Bott periodicity in topological, algebraic and Hermitian K-theory

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This paper is devoted to classical Bott periodicity, its history and more recent extensions in algebraic and Hermitian K-theory. However, it does not aim at completeness. For instance, the variants of Bott periodicity related to bivariant K-theory are described by Cuntz in this handbook. As another example, we dont emphasize here the relation between motivic homotopy theory and Bott periodicity since it is also described by other authors of this handbook. (Grayson, Kahn,...).

1. CLASSICAL BOTT PERIODICITY.

1.1. Bott periodicity [14] was discovered independently from K-theory which started with the work of Grothendieck one year earlier [13]. In order to understand its great impact at the end of the 50's, one should notice that it was (and still is) quite hard to compute homotopy groups of spaces as simple as spheres. For example, it was proved by Serre that $\pi_i(S^n)$ is a finite group for $i \neq n$ and $i \neq 2n-1$ with n even, while $\pi_n(S^n) = \mathbf{Z}$ and $\pi_{2n-1}(S^n)$ is the direct sum of **Z** and a finite group for n even. All these finite groups are unknown in general (note however that $\pi_i(S^n) = 0$ for $i < n$). Since the classical groups O(n) and U(n) are built out of spheres through fibrations

$$O(n) \longrightarrow O(n+1) \longrightarrow S^n$$

$$U(n) \longrightarrow U(n+1) \longrightarrow S^{2n+1}$$

it was thought that computing their homotopy groups would be harder. On the other hand, from these fibrations, it immediatly follows that the homotopy groups of O(n) and U(n) stabilize. More precisely, $\pi_i(U(n)) \cong \pi_i(U(n+1))$ if $n > i/2$ and $\pi_i(O(n)) \cong \pi_i(O(n+1))$ if $n > i+1$. In this range of dimensions and degrees, we shall call $\pi_i(U)$ and $\pi_i(O)$ these stabilized homotopy groups : they are indeed homotopy groups of the "infinite" unitary and orthogonal groups

$$U = \operatorname{colim} U(n) \quad \text{and } O = \operatorname{colim} O(n).$$

1.2. THEOREM [14]. *The homotopy groups $\pi_i(U)$ and $\pi_i(O)$ are periodic of period 2 and 8 respectively. More precisely, there exist homotopy equivalences*[1]

$$U \approx \Omega^2(U) \textit{ and } O \approx \Omega^8(O),$$

where Ω^t denotes the t^{th} iterated loop space.

[1] Through out the paper, we use the symbol ≈ to denote a homotopy equivalence.

1.3. Remarks. Using polar decomposition of matrices, one may replace O(n) and U(n) by the general linear groups $GL_n(\mathbf{R})$ and $GL_n(\mathbf{C})$ which have the same homotopy type respectively. Similarly, one may consider the infinite general linear group $GL(\mathbf{R}) = \mathrm{colim}\ GL_n(\mathbf{R})$ and $GL(\mathbf{C}) = \mathrm{colim}\ GL_n(\mathbf{C})$. Since the homotopy groups of $O \approx GL(\mathbf{R})$ and $U \approx GL(\mathbf{C})$ are periodic, it is enough to compute the first eight ones, which are given by the following table :

i	0	1	2	3	4	5	6	7
$\pi_i(U)$	0	$\mathbf{Z}$	0	$\mathbf{Z}$	0	$\mathbf{Z}$	0	$\mathbf{Z}$
$\pi_i(O)$	$\mathbf{Z}/2$	$\mathbf{Z}/2$	0	$\mathbf{Z}$	0	0	0	$\mathbf{Z}$

In the same paper, Bott gave a more general theorem (in the real case), using the infinite homogeneous spaces related not only to the infinite orthogonal and unitary groups, but also to the infinite symplectic group Sp. More precisely, Sp(n) is the compact Lie group associated to $GL_n(\mathbf{H})$ and Sp = colim Sp(n), which has the same homotopy type as $GL(\mathbf{H}) = \mathrm{colim}\ GL_n(\mathbf{H})$, where **H** is the skew field of quaternions [21].

1.4. THEOREM [14]**.** *We have the following homotopy equivalences* (*where* BG *denotes in general the classifying space of the topological group* G) :

$$\Omega(\mathbf{Z} \times BGL(\mathbf{R})) \approx GL(\mathbf{R})$$
$$\Omega(GL(\mathbf{R})) \approx GL(\mathbf{R})/GL(\mathbf{C})$$
$$\Omega(GL(\mathbf{R})/GL(\mathbf{C})) \approx GL(\mathbf{C})/GL(\mathbf{H})$$
$$\Omega(GL(\mathbf{C})/GL(\mathbf{H})) \approx \mathbf{Z} \times BGL(\mathbf{H})$$
$$\Omega(\mathbf{Z} \times BGL(\mathbf{H})) \approx GL(\mathbf{H})$$
$$\Omega(GL(\mathbf{H})) \approx GL(\mathbf{H})/GL(\mathbf{C})$$
$$\Omega(GL(\mathbf{H})/GL(\mathbf{C})) \approx GL(\mathbf{C})/GL(\mathbf{R})$$
$$\Omega(GL(\mathbf{C})/GL(\mathbf{R})) \approx \mathbf{Z} \times BGL(\mathbf{R}).$$

In particular, we have the homotopy equivalences

$$\Omega^4(BGL(\mathbf{R})) \approx \mathbf{Z} \times BGL(\mathbf{H})$$
$$\Omega^4(BGL(\mathbf{H})) \approx \mathbf{Z} \times BGL(\mathbf{R}).$$

1.5. Theorems 1.2 and 1.4 were not at all easy to prove (note however that $\Omega(BG) \approx G$ is a standard statement). One key ingredient was a heavy use of Morse theory. A detailed proof, starting from a short course in Riemannian geometry may be found in the beautiful book of Milnor [32]. The proof (in the complex case) is based on two lemmas : one first shows that the space of minimal geodesics from I to -I in the special unitary group SU(2m) is homeomorphic to the Grassmannian $G_m(\mathbf{C}^{2m})$. In the other lemma one shows that every non-minimal geodesic from I to -I has index ≥ 2m+2. These lemmas imply the following “unstable” theorem from which Bott periodicity follows easily :

1.6. THEOREM ([32] p. 128). *Let* $G_m(\mathbf{C}^{2m})$ *be the Grassmanniann of m-dimensional subspaces of* $\mathbf{C}^{2m}$. *Then there is an inclusion map from* $G_m(\mathbf{C}^{2m})$ *into the space of paths*[2] *in* SU(2m) *joining* I *and* -I . *For* $i \leq 2m$, *this map induces an isomorphism of homotopy groups*

$$\pi_i(G_m(\mathbf{C}^{2m})) \cong \pi_{i+1}(SU(2m)).$$

1.7. The algebraic topologists felt frustated at that time by a proof using methods of Riemannian geometry in such an essential way. A special seminar [16] held in Paris by Cartan and Moore (1959/1960) was devoted not only to a detailed proof of Bott's theorems, but also to another proof avoiding Morse theory and using more classical methods in algebraic topology. However, this second proof was still too complicated for an average mathematician to grasp (see however the sketch of some elementary proofs of the complex periodicity in the section 3.3 of this paper).

2. INTERPRETATION OF BOTT PERIODICITY VIA K-THEORY

2.1. Two years later, Atiyah and Hirzebruch [6] realized that Bott periodicity was related to the fundamental work of Grothendieck on *algebraic K-theory* [13]. By considering the category of topological vector bundles over a compact space X (instead of algebraic vector bundles), Atiyah and Hirzebruch defined a *topological K-theory* K(X) following the same pattern as Grothendieck. As a new feature however, Atiyah and Hirzebruch managed to define "derived functors" $K^{-n}(X)$ by considering vector bundles over the n^{th} suspension of X_+ (X with one point added outside). There are in fact two K-theories involved, whether one considers real or complex vector bundles. We shall denote them by $K_{\mathbf{R}}$ and $K_{\mathbf{C}}$ respectively if we want to be specific. Theorem 1.2 is then equivalent to the periodicity of the functors K^{-n}. More precisely,

$$K_{\mathbf{C}}^{-n}(X) \cong K_{\mathbf{C}}^{-n-2}(X) \text{ and } K_{\mathbf{R}}^{-n}(X) \cong K_{\mathbf{R}}^{-n-8}(X).$$

These isomorphisms enabled Atiyah and Hirzebruch to extend the definition of K^n for all $n \in \mathbf{Z}$ and define what we now call a "generalized cohomology theory" on the category of compact spaces. Following the same spirit, Atiyah and Bott were able to give a quite elementary proof of the periodicity theorem in the complex case [4].

From the homotopy viewpoint, there exist two Ω–spectra defined by $\mathbf{Z}$ x BGL(k), k = $\mathbf{R}$ or $\mathbf{C}$, and their iterated loop spaces. The periodicity theorems can be rephrased by saying that these Ω–spectra are periodic of period 2 or 8 in the stable homotopy category, according to the type of K-theory involved, depending on whether one considers real or complex vector bundles. For instance, we have the following formula (where [,]' means pointed homotopy classes of maps) :

$$K_k^{-n}(X) \cong [X_+ \wedge S^n, \mathbf{Z} \times BGL(k)]'.$$

[2] Note that this space of paths has the homotopy type of the loop space Ω(SU(2m)).

2.2. As it was noticed by many people in the 60's (Serre, Swan, Bass...), K-theory appears as a "homology theory" on the category of rings. More precisely, let k = **R** or **C** and let us consider a k-vector bundle E over a compact space X. Let A be the Banach algebra C(X) of continuous functions $f : X \longrightarrow k$ (with the Sup norm). If M = Γ(X, E) denotes the vector space of continuous sections $s : X \longrightarrow E$ of the vector bundle E, M is clearly a right A-module if we define s.f to be the continuous section $x \mapsto s(x)f(x)$. Since X is compact, we may find another vector bundle E' such that the Whitney sum E ⊕ E' is trivial, say $X \times k^n$. Therefore, if we set M' = Γ(X, E'), we have $M \oplus M' \cong A^n$ as A-modules, which means that M is a finitely generated projective A-module. The theorem of Serre and Swan [46][29] says precisely that the correspondence $E \mapsto M$ induces a functor from the category $\mathcal{E}(X)$ of vector bundles over X to the category $\mathcal{P}(A)$ of finitely generated projective (right) A-modules, which is an equivalence of categories. In particular, isomorphism classes of vector bundles correspond bijectively to isomorphism classes of finitely generated projective A-modules.

These considerations lead to the following definition of the K-theory of a ring with unit A : we just mimic the definition of K(X) by replacing vector bundles by (finitely generated projective) A-modules. We call this group K(A) by abuse of notation. It is clearly a *covariant* functor on the category of rings (through extension of scalars). We have of course $K(X) \cong K(A)$, when A = C(X), thanks to the equivalence Γ above.

2.3. Similarly to what Atiyah and Hirzebruch did for A = C(X), one would like to define new functors $K_n(A)$, $n \in \mathbf{Z}$, with nice formal properties starting from $K_0(A) = K(A)$. This task is in fact more difficult than it looks for general rings A and we shall concentrate at the beginning on the case when A is a (real or complex) Banach algebra.

Firstly, we extend the definition of K(A) to non-unital algebras (over a commutative base ring k) by "adding a unit" to A. More precisely, we consider the k-module $\tilde{A} = k \oplus A$ provided with the following "twisted" multiplication

$$(\lambda, a).(\lambda', a') = (\lambda.\lambda', \lambda.a' + a.\lambda' + a.a').$$

The ring $\tilde{A}$ now has a unit which is (1, 0). There is an obvious augmentation

$$\tilde{A} \longrightarrow k$$

and K(A) is just defined as the kernel of the induced homomorphism $K(\tilde{A}) \longrightarrow K(k)$. It is easy to see that we recover the previous definition of K(A) if A already has a unit and that the new K-functor is defined for maps between rings not necessarily having a unit. It is less easy to prove that this definition is in fact independent of k ; this follows from the excision property for the functor K_0 [8][33].

Example : if $\mathcal{K}$ is the ideal of compact operators in a k-Hilbert space H (with k = **R** or **C**), then the obvious inclusion from k to $\mathcal{K}$ induces an isomorphism $K(k) \cong K(\mathcal{K}) \cong \mathbf{Z}$. This is a classical result in operator theory (see for instance [39] § 2.2.10 and [29], exercise 6.15).

Secondly, for $n \in \mathbf{N}$, we define $K_n(A)$ as $K(A_n)$, where $A_n = A(\mathbf{R}^n)$ is the Banach algebra of continuous functions f = f(x) from $\mathbf{R}^n$ to A which vanish when x goes to ∞. It is not too difficult to show that $K_{i+1}(A) \cong \text{colim}\ \pi_i(GL_r(A)) \cong \pi_i(GL(A))$, where GL(A) is the direct limit of the $GL_r(A)$ with respect to the obvious inclusions $GL_r(A) \subset GL_{r+1}(A)$ (see for instance the argument in [29], p. 13).

As a fundamental example, let us come back to topology by taking A = C(X), where X is compact. Let $Y = S^n(X_+)$ be the n-suspension of X_+. Then K(Y) is isomorphic to $K_n(A) \oplus \mathbf{Z}$. In order to show this, we notice that C(Y) is isomorphic to $\tilde{C}(X \times \mathbf{R}^n)$. In particular $K(S^n)$ is isomorphic to $K_n(k) \oplus \mathbf{Z}$ (k = **R** or **C**, according to the type of K-theory).

The following theorem, although not explicitly stated in this form in the literature (for the uniqueness part), is a direct consequence of the definitions.

2.4. THEOREM (compare with [29], exercise 6.14)**.** *The functors* $K_n(A)$, $n \in \mathbf{N}$, *and* A *a Banach algebra, are characterized by the following properties*

1) Exactness : *for any exact sequence of Banach algebras (where* A” *has the quotient norm and* A’ *the induced norm)*

$$0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$$

we have an exact sequence of K-*groups*

$$K_{n+1}(A) \longrightarrow K_{n+1}(A'') \longrightarrow K_n(A') \longrightarrow K_n(A) \longrightarrow K_n(A'').$$

2) Homotopy invariance : $K_n(A(I)) \cong K_n(A)$, *where* A(I) *is the ring of continuous functions on the unit interval* I *with values in* A.

3) Normalization : $K_0(A)=K(A)$, the Grothendieck group *defined above.*

2.5. The functors $K_*(A)$ have other nice properties such as the following : a continuous bilinear pairing of Banach algebras

$$A \times C \longrightarrow B$$

induces a “cup-product”

$$K_i(A) \otimes K_j(C) \longrightarrow K_{i+j}(B)$$

which has associative and graded commutative properties [25][29][30]. In particular, if C = k, the field of real or complex numbers, and if A = B is a k-Banach algebra, we have a pairing

$$K_i(A) \otimes K_j(k) \longrightarrow K_{i+j}(A).$$

We can now state the Bott periodicity theorem in the setting of Banach algebras.

2.6. THEOREM (Bott periodicity revisited, according to [25] and [53]).
1) *Let* A *be a* ***complex*** *Banach algebra. Then the group* $K_2(\mathbf{C})$ *is isomorphic to* **Z** *and the cup-product with a generator* u_2 *induces an isomorphism* $\beta_{\mathbf{C}} : K_n(A) \longrightarrow K_{n+2}(A)$.
2) *Let* A *be a* ***real*** *Banach algebra. Then the group* $K_8(\mathbf{R})$ *is isomorphic to* **Z** *and the cup-product with a generator* u_8 *induces an isomorphism* $\beta_{\mathbf{R}} : K_n(A) \longrightarrow K_{n+8}(A)$.

As we said in 2.3, this theorem implies the periodicity of the homotopy groups of the infinite general linear group in the more general setting of a Banach algebra, since $K_n(A) = K(A(\mathbf{R}^n))$ is isomorphic to the homotopy group $\pi_{n-1}(GL(A))$ (see the corollary below). If A = **C** for instance, the group $K(A(\mathbf{R}^n))$ is linked with the classification of *stable* complex vector bundles over the sphere S^n which are determined by homotopy classes of "glueing functions"

$$f : S^{n-1} \longrightarrow GL(A)$$

(see again the argument in [14], p. 13). For a general Banach algebra A, one just has to consider vector bundles over the sphere whose fibers are the A-modules A^r instead of $\mathbf{C}^r$.

2.7. COROLLARY.
a) *If* A *is a* ***complex*** *Banach algebra, we have*

$$\pi_i(GL(A)) \cong \pi_{i+2}(GL(A)) \text{ and } \pi_1(GL(A)) \cong K(A)$$

b) *If* A *is a* ***real*** *Banach algebra, we have*

$$\pi_i(GL(A)) \cong \pi_{i+8}(GL(A)) \text{ and } \pi_7(GL(A)) \cong K(A).$$

Part a) is essentially due to Atiyah and Bott [4], while part b) is due to Wood [53] and the author [25]. In the complex case, we can easily see that the isomorphism $\pi_1(GL(A)) \cong K(A)$ implies the 2-periodicity of the homotopy groups of GL(A). In order to prove this isomorphism, one essentially has to show that any loop in GL(A) can be deformed into a loop of type

$$\theta \mapsto p z + 1 - p$$

where p is an idempotent matrix of a certain size and $z = e^{i\theta}$. This is done via Fourier analysis and stabilization of matrices as explained with full details in [29], following the pattern initiated in [4]. Such an idempotent matrix p is of course associated to a finitely generated projective module. More conceptual proofs will be sketched later.

3. THE ROLE OF CLIFFORD ALGEBRAS.

3.1. One way to understand Bott periodicity in topology is to introduce Clifford algebras as it was pointed out by Atiyah, Bott and Shapiro [5]. Let us denote by C_n the Clifford algebra of V = $\mathbf{R}^n$ associated to the quadratic form $q(v) = (x_1)^2 + ... + (x_n)^2$, with $v = (x_1, ..., x_n)$. We recall that C_n is the quotient of the tensor algebra T(V) by the two-sided ideal generated by the relations $v \otimes v - q(v).1$. It is a finite dimensional semi-simple real algebra of dimension 2^n.
There is a kind of "periodicity" of the C_n considered as $\mathbf{Z}/2$-graded algebras : we have graded algebra isomorphisms[3]

$$C_{n+8} \cong M_{16}(C_n).$$

On the other hand, the complexified Clifford algebras have a 2-periodicity

$$C_{n+2} \otimes_{\mathbf{R}} \mathbf{C} \cong M_2(C_n) \otimes_{\mathbf{R}} \mathbf{C}.$$

These isomorphisms give rise to an "elementary" proof of the eight homotopy equivalences in Theorem 1.4 and the already stated results in Corollary 2.7 [25][53]. Indeed, the afore-mentioned homotopy equivalences can be written in a uniform way (up to connected components) as follows :

$$GL(C_n)/GL(C_{n-1}) \approx \Omega[GL(C_{n+1})/GL(C_n)].$$

In order to avoid the problem with connected components, let us introduce the "classifying space" $\mathcal{K}(A)$ of any (real or complex) Banach algebra A. As a first approximation[4], it is the cartesian product

$$\mathcal{K}(A) = K(A) \times BGL(A)$$

where BGL(A) is the classifying space of the topological group GL(A). This way, we have $\pi_i(\mathcal{K}(A)) = K_i(A)$ for $i \geq 0$. One could also consider the K-theory space $\mathcal{K}(A \otimes C_n)$, where tensor products are taken over $\mathbf{R}$, and the homotopy fiber $\mathcal{F}_n$ of the obvious inclusion map

$$\mathcal{K}(A \otimes C_{n-1}) \longrightarrow \mathcal{K}(A \otimes C_n).$$

We notice that the connected component of this homotopy fiber is precisely the connected component of the homogeneous space $GL(A \otimes C_n)/GL(A \otimes C_{n-1})$. The following theorem now includes all the versions of Bott periodicity quoted so far.

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[3] Where $M_r(B)$ denotes in general the algebra of r x r matrices with coefficients in B.

[4] This description is (*non-canonically*) homotopy equivalent to the *good* definition given later (see 4.3).

3.2. THEOREM. *We have natural homotopy equivalences*

$$\mathcal{F}_n \approx \Omega(\mathcal{F}_{n+1}).$$

3.3. The proof of this theorem (see [25], [53], [39] § 3.5) is still technical and requires easy but tedious lemmas. Therefore, it would be nice to have more conceptual proofs of Bott periodicity, at least in the complex case, as we already said at the end § 2. For the real case, we refer to 5.3 and 7.7.

A first approach (for A = **C**) is an elegant proof of Suslin (unpublished, see also the independent proof of Harris [20]), using the machinery of Γ-spaces due to Segal [41]. Taking into account that the homotopy equivalence $U \cong \Omega(\mathbf{Z} \times BU)$ is a standard statement, the heart of the proof of Bott's periodicity is the homotopy equivalence

$$\mathbf{Z} \times BU \approx \Omega U.$$

Since $X_0 = \mathbf{Z} \times BU$ is a Γ-space, it can be infinitely delooped and there is an explicit recipe to build spaces X_n such that $\Omega^n(X_n) \cong X_0$ [41]. This explicit construction shows that X_1 is *homeomorphic* to the infinite unitary group and this ends the proof ! However, it seems that this proof cannot be generalized to all Banach algebras since the deloopings via the machinery of Γ-spaces are connected : for instance X_2 is *not* homotopically equivalent to $\mathbf{Z} \times BU$.

Another conceptual approach (see [26] [30]), leading in section 4 to *non connected* deloopings of the space $\mathcal{K}(A)$ for any Banach algebra A, is to proceed as follows. Let us assume we have defined $K_n(A)$, not only for $n \in \mathbf{N}$, as we did before, but also for $n \in \mathbf{Z}$. We also assume that the pairing

$$K_i(A) \otimes K_j(C) \longrightarrow K_{i+j}(B)$$

mentioned for i and j ≥ 0, extends to all values of i and j in **Z** and has obvious associative and graded commutative properties. Finally, assume there exists a "negative Bott element" u_{-2} in $K_{-2}(\mathbf{C})$ whose cup-product with u_2 in $K_2(\mathbf{C})$ (as mentioned in 2.6) gives the unit element 1 in $K_0(\mathbf{C}) \cong \mathbf{Z}$.

Under these hypotheses, and assuming that A is a complex Banach algebra, we introduce an inverse homomorphism

$$\beta' : K_{n+2}(A) \longrightarrow K_n(A)$$

of the Bott map

$$\beta : K_n(A) \longrightarrow K_{n+2}(A).$$

It is defined as the cup-product with u_{-2}. It is clear that the composites of β with β' both ways are the identity.

This formal approach may be extended to real Banach algebras as well, although the required elements in $K_8(\mathbf{R})$ and $K_{-8}(\mathbf{R})$ are less easy to construct. One should also compare it to the one described in Cuntz' paper in this handbook, using the machinery of KK-theory (for C*-algebras at least).

4. NEGATIVE K-THEORY AND LAURENT SERIES.

4.1. The price to pay for this last conceptual proof is of course the construction of these "negative" K-groups $K_n(A)$, $n < 0$, which have some independent interest as we shall see later (A is now a *complex* or *real* Banach algebra). This may be done, using the notion of "suspension" of a ring which is in some sense dual to the notion of suspension of a space. More precisely, we define the "cone" CA of a ring A to be the set of all infinite matrices M = (a_{ij}), $i, j \in N$, such that each row and each column only contains a finite and bounded number of non zero elements in A (chosen among a finite number of elements in A). This clearly is a ring for the usual matrix operations. We make CA into a Banach algebra by completing it with respect to the following :

$$\|M\| = \mathrm{Sup}_j \, \Sigma_i \, \left|\|a_{ij}\|\right|$$

(this is just an example ; there are other ways to complete, leading to the same negative K-theory : see [26]). Finally we define $\tilde{A}$, the "stabilization" of A, as the closure of the set of finite matrices[5] in CA. It is a closed 2-sided ideal in CA and the suspension of A - denoted by SA - is the quotient ring $CA/\tilde{A}$.

4.2. DEFINITION/THEOREM [26]. *Let* A *be a Banach algebra. We define the groups* $K_{-n}(A)$ *as* $K(S^nA)$, *where* S^nA *is the* n^{th} *suspension of* A *for* $n > 0$. *Then* $BGL(S^{n+1}A)$ *is a delooping of* $K(S^nA) \times BGL(S^nA)$, *i.e. we have a homotopy equivalence (for any Banach algebra A)*

$$\Omega(BGL(SA)) \approx K(A) \times BGL(A).$$

Accordingly, to any exact sequence of Banach algebras as above

$$0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$$

we can associate an exact sequence of K-*groups*

$$K_{n+1}(A) \longrightarrow K_{n+1}(A'') \longrightarrow K_n(A') \longrightarrow K_n(A) \longrightarrow K_n(A'')$$

for $n \in \mathbf{Z}$.

4.3. As a matter of fact, this definition/theorem gives the *functorial* definition of the K-theory space $\mathcal{K}(A)$ mentioned above : it is nothing but the loop space $\Omega(BGL(SA)) \approx GL(SA)$. If A = **R** or **C**, it is easy to see that this space has the homotopy type of the set of Fredholm operators in a Hilbert space modelled on A(see [2] or [10], appendix A).

[5] A matrix in CA is called finite if all but finitely many of its elements are 0.

4.4. For a better understanding of SA, it may be interesting to notice that the ring of Laurent series $A<t, t^{-1}>$ is a good approximation of the suspension. Any element of $A<t, t^{-1}>$ is a series

$$S = \sum_{n \in \mathbf{Z}} a_n t^n$$

such that $\sum_{n \in \mathbf{Z}} \|a_n\| < +\infty$. There is a ring homomorphism $A<t, t^{-1}> \longrightarrow SA$ which associates to the series above the class of the following infinite matrix

$$\begin{pmatrix} a_0 & a_1 & a_2 & \cdots\cdots \\ a_{-1} & a_0 & a_1 & \cdots\cdots \\ a_{-2} & a_{-1} & a_0 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

For instance, in order to construct the element u_{-2} mentioned in 3.3 , it is enough to describe a finitely generated projective module over the Banach algebra $\mathbf{C} <t, u, t^{-1}, u^{-1}>$, i.e. a non trivial complex vector bundle over the torus $S^1 \times S^1$, as we can see easily, using again the theory of Fourier series [30].

5. BOTT PERIODICITY AND THE ATIYAH-SINGER INDEX THEOREM. KR-THEORY.

5.1. Many variants of topological K-theory have been considered since the 60's, also giving rise to more conceptual proofs of Bott periodicity and generalizing it. One of appealing interest is equivariant K-theory $K_G(X)$, introduced by Atiyah and Segal [40]. Here G is a compact Lie group acting on a compact space X and $K_G(X)$ is the Grothendieck group of the category of (real or complex) G-vector bundles on X. The analog of Bott periodicity in this context is the "Thom isomorphism" : we consider a *complex* G-vector bundle V on X and we would like to compute the *complex* equivariant K-theory of V , defined as the "reduced" K-theory $\tilde{K}_G(V_+)$, where V_+ is the one-point compactification of V. If we denote this group simply by $K_G(V)$, we have an isomorphism (due to Atiyah [3]) for complex equivariant K-theory

$$\beta_{\mathbf{C}} : K_G(X) \longrightarrow K_G(V).$$

In a parallel way, if V is a *real* vector bundle of rank 8m, provided with a spinorial structure[6] Atiyah proved that we also have a "Thom isomorphism" for *real* equivariant K-theory

$$\beta_{\mathbf{R}} : K_G(X) \longrightarrow K_G(V).$$

At this point, we should notice that if G is the trivial group and V a trivial vector bundle, the isomorphisms $\beta_{\mathbf{C}}$ and $\beta_{\mathbf{R}}$ are just restatements of the Bott periodicity theorems as in 1.2.

[6] For a more general statement, using Clifford bundles, see the following paper of the author : Equivariant K-theory of real vector spaces and real projective spaces. Topology and its applications, 122, p. 531-546 (2002).

5.2. The isomorphisms $\beta_{\mathbf{C}}$ and $\beta_{\mathbf{R}}$ are not at all easy to prove. The algebraic ideas sketched in the previous sections are not sufficient (even if G a finite group !). One has to use the full strength of the Atiyah-Singer index theorem (KK-theory in modern terms) in order to construct a map going backwards

$$\beta' : K_G(V) \longrightarrow K_G(X)$$

in the same spirit that led to the construction of the element u_{-2} in § 4 . The difficult part of the theorem is to show that $\beta'.\beta = \mathrm{Id}$. The fact that $\beta.\beta' = \mathrm{Id}$ follows from an ingenious trick due to Atiyah and described in [3].

5.3. The consideration of equivariant vector bundles led Atiyah to another elementary proof (i.e. more algebraic) of *real* Bott periodicity through the introduction of a new theory called KR(X) and defined for any locally compact space X with an involution [1]. In our previous language, the group KR(X) is just the K-theory of the Banach algebra A of continuous functions

$$f : X \longrightarrow \mathbf{C}$$

such that $f(\sigma(x)) = \overline{f(x)}$ (where σ denotes the involution on X) and $f(x) \rightarrow 0$ when $x \rightarrow \infty$.
The basic idea of this "Real" version of Bott periodicity is to prove (using Fourier analysis again) that $KR(X) \cong KR(X \times \mathbf{R}^{1,1})$ where $\mathbf{R}^{p,q}$ denotes in general the Euclidean space $\mathbf{R}^{p+q} = \mathbf{R}^p \oplus \mathbf{R}^q$ with the involution $(x, y) \mapsto (-x, y)$. Using ingenious homeomorphisms between spaces with involution, Atiyah managed to show that this isomorphism implies that $K(A(\mathbf{R}^n))$ is periodic of period 8 with respect to n. As a consequence we recover the classical Bott periodicity (in the real case) if we restrict ourselves to spaces with trivial involution.

6. BOTT PERIODICITY IN ALGEBRAIC K-THEORY.

6.1. Let us now turn our interest to the *algebraic* K-theory of a *discrete* ring A. From now on, we change our notations and write $K_n(A)$ for the Quillen K-groups of a ring with unit A. We recall that for $n > 0$, $K_n(A)$ is the n^{th} homotopy group of $BGL(A)^+$, where $BGL(A)^+$ is obtained from BGL(A) by adding cells of dimensions 2 and 3 in such a way that the fundamental group becomes the quotient of GL(A) by its commutator subgroup, without changing the homology.

If A is a Banach algebra, the K_n-groups considered before will be denoted by $K_n^{top}(A)$ in order to avoid confusion. We also have a definition of $K_n(A)$ for $n < 0$, due to Bass [8] and the author [26][28]. More precisely, $K_{-r}(A) = K(S^rA)$, where SA denotes the suspension of the ring A (Beware : we do *not* take the Banach closure as in 4.1 since A is just a discrete ring). One interpretation of Bott periodicity, due to Bass [8] for $n \le 0$ and Quillen [37] (for all n, assuming A to be regular Noetherian), is the following exact sequence :

$$0 \rightarrow K_{n+1}(A) \longrightarrow K_{n+1}(A[t]) \oplus K_{n+1}(A[t^{-1}]) \longrightarrow K_{n+1}(A[t, t^{-1}]) \longrightarrow K_n(A) \longrightarrow 0$$

[we replace the Laurent series considered in 4.4 by Laurent polynomials].

6.2. However, the natural question to ask is whether we have some kind of "periodicity" for the groups $K_n(A)$, $n \in \mathbf{Z}$. Of course, this question is too naive : for instance, if A is the finite field F_q with q elements, we have $K_{2n-1}(F_q) \cong \mathbf{Z}/(q^n - 1)\mathbf{Z}$, as proved by Quillen [36]. The situation is much better if we consider algebraic K-theory with finite coefficients, introduced by Browder and the author in the 70's [15] [42]. Since we know that algebraic K-theory is represented by a spectrum $\mathbb{K}(A)$ defined through the spaces $BGL(S^nA)^+$, as proved by Wagoner [51], there is a well known procedure in algebraic topology for constructing an associated mod. n-spectrum for any positive integer n > 1. The homotopy groups of this spectrum are the K-theory groups of A with coefficients in $\mathbf{Z}/n$.

An alternative (dual) approach is to consider the Puppe sequence associated to a self-map of degree n of the sphere S^r

$$S^r \longrightarrow S^r \longrightarrow M(n,r) \longrightarrow S^{r+1} \longrightarrow S^{r+1}.$$

The space M(n, r), known as a "Moore space", has two cells of dimensions r and r+1 respectively, the second cell being attached to the first by a map of degree n. If n = 2 and r = 1 for instance, M(n, r) is the real projective space of dimension 2. We now define K-theory with coefficients in $\mathbf{Z}/n$, denoted by $K_{r+1}(A ; \mathbf{Z}/n)$, for $r \geq 0$, as the group of pointed homotopy classes of maps from M(n,r) to $\mathcal{K}(A)$. From the classical Puppe sequence in algebraic topology, we get an exact sequence

$$K_{r+1}(A) \longrightarrow K_{r+1}(A) \longrightarrow K_{r+1}(A ; \mathbf{Z}/n) \longrightarrow K_r(A) \longrightarrow K_r(A)$$

where the arrows between the groups $K_i(A)$ are multiplication by n. Since $K_{r+t+1}(S^tA; \mathbf{Z}/n)$ is canonically isomorphic to $K_{r+1}(A ; \mathbf{Z}/n)$, we may extend this definition of $K_{r+1}(A ; \mathbf{Z}/n)$ to all values of $r \in \mathbf{Z}$, by putting $K_{r+1}(A ; \mathbf{Z}/n) = K_{r+t+1}(S^tA; \mathbf{Z}/n)$ for t large enough so that $r + t \geq 0$. With certain restrictions on n or A (we omit the details), this K-theory mod. n may be provided with a ring structure, having some nice properties [15].

Here is a fundamental theorem of Suslin [43] which is the true analog of complex Bott periodicity in algebraic K-theory.

6.3. THEOREM. *Let* F *be an algebraically closed field and let* n *be an integer prime to the characteristic of* F. *Then there is a canonical isomorphism between graded rings*

$$K_{2r}(F ; \mathbf{Z}/n) \cong (\mu_n)^{\otimes r} \text{ and } K_{2r+1}(F ; \mathbf{Z}/n) = 0,$$

where μ_n *denotes the group of* n^{th} *roots of unity in* F (*with* $r \geq 0$).

6.4. Starting with this theorem one may naturally ask if there is a way to compute $K_*(F ; \mathbf{Z}/n)$ for an arbitrary field F (not necessarily algebraically closed). If F is the field of real numbers, and if we work in topological K-theory instead, we know that it is not an easy task, since we get an 8-periodicity which looks mysterious compared to the 2-periodicity of the complex case. All these types of questions are in fact related to a "homotopy limit problem"[7]. More precisely, let us define in general $\mathbb{K}(A, n)$ as the spectrum of the K-theory of A mod. n as in 6.2. It is easy to show that $\mathbb{K}(F, n)$ is the fixed point set of $\mathbb{K}(\overline{F}, n)$, where $\overline{F}$ denotes the separable closure of F, with respect to the action of the Galois group G (which is a profinite group). We have a fundamental map

$$\phi : \mathbb{K}(F, n) = \mathbb{K}(\overline{F}, n)^G \longrightarrow \mathbb{K}(\overline{F}, n)^{hG} .$$

where $\mathbb{K}(\overline{F}, n)^{hG}$ is the "homotopy fixed point set" of $\mathbb{K}(\overline{F}, n)$, i.e. the set of equivariant maps $EG \longrightarrow \mathbb{K}(\overline{F}, n)$ where EG is the "universal" principal G-bundle over BG. Let us denote by $K_r^{et}(F ; \mathbf{Z}/n)$ the r^{th} homotopy group of this space of equivariant maps (which we call "étale K-theory" groups). According to 6.3 and the general theory of homotopy fixed point sets, there is a spectral sequence[8]

$$E_2^{p,q} = H^p(G; \mu_n^{\otimes(q/2)}) \Rightarrow K_{q-p}^{ét}(F ; \mathbf{Z}/n).$$

Let us assume now that the characteristic of the field does not divide n. A version of the "Lichtenbaum-Quillen conjecture"[9] is that ϕ induces an isomorphism on homotopy groups π_r for $r > d_n$, where d_n is the n-cohomological dimension of G. In other words, the canonical map

$$K_r(F ; \mathbf{Z}/n) \longrightarrow K_r^{et}(F ; \mathbf{Z}/n)$$

should be an isomorphism for $r > d_n$. The surjectivity of this map was investigated and proved in many cases by Soulé [42], Dwyer-Friedlander-Snaith-Thomason [18] in the 80's.

6.5. In order to compare more systematically algebraic K-theory and étale K-theory (which is periodic, as we shall see), there is an elegant approach, initiated by Thomason [47]. If we stick to fields and to an odd prime p (for the sake of simplicity), there is a "Bott element" β belonging to the group $K_{2(p-1)}(F ; \mathbf{Z}/p)$ as it was first shown by Browder and the author in the 70's [15] [42] . One notice that the usual Bott element $u_2 \in K_2(\mathbf{C} ; \mathbf{Z}/p)$ can be lifted to an element u of the group $K_2(A ; \mathbf{Z}/p)$ via the homomorphism induced by the ring map $A \longrightarrow \mathbf{C}$, where A is the ring of p- cyclotomic integers. By a transfer argument, one then shows that u^{p-1} is the image of a canonical element β of the group $K_{2(p-1)}(\mathbf{Z} ; \mathbf{Z}/p)$ through the standard homomorphism $K_{2(p-1)}(\mathbf{Z} ; \mathbf{Z}/p) \longrightarrow K_{2(p-1)}(A ; \mathbf{Z}/p)$. By abuse of notation, we shall also call β its image in $K_{2(p-1)}(F ; \mathbf{Z}/p)$ and $\beta_{ét}$ its image in $K_{2(p-1)}^{ét}(F ; \mathbf{Z}/p)$. The important remark here is that $\beta_{ét}$ is invertible in the étale K-theory ring (which is a way to state that étale K-theory is periodic of period 2(p - 1)). Therefore, there is a factorisation

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7 See the paper of Thomason : The homotopy limit problem. Contemporary Mathematics Vol. 19, 407-419 (1983).

8 Note that $\pi_q(\mathbb{K}(\overline{F}, n) = \mu_n^{\otimes(q/2)}$ as stated in 6.3 where we put $\mu_n^{\otimes(q/2)} = 0$ for q odd.

9 Many mathematiciens contributed to this formulation which is quite different than the original one by Lichtenbaum and Quillen. We should mention the following names : Dwyer, Friedlander, Mitchell, Snaith, Thomason... For an overview, see for instance [47], p. 516-523.

$$\begin{array}{ccc} K_*(F\,;\mathbf{Z}/p) & \longrightarrow & K_*^{et}(F\,;\mathbf{Z}/p) \\ & \searrow \quad \nearrow & \\ & K_*(F\,;\mathbf{Z}/p)[\beta^{-1}]. & \end{array}$$

We can now state the main result of Thomason :

6.6. THEOREM [47]. *Let us assume that* F *is of finite p-étale dimension and moreover satisfies the (mild) conditions of Theorem 4.1 in* [47]. *Then, the map*

$$K_*(F\,;\mathbf{Z}/p)[\beta^{-1}] \longrightarrow K_*^{ét}(F\,;\mathbf{Z}/p)$$

defined above is an isomorphism.

6.7. In order to make more progress in the proof of the Lichtenbaum-Quillen conjecture, Bloch and Kato formulated in the 90's another conjecture which is now central to the current research in algebraic K-theory. This conjecture states that for any integer n, the Galois symbol from Milnor's K-theory mod.n [33] to the corresponding Galois cohomology

$$K_r^M(F)/n \longrightarrow H^r_{ét}(F\,;\mu_n^{\otimes r}) = H^r(G\,;\mu_n^{\otimes r})$$

is an isomorphism. This conjecture was first proved by Voevodsky for $n = 2^k$: it is the classical Milnor's conjecture [49].

After this fundamental work of Voevodsky, putting as another ingredient a spectral sequence first established by Bloch and Lichtenbaum [11], many authors were able to solve the Lichtenbaum-Quillen conjecture for $n = 2^k$. We should mention Kahn [24], Rognes and Weibel [38], Ostvaer and Rosenschon (to appear).

At the present time (August 2003), there is some work in progress by Rost (unpublished) and Voevodsky [50] giving some evidence that the Bloch-Kato conjecture should also be true for all values of n. Assuming this work accomplished, the Lichtenbaum-Quillen conjecture will then be proved in general !

There is another interesting consequence of the Bloch-Kato conjecture which is worth mentioning : we should have a "motivic" spectral sequence (first conjectured by Beilinson), different from the one written in 6.4 :

$$E_2^{p,q} \Rightarrow K_{q-p}(F\,;\mathbf{Z}/n).$$

Here, the term E_2 of the spectral sequence is the following :

$$E_2^{p,q} = H^p(G;\mu_n^{\otimes(q/2)}) \text{ for } p \leq q/2 \text{ and}$$

$$E_2^{p,q} = 0 \text{ for } p > q/2.$$

This spectral sequence should degenerate in many cases, for instance if n is odd or if F is an exceptional field ([24] p. 102).

6.8. Example. If F is a number field and if n is odd, we have $d_n = 1$ with the notations of 6.4. In this case, the degenerating spectral sequence above shows a direct link between algebraic K-theory and Galois cohomology, quite interesting in Number Theory.

6.9. The Bloch-Kato conjecture (if proved in general) also sheds a new light on Thomason's localisation map considered in 6.6 :

$$K_r(F\,;\mathbf{Z}/p) \longrightarrow K_r(F\,;\mathbf{Z}/p)[\beta^{-1}].$$

For instance, we can state Kahn's Theorem 2, ([24] p. 100) which gives quite general conditions of injectivity or surjectivity for this map in a certain range of degrees, not only for fields, but also for finite dimensional schemes.

6.10. We should notice in passing that a topological analog of the Lichtenbaum-Quillen conjecture is true : in this framework, one should replace the classifying space of algebraic K-theory by the classifying space of (complex) topological K-theory $\mathcal{K}^{top}(A \otimes_{\mathbf{R}} \mathbf{C})$, where A is a *real* Banach algebra and $\mathbf{Z}/2$ acts by complex conjugation (compare with [44]). Then the fixed point set (i.e. the classifying space of the topological K-theory of A) has the homotopy type of the homotopy fixed point set [31].

6.11. Despite these recent breakthroughs, the groups $K_r(A)$ are still difficult to compute *explicitly,* even for rings as simple as the ring of integers in a number field (although we know these groups rationally from the work of Borel [12] on the rational cohomology of arithmetic groups). However, thanks to the work of Bökstedt, Rognes and Weibel [38], we can at least compute the 2-primary torsion of $K_r(\mathbf{Z})$ through the following homotopy cartesian square

$$\begin{array}{ccc} BGL(\mathbf{Z}[1/2])^+_{\#} & \longrightarrow & BGL(\mathbf{R})_{\#} \\ \downarrow & & \downarrow \\ BGL(\mathbf{F}_3)^+_{\#} & \longrightarrow & BGL(\mathbf{C})_{\#}. \end{array}$$

Here the symbol # means 2-adic completion, while BGL(**R**) and BGL(**C**) denote the classifying spaces of the *topological* groups GL(**R**) and GL(**C**) respectively. From this homotopy cartesian square, Rognes and Weibel [38] obtained the following results (modulo a finite odd torsion group and with $n > 0$ for the first 2 groups and $n \geq 0$ for the others) :

$K_{8n}(\mathbf{Z}) = 0$
$K_{8n+1}(\mathbf{Z}) = \mathbf{Z} \oplus \mathbf{Z}/2$
$K_{8n+2}(\mathbf{Z}) = \mathbf{Z}/2$
$K_{8n+3}(\mathbf{Z}) = \mathbf{Z}/16$

$K_{8n+4}(\mathbf{Z}) = 0$
$K_{8n+5}(\mathbf{Z}) = \mathbf{Z}$
$K_{8n+6}(\mathbf{Z}) = 0$
$K_{8n+7}(\mathbf{Z}) = \mathbf{Z}/2^{r+2}$ where 2^r is the 2 primary component of the number 4n+4.

There is a (non published) conjecture of S.A. Mitchell about the groups $K_i(\mathbf{Z})$ in general (including the odd torsion), for $i \geq 2$. Let us write the k^{th} Bernoulli number ([29] p. 297 and 299) as an irreducible fraction c_k/d_k. Then we should have the following explicit computations of $K_r(\mathbf{Z})$ for $r \geq 2$, with $k = [\frac{r}{4}] + 1$:

$K_{8n}(\mathbf{Z}) = 0$
$K_{8n+1}(\mathbf{Z}) = \mathbf{Z} \oplus \mathbf{Z}/2$
$K_{8n+2}(\mathbf{Z}) = \mathbf{Z}/2 \oplus \mathbf{Z}/c_k$ (k = 2n + 1)
$K_{8n+3}(\mathbf{Z}) = \mathbf{Z}/8k.d_k$ (k = 2n + 1)
$K_{8n+4}(\mathbf{Z}) = 0$
$K_{8n+5}(\mathbf{Z}) = \mathbf{Z}$
$K_{8n+6}(\mathbf{Z}) = \mathbf{Z}/c_k$ (k = 2n + 2)
$K_{8n+7}(\mathbf{Z}) = \mathbf{Z}/4k.d_k$ (k = 2n + 2).

Example. If we want to compute $K_{22}(\mathbf{Z})$ and $K_{23}(\mathbf{Z})$, we write

$$22 = 8.2 + 6 \text{ and } 23 = 8.2 + 7$$

Hence k = 6 with an associated Bernoulli number $B_6 = 691/2730$. Therefore, according to the conjecture, we should have (compare with [42]) :

$$K_{22}(\mathbf{Z}) \cong \mathbf{Z}/691 \text{ and } K_{23}(\mathbf{Z}) \cong \mathbf{Z}/65\,520$$

Note that for the groups $K_r(\mathbf{Z})$ for $r \leq 6$, complete results (not just conjectures) are found among other theorems in the reference [19]. Here they are (the new ones are K_5 and K_6) :

$K_0(\mathbf{Z}) = \mathbf{Z}$
$K_1(\mathbf{Z}) = \mathbf{Z}/2$
$K_2(\mathbf{Z}) = \mathbf{Z}/2$ (Milnor)
$K_3(\mathbf{Z}) = \mathbf{Z}/48$ (Lee and Szczarba)
$K_4(\mathbf{Z}) = 0$ (Rognes)
$K_5(\mathbf{Z}) = \mathbf{Z}$
$K_6(\mathbf{Z}) = 0$.

6.12. Other rings with periodic algebraic K-theory, arising from a completely different point of view are complex *stable* C* algebras A : by definition, they are isomorphic to their completed tensor product with the algebra $\mathcal{K}$ of compact operators in a complex Hilbert space. For example, the C* algebra associated to a foliation is stable [17]. Another example is the ring of continuous functions from a compact space X to the ring $\mathcal{K}$. These algebras are not unital but, as proved by Suslin and Wodzicki, they satisfy excision. A direct consequence of this excision property is the following theorem, conjectured by the author in the 70's (the tentative proof was based on the scheme described at the end of 3.3, assuming excision).

6.13. THEOREM [45]**.** *Let* A *be a complex stable* C*-*algebra. Then the obvious map*

$$K_n(A) \longrightarrow K_n^{top}(A)$$

is an isomorphism. In particular, we have Bott periodicity

$$K_n(A) \approx K_{n+2}(A)$$

for the algebraic K-theory groups.

M. Wodzicki can provide many other examples of ideals $\mathfrak{J}$ in the ring $\mathfrak{B}(H)$ of bounded operators in a Hilbert space such that the tensor product (completed of not) $B \otimes \mathfrak{J}$, for B any complex algebra, has a periodic ALGEBRAIC K-theory. These ideals are characterized by the fact that $\mathfrak{J} = \mathfrak{J}^2$ and that the commutator subgroup $[\mathfrak{B}(H), \mathfrak{J}]$ coincides with $\mathfrak{J}$.

7. BOTT PERIODICITY IN HERMITIAN K-THEORY.

7.1. As we have seen, usual K-theory is deeply linked with the general linear group. One might also consider what happens for the other classical groups. Not only is it desirable, but this setting turns out to be quite suitable for a generalization of Bott periodicity and the computation of the homology of *discrete* orthogonal and symplectic groups in terms of classical Witt groups.

The starting point is a ring A with an antiinvolution $a \mapsto \bar{a}$, together with an element ε in the center of A such that $\varepsilon\bar{\varepsilon} = 1$. In most examples, $\varepsilon = \pm 1$. For reasons appearing later (see 7.6), we also assume the existence of an element λ in the center of A such that $\lambda + \bar{\lambda} = 1$ (if 2 is invertible in A, we may choose λ = 1/2). If M is a *right* f.g.p. (finitely generated projective) module over A, we define its dual M* as the group of **Z**-linear maps $f : M \longrightarrow A$ such that $f(m.a) = \bar{a}.f(m)$ for $m \in M$ and $a \in A$. It is again a right f.g.p. A-module if we put (f.b)(m) = f(m).b for $b \in A$. An ε-hermitian form on M is an A-linear map $\tilde{\phi} : M \longrightarrow M^*$ satisfying some conditions of ε-symmetry ($\tilde{\phi} = \varepsilon\,\tilde{\phi}^*$ as written below). More precisely, it is given by a **Z**-bilinear map

$$\phi : M \times M \longrightarrow A$$

such that

$$\phi(ma, m'b) = \bar{a}\, \phi(m, m')b$$
$$\phi(m', m) = \varepsilon\ \overline{\phi(m, m')}$$

with obvious notations. Such a ϕ is called an ε-*hermitian form* and (M, ϕ) is an ε-*hermitian module*. The map

$$\tilde{\phi} : m' \mapsto [m \mapsto \phi(m, m')]$$

does define a morphism from M to M* and we say that ϕ is non-degenerate if $\tilde{\phi}$ is an isomorphism.

Fundamental example (the hyperbolic module). Let N be a f.g.p. module and M = N ⊕ N*. A non-degenerate ε-hermitian form ϕ on M is given by the following formula

$$\phi((x, f), (x', f')) = \overline{f(x')} + \varepsilon f'(x).$$

We denote this module by H(N). If $N = A^n$, we may identify N with its dual via the map $y \mapsto f_y$ with $f_y(x) = \bar{x}\, y$. The hermitian form on $A^n \oplus A^n$ may then be written as

$$\phi((x, y), (x', y')) = \bar{y}\, x' + \varepsilon\ \bar{x}\, y'.$$

7.2. There is an obvious definition of direct sum for non-degenerate ε-hermitian modules. We write ${}_\varepsilon L(A)$ or ${}_\varepsilon L_0(A)$ for the Grothendieck group constructed from such modules[10].

Examples : let A be the ring of continuous functions on a compact space X with complex values. If A is provided with the trivial involution, ${}_1L(A)$ is isomorphic to the *real* topological K-theory of X while ${}_{-1}L(A)$ is isomorphic to its quaternionic topological K-theory (see e.g. [29], p. 106, exercise 6.8).

7.3. In the Hermitian case, the analog of the general linear group is the ε-orthogonal group which is the group of automorphisms of $H(A^n)$, denoted by ${}_\varepsilon O_{n,n}(A)$: its elements may be described concretely in terms of 2n x 2n matrices

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

such that M*M = MM* = I , where

$$M^* = \begin{pmatrix} {}^t\bar{d} & \varepsilon\, {}^t\bar{b} \\ \bar{\varepsilon}\, {}^t\bar{c} & {}^t\bar{a} \end{pmatrix}$$

[10] We use the letter L which is quite convenient, but the reader should not mix up the present definition with the definition of the surgery groups, also denoted by the letter L (see the papers of Lück/Reich, Rosenberg and Williams in this handbook).

Example : if A = the field of real numbers **R**, ${}_1O_{n,n}(A)$ is the classical group O(n, n) which has the homotopy type of O(n) x O(n). By contrast, ${}_{-1}O_{n,n}(A)$ is the classical group Sp(2n, **R**) which has the homotopy type of the unitary group U(n) [21].
The infinite orthogonal group

$${}_\varepsilon O(A) = \lim {}_\varepsilon O_{n,n}(A)$$

has a commutator subgroup which is perfect (similarly to GL). Therefore, we can perform the + construction of Quillen as in [37].

7.4. DEFINITION. *The higher Hermitian* K-*theory of a ring* A *(for n > 0) is defined as*

$${}_\varepsilon L_n(A) = \pi_n(B_\varepsilon O(A)^+).$$

Example. Let F be a field of characteristic different from 2 provided with the trivial involution. Then ${}_\varepsilon L_1(F) = 0$ if $\varepsilon = -1$ and ${}_\varepsilon L_1((F) = \mathbf{Z}/2 \times F^*/F^{*2}$ if $\varepsilon = +1$ (see e.g. [9]).

Notation. We write

$$\mathcal{K}(A) = K(A) \times BGL(A)^+$$

for the classifiying space of algebraic K-theory - as before - and

$${}_\varepsilon\mathcal{L}(A) = {}_\varepsilon L(A) \times B_\varepsilon O(A)^+$$

for the classifying space of Hermitian K-theory.

7.5. There are two interesting functors between Hermitian K-theory and algebraic K-theory. One is the forgetful functor from modules with hermitian forms to modules (with no forms) and the other one from modules to modules with forms, sending N to H(N), the hyperbolic module associated to N. These functors induce two maps

$$F : {}_\varepsilon\mathcal{L}(A) \longrightarrow \mathcal{K}(A) \quad \text{and} \quad H : \mathcal{K}(A) \longrightarrow {}_\varepsilon\mathcal{L}(A).$$

We define ${}_\varepsilon\mathcal{V}(A)$ as the homotopy fiber of F and ${}_\varepsilon\mathcal{U}(A)$ as the homotopy fiber of H. We thus define two "relative" theories :

$${}_\varepsilon V_n(A) = \pi_n({}_\varepsilon\mathcal{V}(A)) \text{ and } {}_\varepsilon U_n(A) = \pi_n({}_\varepsilon\mathcal{U}(A)).$$

7.6. THEOREM (the fundamental theorem of Hermitian K-theory [30])**.** *Let* A *be a* ***discrete*** *ring with the hypotheses in 7.1. Then, there is a natural homotopy equivalence between* ${}_\varepsilon\mathcal{V}(A)$ *and the loop space of* ${}_{-\varepsilon}\mathcal{U}(A)$. *In particular,*

$${}_\varepsilon V_n(A) \cong {}_{-\varepsilon}U_{n+1}(A).$$

Moreover, if we work within the framework of Banach algebras with an antiinvolution, the same statement is valid for the topological analogs (i.e. replacing $BGL(A)^+$ *by* $BGL(A)^{top}$ *and* $B_\varepsilon O(A)^+$ *by* $B_\varepsilon O(A)^{top}$).

7.7. In order to get a feeling for this theorem, it is worthwhile to see what we get for the classical examples A = **R**, **C** or **H** , with their usual topology and various antiinvolutions. Note in general that the connected component of ${}_\varepsilon\mathcal{V}(A)$ (resp. ${}_{-\varepsilon}\mathcal{U}(A)$) is the connected component of the homogeneous space $GL(A)/{}_\varepsilon O(A)$ (resp. ${}_{-\varepsilon}O(A)/GL(A)$). For instance, if A = **R**, ε = -1, we get the homogeneous spaces

$$GL(\mathbf{R})/{}_{-1}O(\mathbf{R}) \text{ and } {}_1O(\mathbf{R})/GL(\mathbf{R})$$

which have the homotopy type of GL(**R**)/GL(**C**), and GL(**R**) respectively [21]. In this case, Theorem 7.6 implies that GL(**R**)/GL(**C**) has the homotopy type of the loop space ΩGL(**R**), one of the eight homotopy equivalences of Bott (compare with 1.4). It is a pleasant exercise to recover the remaining seven homotopy equivalences by dealing with other classical groups and various inclusions between them. Since the list of classical groups is finite, it is "reasonable" to expect some periodicity...

7.8. An advantage of this viewpoint (compared to the Clifford algebra approach in § 3 for instance) is the context of this theorem, valid in the *discrete* case, which implies Bott periodicity for "discrete" Hermitian K-theory. For instance, if we consider the higher Witt groups

$${}_\varepsilon W_n(A) = \text{Coker}\,(K_n(A) \longrightarrow {}_\varepsilon L_n(A)),$$

the fundamental theorem implies a periodicity isomorphism (modulo 2 torsion)[11] :

$${}_\varepsilon W_n(A) \cong {}_{-\varepsilon}W_{n-2}(A).$$

From this result we get some information about the homology of the *discrete* group ${}_\varepsilon O(A)$ (at least rationally) [30]. If we denote by ${}_\varepsilon\mathbf{W}(A)$ the periodic graded vector space $\bigoplus_n {}_\varepsilon W_n(A) \otimes_{\mathbf{Z}} \mathbf{Q}$, we find that the homology with rational coefficients $H_*({}_\varepsilon O(A) ; \mathbf{Q})$ may be written as the tensor product of the symmetric algebra of ${}_\varepsilon\mathbf{W}(A)$ with a graded vector space[12] $T_*(A)$ such that $T_0(A) = \mathbf{Q}$. This result is of course related to the classical theorems of Borel [12] , when A is the ring of S-integers in a number field.

7.9. In another direction, if A is a regular noetherian ring with 2 invertible in A, the isomorphism ${}_\varepsilon W_n(A) \approx {}_{-\varepsilon}W_{n-2}(A)$ is true for $n \leq 0$ with no restriction about the 2-torsion. This implies a 4-periodicity of these groups. As it was pointed in [23], these 4-periodic Witt groups are

[11] As proved in [30], the hypotheses in 7.1 are no longer necessary for this statement and A might be an arbitrary ring.

[12] $T^*(A)$ is the symmetric algebra of $K^+(A) \otimes_{\mathbf{Z}} \mathbf{Q}$, where $K^+(A)$ is the part of K-theory which is invariant under the contragredient isomorphism (it is induced by the map sending a matrix M to ${}^t\overline{M}^{-1}$).

isomorphic to Balmer's triangular Witt groups in this case [7]. Note these Balmer's Witt groups are isomorphic to surgery groups as it was proved by Walter [52], even if the ring A is not regular. It would be interesting to relate - for non regular rings - the negative Witt groups of our paper to the classical surgery groups .

7.10. Finally, we should remark that the 2-primary torsion of the Hermitian K-theory of the ring **Z**[1/2] can be computed the same way as the 2-primary torsion of the algebraic K-theory of **Z** (compare with 6.11), thanks to the following homotopy cartesian square proved in [10] :

$$\begin{array}{ccc} B_\varepsilon O(\mathbf{Z}[1/2])^+_\# & \longrightarrow & B_\varepsilon O\mathbf{R})_\# \\ \downarrow & & \downarrow \\ B_\varepsilon O(\mathbf{F}_3)^+_\# & \longrightarrow & B_\varepsilon O(\mathbf{C})_\#. \end{array}$$

From this diagram, we deduce the following 2-adic computations of the groups ${}_\varepsilon L_i = {}_\varepsilon L_i(\mathbf{Z}[1/2])$ for i ≥ 2, i an integer mod. 8, and ε = ±1, in comparizon with the table of the $K_i = K_i(\mathbf{Z})$ in 6.11 (where 2^t is again the 2-primary component of i + 1) :

i	0	1	2	3	4	5	6	7
K_i	0	$\mathbf{Z} \oplus \mathbf{Z}/2$	$\mathbf{Z}/2$	$\mathbf{Z}/16$	0	$\mathbf{Z}$	0	$\mathbf{Z}/2^{t+1}$
${}_1L_i$	$\mathbf{Z} \oplus \mathbf{Z}/2$	$(\mathbf{Z}/2)^3$	$(\mathbf{Z}/2)^2$	$\mathbf{Z}/8$	$\mathbf{Z}$	0	0	$\mathbf{Z}/2^{t+1}$
${}_{-1}L_i$	0	0	$\mathbf{Z}$	$\mathbf{Z}/16$	$\mathbf{Z}/2$	$\mathbf{Z}/2$	$\mathbf{Z}$	$\mathbf{Z}/2^{t+1}$

8. CONCLUSION

Let us add a few words of conclusion about the relation between motivic ideas and Bott periodicity, although other papers in this handbook will develop this analogy with more details (see also 6.4).

Morel and Voevodsky [35] have proved that algebraic K-theory is representable by an infinite Grassmannian in the unstable motivic homotopy category. Moreover, Voevodsky [48] has shown that this, together with Quillen's computation of the K-theory of the projective line, implies that algebraic K-theory is representable in the stable motivic homotopy category by a motivic (2,1)-periodic Ω-spectrum. This mysterious "periodicity" is linked with two algebraic analogs of the circle, already considered in [27] and [28]: one circle is the scheme of the subring of A[t] consisting of polynomials P(t) such that P(0) = P(1). The second one is the multiplicative group G_m which is the scheme of the ring $A[t, t^{-1}]$, already considered in 6.1. The "smash product" (in the homotopy category of schemes) of these two models is the projective line $\mathbf{P}^1$. This (2,1)-periodicity referred to above is a consequence of Quillen's computation of the K-theory of the projective line [34].

In the same spirit, Hornbostel [22] has shown that Hermitian K-theory is representable in the unstable motivic homotopy category. Combining this with 7.6 and the localisation theorem for Hermitian K-theory of Hornbostel-Schlichting [23] applied to A[t, t^{-1}], Hornbostel deduces that Hermitian K-theory is representable by a motivic (8-4)-periodic Ω-spectrum in the stable homotopy category. This periodicity is linked with the computation of the Hermitian K-theory of the smash product of P^1 four times with itself (which is S^8 from a homotopy viewpoint).

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