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A new subclass of the meromorphic harmonic γ-starlike functions

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Dedicated to Professor H. M. Srivastava on the Occasion of his Seventieth Birth Anniversary

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ABSTRACT

Recently Bostancı and Öztürk defined a new operator M^n for meromorphic harmonic functions. They introduced new classes of meromorphic harmonic starlike functions in $\widetilde{U} = \{z : |z| > 1\}$ using operator M^n. In this work, we have generalized these classes to meromorphic harmonic γ-starlike functions. Also, we have examined some properties of these classes.

1. Introduction

A continuous function $f = u + iv$ is a complex valued harmonic function in a complex domain D if both u and v are real harmonic in D. In any simply connected domain $D \subset \mathbb{C}$, we can write $f = h + \bar{g}$, where h and g are analytic in D. A necessary and sufficient condition for f to be locally univalent and sense preserving in D is that $|h'(z)| > |g'(z)|$ in D (see [3]). There are numerous papers on univalent harmonic functions defined on the domain $\widetilde{U} = \{z : |z| < 1\}$ (see [3,1,5,6]). In [4] Hengartner and Schober investigated functions harmonic in the exterior of the unit disc $\widetilde{U} = \{z : |z| > 1\}$. They showed that a complex valued, harmonic, sense preserving, univalent mapping f, defined on $\widetilde{U}$ and satisfying $f(\infty) = \infty$, must admit the representation

$$f(z) = h(z) + \overline{g(z)} + A\log|z|, \tag{1}$$

where

$$h(z) = \alpha z + \sum_{k=1}^{\infty} a_k z^{-k} \quad \text{and} \quad g(z) = \beta z + \sum_{k=1}^{\infty} b_k z^{-k}. \tag{2}$$

$0 \leqslant |\beta| < |\alpha|,\ A \in \mathbb{C}$ and $a = \bar{f}_{\bar{z}}/f_z$ is analytic and satisfies $|a(z)| < 1$ for $z \in \widetilde{U}$. After this work, Jahangiri and Silverman [7], gave sufficient coefficient conditions for which functions of the form (1) will be univalent. Under certain restrictions, they also gave necessary and sufficient coefficient conditions for functions to be harmonic and starlike. In [7], the following theorem, which we shall use in this work, is also proved.

Theorem 1.1. *Let $f(z) = h(z) + \overline{g(z)} + A\log|z|$ where h(z) and g(z) are of the form Eq. (2). If*

$$\sum_{k=1}^{\infty} k(|a_k| + |b_k|) \leqslant |\alpha| - |\beta| - |A|, \tag{3}$$

then f(z) is sense preserving and univalent in $\widetilde{U}$.

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Recently, Bostanci and Öztürk [2], have defined the following operator M^n for meromorphic harmonic functions $f = h + \bar{g}$ where h and g are of the form (2), as follows:

$$M^0 f(z) = f(z), \quad M^1 f(z) = Mf(z) = \frac{\overline{(z^2 g(z))'}}{z} - z^3\left(\frac{h(z)}{z^2}\right)'$$

and for $n = 2, \ldots$

$$M^n f(z) = M(M^{n-1} f(z)).$$

Hence, they obtain for $n = 0, 1, \ldots$

$$M^n f(z) = \alpha z + \sum_{k=1}^{\infty}(k+2)^n a_k z^{-k} + \overline{3^n \beta z + (-1)^n \sum_{k=1}^{\infty}(k-2)^n b_k z^{-k}}.$$

In [2], they defined the classes $MH^*(n)$ and $\overline{MH}^*(n)$. Also they investigated some properties of these classes such as coefficient estimates and distortion theorems.

In this work, we extended this results to the general cases $MH^*(n,\gamma)$ and $\overline{MH}^*(n,\gamma)$, $0 \leqslant \gamma < (\alpha + 3^n\beta)/(\alpha - 3^n\beta)$. Using the operator M^n, we now introduce the following classes:

Let $MH^*(n,\gamma)$ denote the class of meromorphic harmonic, sense preserving, univalent functions that satisfy the following condition.

$$\mathrm{Re}\left\{2 - \frac{M^{n+1} f(z)}{M^n f(z)}\right\} > \gamma, \quad z \in \widetilde{U}, \quad n \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}. \tag{4}$$

Also, let $\overline{MH}^*(n,\gamma)$ be the subclass of $MH^*(n,\gamma)$ which consists of meromorphic harmonic functions of the form

$$f_n(z) = h(z) + \overline{g_n(z)} = -\alpha z - \sum_{k=1}^{\infty} a_k z^{-k} + \overline{\beta z - (-1)^n \sum_{k=1}^{\infty} b_k z^{-k}}, \tag{5}$$

where $\alpha > 3^n\beta \geqslant 0$, $a_k \geqslant 0$, $b_k \geqslant 0$ and $b_2 \leqslant (\alpha - \beta)/2$.

For $0 \leqslant \gamma_1 < \gamma_2 < (\alpha + 3^n\beta)/(\alpha - 3^n\beta)$, $MH^*(n,\gamma_2) \subset MH^*(n,\gamma_1) \subset MH^*(n,0)$ and $\overline{MH}^*(n,\gamma_2) \subset \overline{MH}^*(n,\gamma_1) \subset \overline{MH}^*(n,0)$.

Notice that if we take $n = 0$ in the inequality Eq. (4), then we obtain

$$\gamma < \mathrm{Re}\left\{\frac{zh'(z) - \overline{zg'(z)}}{h(z) + \overline{g(z)}}\right\} = \mathrm{Im}\frac{\partial}{\partial\theta}(\log f(re^{i\theta})) = \frac{\partial}{\partial\theta}\arg(f(re^{i\theta})), \tag{6}$$

where $z = re^{i\theta}$, $0 \leqslant \theta < 2\pi$, $r > 1$. The inequality (6) is a necessary and sufficient condition for functions f of the form (1) to be γ-starlike in $\widetilde{U}$. This classification in (6) for harmonic univalent functions was first used by Jahangiri, [8].

Also, specializing the parameters n, α, β and γ, we have.

(i) $MH^*(n,0) = MH^*(n)$ (see [2]);
(ii) $MH^*(0,0) = H_0^*$ denotes the subclass of harmonic sense preserving functions f that are starlike in $\widetilde{U}$ and $\overline{MH}^*(0,0) = TH_0^*$ (see [7]);
(iii) If we substitute $\alpha = 1$, $\beta = 0$, then we have $MH^*(0,0) = \Sigma_H^*(0)$, (see [9]).

2. Coefficient inequalities

In this section we obtain coefficient bounds. Our first theorem gives a sufficient coefficient condition for the class $MH^*(n,\gamma)$.

Theorem 2.1. *If $f(z) = h(z) + \overline{g(z)}$ where $|b_2| \leqslant (|\alpha| - |\beta|)/2$, h(z) and g(z) of the form (2) and the condition*

$$\sum_{k=1}^{\infty}(k+\gamma)(k+2)^n|a_k| + \sum_{k=3}^{\infty}(k-\gamma)(k-2)^n|b_k| + |b_1| \leqslant (1-\gamma)|\alpha| - (1+\gamma)3^n|\beta| \tag{7}$$

is satisfied then f(z) is univalent, sense preserving in $\widetilde{U}$ and $f(z) \in MH^*(n,\gamma)$.

Proof. In view of Theorem 1.1, $f(z)$ is sense preserving and univalent in $\widetilde{U}$. Now it remains to show that the condition (7) is sufficient for f to be in $MH^*(n,\gamma)$. We use the fact that $\mathrm{Re}\,\zeta \geqslant 0$ if and only if $|1+\zeta| \geqslant |1-\zeta|$ in $\widetilde{U}$. Therefore, it is sufficient to show that

$$|p_n(z) + 1| > |p_n(z) - 1|, \quad z \in \widetilde{U}, \tag{8}$$

where

$$p_n(z) = \frac{(2-\gamma)M^n f(z) - M^{n+1} f(z)}{M^n f(z)}.$$

From (8), we must have

$$\frac{\left|(3-\gamma)M^n f(z) - M^{n+1} f(z)\right|}{\left|M^n f(z)\right|} - \frac{\left|(1-\gamma)M^n f(z) - M^{n+1} f(z)\right|}{\left|M^n f(z)\right|} > 0. \tag{9}$$

Substituting for $M^n f(z)$ and $M^{n+1} f(z)$ in (9), we obtain

$$\begin{aligned}
&\left|(3-\gamma)M^n f(z) - M^{n+1} f(z)\right| - \left|(1-\gamma)M^n f(z) - M^{n+1} f(z)\right| \\
&= \left|-(2-\gamma)\alpha z + \sum_{k=1}^{\infty}(k+\gamma-1)(k+2)^n a_k z^{-k} - \gamma 3^n \overline{\beta z} - (-1)^n \overline{\sum_{k=1}^{\infty}(k+1-\gamma)(k-2)^n b_k z^{-k}}\right| \\
&\quad - \left|\gamma\alpha z + \sum_{k=1}^{\infty}(k+\gamma+1)(k+2)^n a_k z^{-k} + (2+\gamma)3^n \overline{\beta z} - (-1)^n \overline{\sum_{k=1}^{\infty}(k-1-\gamma)(k-2)^n b_k z^{-k}}\right| \\
&\geqslant (2-\gamma)|\alpha||z| - \sum_{k=1}^{\infty}(k+\gamma-1)(k+2)^n |a_k||z|^{-k} - \gamma 3^n |\beta||z| - \sum_{k=3}^{\infty}(k+1-\gamma)(k-2)^n |b_k||z|^{-k} \\
&\quad - (2-\gamma)|b_1||z|^{-1} - \gamma|\alpha||z| - \sum_{k=1}^{\infty}(k+\gamma+1)(k+2)^n |a_k||z|^{-k} - (2+\gamma)3^n|\beta||z| \\
&\quad - \sum_{k=3}^{\infty}(k-1-\gamma)(k-2)^n |b_k||z|^{-k} - \gamma|b_1||z|^{-1} \\
&= 2(1-\gamma)|\alpha||z| - \sum_{k=1}^{\infty} 2(k+\gamma)(k+2)^n |a_k||z|^{-k} - 2(1+\gamma)3^n|\beta||z| - 2|b_1||z|^{-1} - \sum_{k=3}^{\infty} 2(k-\gamma)(k-2)^n |b_k||z|^{-k} \\
&= 2|z|\left\{(1-\gamma)|\alpha| - \sum_{k=1}^{\infty}(k+\gamma)(k+2)^n |a_k||z|^{-k-1} - (1+\gamma)3^n|\beta| - |b_1||z|^{-2} - \sum_{k=3}^{\infty}(k-\gamma)(k-2)^n |b_k||z|^{-k-1}\right\} \\
&\geqslant 2\left\{(1-\gamma)|\alpha| - \sum_{k=1}^{\infty}(k+\gamma)(k+2)^n |a_k| - (1+\gamma)3^n|\beta| - |b_1| - \sum_{k=3}^{\infty}(k-\gamma)(k-2)^n |b_k|\right\} \geqslant 0, \quad \text{by (7)} \quad \square.
\end{aligned}$$

We next show that the above sufficient condition for $MH^*(n,\gamma)$ is also necessary for functions in $\overline{MH}^*(n,\gamma)$.

Theorem 2.2. *Let the functions $f_n(z)$ be defined by Eq. (5) and $b_2 \leqslant (\alpha-\beta)/2$. A necessary and sufficient condition for $f_n(z) \in \overline{MH}^*(n,\gamma)$ is that*

$$\sum_{k=1}^{\infty}(k+\gamma)(k+2)^n a_k + \sum_{k=3}^{\infty}(k-\gamma)(k-2)^n b_k + (1-\gamma)b_1 \leqslant (1-\gamma)\alpha - (1+\gamma)3^n\beta. \tag{10}$$

The estimate (10) is sharp and the equality is attained for the function

$$f_{2n,k}(z) = -\alpha z - \frac{(1-\gamma)\alpha - (1+\gamma)3^{2n}\beta}{2(k+\gamma)(k+2)^{2n}} z^{-k} + \beta\bar{z} - \frac{(1-\gamma)\alpha - (1+\gamma)3^{2n}\beta}{2(k-\gamma)(k-2)^{2n}} \bar{z}^{-k},$$

for $k \geqslant 3$.

Proof. In view of Theorem 2.1, it is sufficient to prove the "only if" part, since $\overline{MH}^*(n,\gamma) \subset MH^*(n,\gamma)$ and

$$\sum_{k=1}^{\infty}(k+\gamma)(k+2)^n a_k + \sum_{k=3}^{\infty}(k-\gamma)(k-2)^n b_k + (1-\gamma)b_1 \leqslant \sum_{k=1}^{\infty}(k+\gamma)(k+2)^n a_k + \sum_{k=3}^{\infty}(k-\gamma)(k-2)^n b_k + b_1.$$

Assume that $f_n(z) \in \overline{MH}^*(n,\gamma)$. Let z be a complex number. If $\mathrm{Re}(z) > 0$, then $\mathrm{Re}(1/z) > 0$. Thus from (4) we obtain

$$0 < \mathrm{Re}\left\{\frac{M^n f(z)}{(2-\gamma)M^n f(z) - M^{n+1} f(z)}\right\} \leqslant \left|\frac{M^n f(z)}{(2-\gamma)M^n f(z) - M^{n+1} f(z)}\right|$$
$$= \left|\frac{-\alpha z - \sum_{k=1}^{\infty}(k+2)^n a_k z^{-k} + \overline{3^n \beta z - (-1)^n \sum_{k=1}^{\infty}(k-2)^n b_k z^{-k}}}{-(1-\gamma)\alpha z + \sum_{k=1}^{\infty}(k+\gamma)(k+2)^n a_k z^{-k} - (1+\gamma)3^n \beta \bar{z} - (-1)^n \sum_{k=1}^{\infty}(k-\gamma)(k-2)^n b_k \bar{z}^{-k}}\right|$$
$$\leqslant \frac{\alpha|z| + \sum_{k=1}^{\infty}(k+2)^n a_k |z|^{-k} + 3^n \beta |z| + \sum_{k=3}^{\infty}(k-2)^n b_k |z|^{-k} + |b_1||z|^{-1}}{(1-\gamma)\alpha|z| - \sum_{k=1}^{\infty}(k+\gamma)(k+2)^n a_k |z|^{-k} - (1+\gamma)3^n \beta |z| - \sum_{k=3}^{\infty}(k-\gamma)(k-2)^n b_k |z|^{-k} - (1-\gamma) b_1 |z|^{-1}}$$
$$< \frac{\alpha + \sum_{k=1}^{\infty}(k+2)^n a_k + 3^n \beta + \sum_{k=2}^{\infty}(k-2)^n b_k + b_1}{(1-\gamma)\alpha - \sum_{k=1}^{\infty}(k+\gamma)(k+2)^n a_k - (1+\gamma)3^n \beta - \sum_{k=3}^{\infty}(k-\gamma)(k-2)^n b_k - (1-\gamma) b_1}. \quad (11)$$

From (11), we find

$$\sum_{k=1}^{\infty}(k+\gamma)(k+2)^n a_k + \sum_{k=3}^{\infty}(k-\gamma)(k-2)^n b_k + (1-\gamma) b_1 \leqslant (1-\gamma)\alpha - (1+\gamma)3^n \beta. \quad \square$$

3. Convex combinations

In this section, we show that the class $\overline{MH}^*(n,\gamma)$ is invariant under convex combinations of its members.

Theorem 3.1. *The class $\overline{MH}^*(n,\gamma)$ is a convex set.*

Proof. For $i = 1,2,\ldots$ suppose that $f_{n,i}(z) \in \overline{MH}^*(n,\gamma)$ where $f_{n,i}(z)$ is given by

$$f_{n,i}(z) = -\alpha_i z - \sum_{k=1}^{\infty} a_{k,i} z^{-k} + \overline{\beta_i z - (-1)^n \sum_{k=1}^{\infty} b_{k,i} z^{-k}}.$$

Then, by Theorem 2.2,

$$\sum_{k=1}^{\infty}(k+\gamma)(k+2)^n a_{k,i} + \sum_{k=3}^{\infty}(k-\gamma)(k-2)^n b_{k,i} + (1-\gamma) b_{1,i} \leqslant (1-\gamma)\alpha_i - (1+\gamma)3^n \beta_i. \quad (12)$$

For $\sum_{i=1}^{\infty} t_i = 1,\ 0 \leqslant t_i \leqslant 1$, the convex combination of $f_{n,i}$ may be written as

$$\sum_{i=1}^{\infty} t_i f_{n,i}(z) = -\left(\sum_{i=1}^{\infty} t_i \alpha_i\right) z - \sum_{k=1}^{\infty}\left(\sum_{i=1}^{\infty} t_i a_{k,i}\right) z^{-k} + \left(\sum_{i=1}^{\infty} t_i \beta_i\right)\bar{z} - (-1)^n \sum_{k=1}^{\infty}\left(\sum_{i=1}^{\infty} t_i b_{k,i}\right)\bar{z}^{-k}.$$

Hence, $\sum_{i=1}^{\infty} t_i f_{n,i}(z) \in \overline{MH}^*(n,\gamma)$ since

$$\sum_{k=1}^{\infty}(k+\gamma)(k+2)^n\left(\sum_{i=1}^{\infty} t_i a_{k,i}\right) + \sum_{k=3}^{\infty}(k-\gamma)(k-2)^n\left(\sum_{i=1}^{\infty} t_i b_{k,i}\right) + (1-\gamma)\left(\sum_{i=1}^{\infty} t_i b_{1,i}\right)$$
$$= \sum_{i=1}^{\infty} t_i\left[\sum_{k=1}^{\infty}(k+\gamma)(k+2)^n a_{k,i} + \sum_{k=3}^{\infty}(k-\gamma)(k-2)^n b_{k,i} + (1-\gamma) b_{1,i}\right] \leqslant \sum_{i=1}^{\infty} t_i\left[(1-\gamma)\alpha_i - (1+\gamma)3^n \beta_i\right]$$
$$= (1-\gamma)\left(\sum_{i=1}^{\infty} t_i \alpha_i\right) - (1+\gamma)3^n\left(\sum_{i=1}^{\infty} t_i \beta_i\right). \quad \square$$

4. A distortion theorem and extreme points

Theorem 4.1. *Let the function $f_n(z)$ be in the class $\overline{MH}^*(n,\gamma)$. Then, for $|z| = r > 1$, we have*

$$(\alpha-\beta)r - \left[(1-\gamma)\alpha - (1+\gamma)3^n\beta\right] r^{-1} \leqslant |f_n(z)| \leqslant (\alpha+\beta)r + \left[(1-\gamma)\alpha - (1+\gamma)3^n\beta\right] r^{-1}.$$

Proof. Let $f_n(z) \in \overline{MH}^*(n,\gamma)$. Taking the absolute value of $f_n(z)$, we obtain

$$|f_n(z)| = \left|-\alpha z - \sum_{k=1}^{\infty} a_k z^{-k} + \overline{\beta z - (-1)^n \sum_{k=1}^{\infty} b_k z^{-k}}\right| \leqslant \alpha r + \beta r + \sum_{k=1}^{\infty}(a_k + b_k) r^{-k} \leqslant \alpha r + \beta r + \sum_{k=1}^{\infty}(a_k + b_k) r^{-1}$$
$$\leqslant \alpha r + \beta r + r^{-1}\left\{\sum_{k=1}^{\infty}(k+\gamma)(k+2)^n a_k + \sum_{k=3}^{\infty}(k-\gamma)(k-2)^n b_k + (1-\gamma) b_1\right\}$$
$$\leqslant (\alpha+\beta)r + \left[(1-\gamma)\alpha - (1+\gamma)3^n\beta\right] r^{-1}.$$

The proof for the bound on the right hand is similar to that given above and we omit it. □

Corollary 4.2. *Let $0 \leqslant \gamma < (\alpha + 3^n\beta)/(\alpha - 3^n\beta)$ and $n \in \mathbb{N}_0$. Then $\overline{MH}^*(n+1,\gamma) \subset \overline{MH}^*(n,\gamma)$.*

For $f_n = h + \bar{g}_n$ the family $\overline{MH}^*(n,\gamma)$ is locally uniformly bounded as in (5). In this section, we examine the extreme points for functions in $\overline{MH}^*(n,\gamma)$, for each fixed α when f_n is defined by (5). This family is still not compact under the topology of locally uniform convergence. To see this, observe that for $m = 1, 2, \ldots,$

$$f_{n,m}(z) = -\alpha z + \frac{\alpha m}{m+1}\bar{z} \in \overline{MH}^*(n,\gamma)$$

but

$$\lim_{m\to\infty} f_{n,m}(z) = -\alpha z + \alpha\bar{z} \notin \overline{MH}^*(n,\gamma).$$

Nevertheless, we can still use the coefficient bounds in Theorem 2.2 to determine the extreme points of the closed convex hull of $\overline{MH}^*(n,\gamma)$ $clco\overline{MH}^*(n,\gamma)$.

Theorem 4.3. *Let for $z \in \widetilde{U}$,*

$$\begin{aligned} h_{n,0}(z) &= -z, \quad g_{n,0}(z) = -z + \frac{(1-\gamma)\bar{z}}{(1+\gamma)3^n}, \\ g_{n,1}(z) &= -z - (-1)^n\bar{z}^{-1}, \quad g_{n,2}(z) = -z - \frac{(-1)^n}{2}\bar{z}^{-2}, \\ h_{n,k}(z) &= -z - \frac{1-\gamma}{(k+\gamma)(k+2)^n}z^{-k}, \quad k \geqslant 1 \end{aligned}$$

and

$$g_{n,k}(z) = -z - \frac{1-\gamma}{(k-\gamma)(k-2)^n}\bar{z}^{-k}, \quad k \geqslant 3.$$

Then $f_n(z) \in clco\overline{MH}^*(n,\gamma)$ if and only if it can be expressed in the form

$$f_n(z) = \sum_{k=0}^{\infty}\left[x_k h_{n,k}(z) + y_k g_{n,k}(z)\right],$$

where

$$x_k \geqslant 0, \quad y_k \geqslant 0 \quad \text{and} \quad \sum_{k=0}^{\infty}(x_k + y_k) = \alpha.$$

Proof. Let

$$f_n(z) = \sum_{k=0}^{\infty}\left[x_k h_{n,k}(z) + y_k g_{n,k}(z)\right],$$

with

$$x_k \geqslant 0, \quad y_k \geqslant 0 \quad \text{and} \quad \sum_{k=0}^{\infty}(x_k + y_k) = \alpha.$$

Then we have

$$\begin{aligned} f_n(z) &= \sum_{k=0}^{\infty}\left[x_k h_{n,k}(z) + y_k g_{n,k}(z)\right] \\ &= x_0 h_{n,0}(z) + \sum_{k=1}^{\infty} x_k\left(-z - \frac{1}{(k+\gamma)(k+2)^n}z^{-k}\right) + y_0 g_{n,0}(z) + y_1 g_{n,1}(z) + y_2 g_{n,2}(z) \\ &\quad + \sum_{k=3}^{\infty} y_k\left(-z - \frac{1}{(k-\gamma)(k-2)^n}\bar{z}^{-k}\right) \\ &= -\sum_{k=0}^{\infty}(x_k + y_k)z - \sum_{k=1}^{\infty}\frac{x_k}{(k+\gamma)(k+2)^n}z^{-k} + \frac{y_0}{3^n}\bar{z} - y_1\frac{\bar{z}^{-1}}{1-\gamma} - \frac{y_2}{2}\bar{z}^{-2} - \sum_{k=3}^{\infty}\frac{y_k}{(k-\gamma)(k-2)^n}\bar{z}^{-k} \\ &= -\alpha z - \sum_{k=1}^{\infty}\frac{x_k}{(k+\gamma)(k+2)^n}z^{-k} + \frac{y_0}{3^n}\bar{z} - y_1\frac{\bar{z}^{-1}}{1-\gamma} - \frac{y_2}{2}\bar{z}^{-2} - \sum_{k=3}^{\infty}\frac{y_k}{(k-\gamma)(k-2)^n}\bar{z}^{-k}. \end{aligned}$$

Since

$$\sum_{k=1}^{\infty}(k+\gamma)(k+2)^n\frac{x_k}{(k+\gamma)(k+2)^n}+\sum_{k=3}^{\infty}(k-\gamma)(k-2)^n\frac{y_k}{(k-\gamma)(k-2)^n}+(1-\gamma)y_1=\left(\sum_{k=1}^{\infty}x_k+y_1+\sum_{k=3}^{\infty}y_k\right)$$
$$=\alpha-y_0-x_0-y_2\leqslant\alpha-3^n\frac{y_0}{3^n}(1-\gamma)\alpha'-(1+\gamma)3^n\beta'$$

by Theorem 2.2, $f_n(z)\in clco\overline{MH}^*(n,\gamma)$. Conversely, we suppose that $f_n(z)\in clco\overline{MH}^*(n,\gamma)$; then we may write

$$f_n(z)=h(z)+\overline{g_n(z)}=-\alpha z-\sum_{k=1}^{\infty}a_kz^{-k}+\overline{\beta z-(-1)^n\sum_{k=1}^{\infty}b_kz^{-k}},$$

where $\alpha > 3^n\beta\geqslant 0$, $a_k\geqslant 0$, $b_k\geqslant 0$. We set

$$a_k=\frac{x_k}{(k+\gamma)(k+2)^n},\quad k=1,2,\ldots,\ \beta=\frac{y_0}{3^n}\ b_1=y_1,$$
$$b_k=\frac{y_k}{(k-\gamma)(k-2)^n},\quad k=3,4\ldots\ b_2=\frac{y_2}{2}.$$

Hence, we obtain

$$\begin{aligned}f_n(z)&=h(z)+\overline{g_n(z)}=-\alpha z-\sum_{k=1}^{\infty}a_kz^{-k}+\overline{\beta z-(-1)^n\sum_{k=1}^{\infty}b_kz^{-k}}\\&=-\sum_{k=0}^{\infty}(x_k+y_k)z+\sum_{k=1}^{\infty}\frac{x_k}{(k+\gamma)(k+2)^n}z^{-k}+\frac{y_0}{3^n}\bar{z}-y_1\frac{\bar{z}^{-1}}{(1-\gamma)}-\frac{y_2}{2}\bar{z}^{-2}+\sum_{k=3}^{\infty}\frac{y_k}{(k-\gamma)(k-2)^n}\bar{z}^{-k}\\&=-x_0z+\sum_{k=1}^{\infty}x_k\left(-z-\frac{1}{(k+\gamma)(k+2)^n}z^{-k}\right)+y_0\left(-z+\frac{\bar{z}}{3^n}\right)+y_1\left(-z-\bar{z}^{-1}\right)+y_2\left(-z-\frac{1}{2}\bar{z}^{-2}\right)\\&\quad+\sum_{k=3}^{\infty}y_k\left(-z-\frac{1}{(k-\gamma)(k-2)^n}\bar{z}^{-k}\right)\\&=\sum_{k=0}^{\infty}\left[x_kh_{n,k}(z)+y_kg_{n,k}(z)\right].\end{aligned}$$

Hence, the proof is completed. □

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