.3 + 1.5z + 3z^2 + (1.8 + \gamma)z^3$. Fig. 1 displays the roots of $P(z)$ (represented by •) and $S(z)$ for $\gamma = 1$ (represented by $*$). Note that one of the roots of $P(z)$ lies on the unit circle. The conditions of Theorem 12 are seen to be satisfied when r is selected to be 0.55 and the roots of the perturbed polynomial $S(z)$ for $\gamma = 1$ are observed to lie within the unit circle.

Example 14. Let us consider the polynomial $S(z) = 0.04 + 0.3z + z^2 + 2.5z^3 + 5z^4 + (3.33 + \gamma)z^5$. Fig. 2 displays the roots of $P(z)$ (represented by •) and $S(z)$ for $\gamma = 1.57$ (represented by $*$). The conditions of Theorem 12 are seen to be satisfied when r is selected to be $\frac{3}{5}$, and the roots of the perturbed polynomial $S(z)$ (with $\gamma = 1.57$) are observed to lie inside the unit circle.

Example 15. Let us consider the polynomial $S(z) = 0.16 + 0.2z + 0.9z^2 + 2.2z^3 + 3.75z^4 + 6z^5 + (5.1 + \gamma)z^6$. Fig. 3 displays the roots of $P(z)$ (represented by •) and $S(z)$ for $\gamma = 0.88$ (represented by $*$). The conditions of Theorem 12 are seen to be satisfied when r is chosen to be $\frac{5}{6}$ and the roots of perturbed polynomial $S(z)$ (with $\gamma = 0.88$) are observed to lie inside the unit circle.

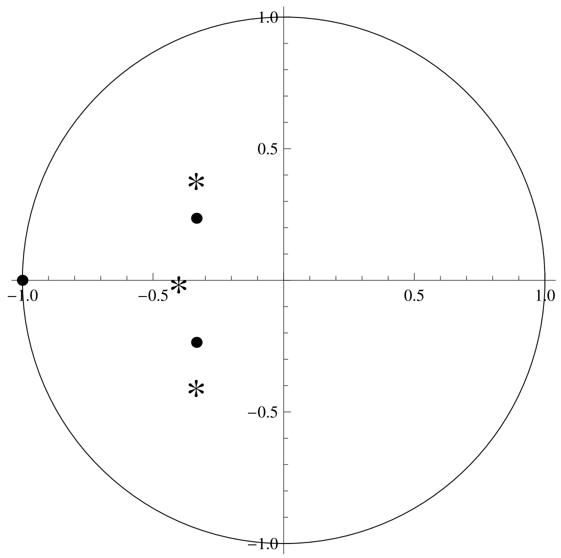

Fig. 1. Roots of $S(z) = 0.3 + 1.5z + 3z^2 + (1.8 + \gamma)z^3$ for $\gamma = 0$ (black dots) and $\gamma = 1$ (stars).

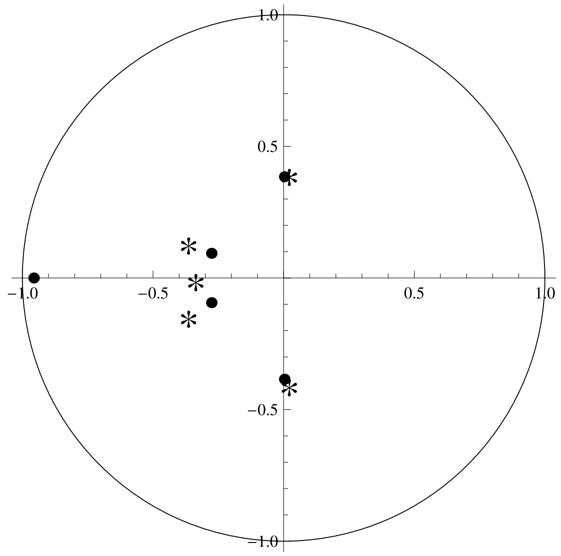

Fig. 2. Roots of $S(z) = 0.04 + 0.3z + z^2 + 2.5z^3 + 5z^4 + (3.33 + \gamma)z^5$ for $\gamma = 0$ (black dots) and $\gamma = 1.57$ (stars).

5. Application to the study of stability of Brown (K, L) methods

The (K, L) methods of Brown are defined by

$$\sum_{i=0}^{K} \alpha_i y_{n+i} = \sum_{j=1}^{L} h^j \beta_j f_{n+K}^{(j-1)},$$

where the coefficients α_i and β_j are chosen to maximize the precision of the method and are given by (see Jeltsch and Kratz [3])

$$\alpha_i = (-1)^{K-i} \binom{K}{i} (K-i)^{-L}, \quad i = 0, 1, \ldots, K-1,$$
$$\alpha_K = -(\alpha_0 + \alpha_1 + \cdots + \alpha_{K-1}),$$
$$\beta_j = \frac{(-1)^j}{j!} \sum_{i=0}^{K-1} (-1)^{K-i} \binom{K}{i} (K-i)^{j-L}, \quad j = 1, 2, \ldots, L.$$

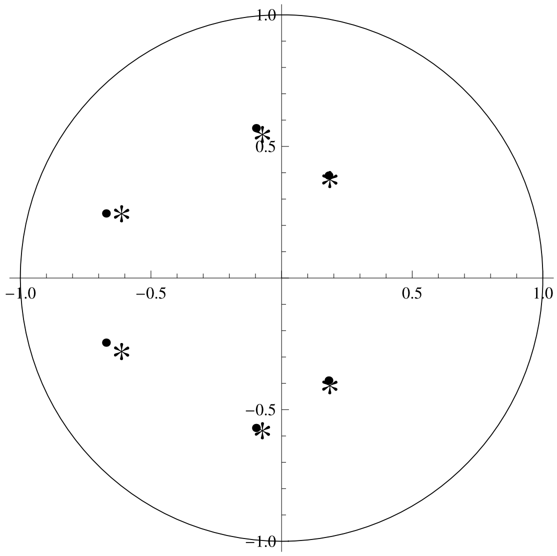

Fig. 3. Roots of $S(z) = 0.16 + 0.2z + 0.9z^2 + 2.2z^3 + 3.75z^4 + 6z^5 + (5.1 + \gamma)z^6$ for $\gamma = 0$ (black dots) and $\gamma = 0.88$ (stars).

When $L = 1$, the $(K, 1)$ class of Brown methods are none other than the well-known Backward Difference Formulae (BDF). An interesting result in Meneguette [5] concerning the A_0-stability of the (K, L) method shows that it is related to the polynomial

$$\rho_\gamma(z) = (\alpha_K + \gamma)z^K + \alpha_{K-1}z^{K-1} + \cdots + \alpha_0,$$

where $\gamma > 0$ and to the first characteristic polynomial that supplies zero-stability:

$$\rho_0(z) = \alpha_K z^K + \alpha_{K-1}z^{K-1} + \cdots + \alpha_0 = \rho(z).$$

That is, if $\rho_\gamma(z)$ has all the zeros inside or on the unit circle, a Brown (K, L) method is A_0-stable.

For more details about A_0-stability, zero-stability, strong-stability and stiff-stability, see Jeltsch [2].

Meneguette [5] also proves the following theorem concerning the coefficients of the first characteristic polynomial $\rho(z)$ of a Brown (K, L) method.

Theorem 16. *The coefficients of the first characteristic polynomial $\rho(z) = \sum_{i=0}^{K} \alpha_i z^i$ associated with a Brown (K, L) method satisfy*:

1. *For each $K = 1, 2, \ldots, K_L$, we have $0 < |\alpha_0| < |\alpha_1| < \cdots < |\alpha_{K-1}|$ and $|\alpha_K| < |\alpha_{K-1}|$, where*

$$K_L = \min\left\{2^{L+1} + 1, \frac{3^{L+1}}{2^L} + 2, \ldots, \frac{(L+1)^{L+1}}{L^L} + L\right\}.$$

2. *For each $K = 2, 3, \ldots, K_L^*$, we have $2|\alpha_j| < |\alpha_{j+1}|$, $j = 0, 1, \ldots, K - 2$, and $2|\alpha_K| > |\alpha_{K-1}|$, where*

$$K_L^* = \min\left\{2^L + 1, \frac{3^{L+1}}{2^{L+1}} + 2, \ldots, \frac{(L+1)^{L+1}}{2L^L} + L\right\}.$$

As $K_L > K_L^*$, the first condition of Theorem 16 is certainly satisfied for each $K = 1, 2, \ldots, K_L^*$.

As an application of Theorem 12 we shall now prove two results concerning the stability of a class of Brown (K, L) methods.

Theorem 17. *For $K \leqslant K_L^*$, every zero-stable Brown (K, L) method is also A_0-stable.*

Proof. If the Brown (K, L) method is zero-stable then $\rho(z)$ has all the zeros inside or on the unit circle. For the large subclass of Brown (K, L) methods, with $K \leqslant K_L^*$, from Theorem 16, the coefficients of $\rho(z)$ satisfy

$$0 < |\alpha_0| < |\alpha_1| < \cdots < |\alpha_{K-1}| \quad \text{and} \quad |\alpha_K| < |\alpha_{K-1}|,$$
$$|\alpha_j| < \frac{1}{2}|\alpha_{j+1}|, \quad j = 0, 1, \ldots, K - 2, \quad \text{and} \quad |\alpha_K| > \frac{1}{2}|\alpha_{K-1}|.$$

For this subclass the assumptions of Theorem 12 are satisfied with $r = \frac{1}{2}$. Then, $\rho_\gamma(z)$ has all its zeros inside the unit circle and, consequently, the Brown (K, L) method is A_0-stable. □

In [2], Jeltsch proved that Brown (K, L) methods are stiffly-stable if and only if they are A_0-stable and strongly-stable. In addition, he conjectured in the last paragraph of his paper that a zero-stable Brown method is A_0-stable and stiffly-stable. Since for $K \leqslant K_L^*$, every zero-stable Brown method is A_0-stable, the assumption of A_0-stability could be dropped, that is, Brown methods are stiffly-stable if and only if they are zero-stable.

Theorem 18. *For $K \leqslant K_L^*$, a Brown (K, L) method is zero-stable if and only if it is stiffly-stable.*

Proof. If a Brown (K, L) method is stiffly-stable then it is A_0-stable and strongly-stable. Consequently, it is zero-stable.

On the other hand, for $K \leqslant K_L^*$, a zero-stable Brown (K, L) method is also A_0-stable (from Theorem 17) and strongly-stable (see Meneguette [5]). Then, it is stiffly-stable. □

6. Conclusion

This note has provided further evidence of the validity of Conjecture 2 through the proof of a partial result. Although this conjecture is in the area of the geometry of polynomials it clearly has applications to the theory of numerical methods. Indeed, one of the partial results supplied in this note, for instance when the coefficients satisfy the conditions of Theorem 12, has provided a partial answer to Jeltsch's conjecture: the large subclass of zero-stable Brown (K, L) methods with $K \leqslant K_L^*$ are in fact stiffly-stable. Although Conjecture 2 remains unresolved, substantial experimentation suggests that it is, nonetheless, true.

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