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Explicit construction of general multivariate Padé approximants to an Appell function

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Properties of Padé approximants to the Gauss hypergeometric function ${}_2F_1(a, b; c; z)$ have been studied in several papers and some of these properties have been generalized to several variables in [6]. In this paper we derive explicit formulae for the general multivariate Padé approximants to the Appell function $F_1(a, 1, 1; a+1; x, y) = \sum_{i,j=0}^{\infty}(ax^i y^j/(i+j+a))$, where a is a positive integer. In particular, we prove that the denominator of the constructed approximant of partial degree n in x and y is given by $q(x, y) = (-1)^n\binom{m+n+a}{n}F_1(-m-a, -n, -n; -m-n-a; x, y)$, where the integer m, which defines the degree of the numerator, satisfies $m \geqslant n+1$ and $m+a \geqslant 2n$. This formula generalizes the univariate explicit form for the Padé denominator of ${}_2F_1(a, 1; c; z)$, which holds for $c > a > 0$ and only in half of the Padé table. From the explicit formulae for the general multivariate Padé approximants, we can deduce the normality of a particular multivariate Padé table.

Keywords: Padé approximant, hypergeometric function, multivariate

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1. Introduction

The study of generalized hypergeometric functions of several variables has been extensive, due to their frequent occurrence in the solutions of statistical and physical problems. In this paper we focus on the first Appell function $F_1(a, b, b'; c; x, y)$ as given below.

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For any positive integer i, let

$$(a)_i := \begin{cases} a(a+1)(a+2)\cdots(a+i-1), & i \geqslant 1, \\ 1, & i = 0. \end{cases} \tag{1.1}$$

Then the Gauss or ordinary hypergeometric function is given by

$${}_2F_1(a, b; c; z) := \sum_{i=0}^{\infty} \frac{(a)_i (b)_i}{(c)_i i!} z^i, \tag{1.2}$$

where the parameters a, b, c and z may be real or complex. The natural generalizations of the Gauss hypergeometric function to two variables are the following four functions, each called Appell function (see [12] for more details):

$$F_1(a, b, b'; c; x, y) = \sum_{i,j=0}^{\infty} \frac{(a)_{i+j}(b)_i(b')_j x^i y^j}{(c)_{i+j} i! j!};$$

$$F_2(a, b, b'; c, c'; x, y) = \sum_{i,j=0}^{\infty} \frac{(a)_{i+j}(b)_i(b')_j x^i y^j}{(c)_i (c')_j i! j!};$$

$$F_3(a, a', b, b'; c; x, y) = \sum_{i,j=0}^{\infty} \frac{(a)_i(a')_j(b)_i(b')_j x^i y^j}{(c)_{i+j} i! j!};$$

$$F_4(a, b; c, c'; x, y) = \sum_{i,j=0}^{\infty} \frac{(a)_{i+j}(b)_{i+j} x^i y^j}{(c)_i (c')_j i! j!}.$$

All four Appell functions reduce to the Gauss function if one of the variables is equal to zero.

Properties of Padé approximants to the Gauss function ${}_2F_1(a, 1; c; z)$, where $c > a > 0$, have been given in several papers [8,11,13,15]. Among these we find the following explicit formula for the Padé denominator. Let us denote the Padé approximant of degree m in the numerator and n in the denominator by $p(z)/q(z)$. Then if c is not a negative integer and if $n \leqslant m + 1$, the denominator of the Padé approximant is given by

$$q(z) = {}_2F_1(-a - m, -n; -c - m - n + 1; z).$$

Also, the table of Padé approximants to the Gauss function ${}_2F_1(a, 1; c; z)$, where $c > a > 0$, has been proven to be normal, meaning that every Padé approximant occurs only once in the entire table.

In this paper, our goal is to find explicit formulae for some general multivariate Padé approximants to the Appell function F_1 when $b = b' = 1$ and $c = a + 1$, i.e. to the Appell function

$$F_1(a, 1, 1; a + 1; x, y) = \sum_{i,j=0}^{\infty} \frac{a x^i y^j}{i + j + a}. \tag{1.3}$$

To this end we first explicitly construct the general multivariate Padé approximants to the q analogue of $F_1(a, 1, 1; a+1; x, y)$, namely

$$L_q(x, y) := \sum_{i,j=0}^{\infty} \frac{(q^a - 1)x^i y^j}{q^{i+j+a} - 1},$$

where $|q| > 1$, $|x|, |y| < |q|$, and $a \geqslant 1$ is an integer, by using the residue theorem and the functional equation method (see [2,16–18] for more applications of this method). Then, under suitable conditions, we find the limit of the Padé approximant to $L_q(x, y)$ when q approaches one, which equals the general multivariate Padé approximant to the Appell function $F_1(a, 1, 1; a+1; x, y)$. When considering the table of general multivariate Padé approximants that can be constructed using this procedure, we can prove that this table is normal.

Let

$$F(x, y) := \sum_{(i,j)\in\mathbb{N}^2} c_{ij} x^i y^j, \quad c_{ij} \in \mathbb{C}, \tag{1.4}$$

be a formal power series, and let M, N, E be index sets in $\mathbb{N} \times N =: N^2$. An *$(M, N)$ general multivariate Padé approximant to $F(x, y)$ on the lattice E* is a rational function

$$[M/N]_E(x, y) := \frac{P(x, y)}{Q(x, y)}, \tag{1.5}$$

where the polynomials

$$P(x, y) := \sum_{(i,j)\in M} a_{ij} x^i y^j, \quad a_{ij} \in \mathbb{C},$$
$$Q(x, y) := \sum_{(i,j)\in N} b_{ij} x^i y^j, \quad b_{ij} \in \mathbb{C},$$

are such that

$$(FQ - P)(x, y) = \sum_{(i,j)\in\mathbb{N}^2\setminus E} d_{ij} x^i y^j, \quad d_{ij} \in \mathbb{C}, \tag{1.6}$$

with

$$M \subseteq E, \tag{1.7}$$
$$\#(E\setminus M) \geqslant \#N - 1 \tag{1.8}$$

and E satisfies the inclusion property

$$(i, j) \in E,\ 0 \leqslant k \leqslant i,\ 0 \leqslant l \leqslant j \quad \Longrightarrow \quad (k, l) \in E. \tag{1.9}$$

Equation (1.6) translates to the linear system of equations

$$d_{ij} = 0, \quad (i, j) \in E. \tag{1.10}$$

Using condition (1.7), we can split the system of equations (1.10) in an inhomogeneous linear system defining the numerator coefficients a_{ij},

$$\sum_{\mu=0}^{i}\sum_{\nu=0}^{j} c_{\mu\nu} b_{i-\mu,j-\nu} = a_{ij}, \quad (i,j) \in M, \tag{1.11}$$

and a homogeneous linear system defining the denominator coefficients b_{ij},

$$\sum_{\mu=0}^{i}\sum_{\nu=0}^{j} c_{\mu\nu} b_{i-\mu,j-\nu} = 0, \quad (i,j) \in E\backslash M, \tag{1.12}$$

where $b_{kl} = 0$ for $(k,l) \notin N$. Condition (1.9) takes care of the Padé approximation property, provided $Q(0,0) \neq 0$, namely

$$\left(F - \frac{P}{Q}\right)(x,y) = \sum_{(i,j)\in\mathbb{N}^2\backslash E} e_{ij} x^i y^j, \quad e_{ij} \in \mathbb{C}.$$

It is clear that a nontrivial general multivariate Padé approximant always exists if the equal sign applies in condition (1.8), and that it will be unique up to a constant factor in the numerator and denominator if the coefficient matrix of the linear system (1.12) has maximal rank $\#N - 1$. If the rank of the coefficient matrix of (1.12) is less than the maximal rank, then multiple solutions of $Q(x,y)$ and $P(x,y)$ exist and we refer to [1] for a detailed discussion of this situation. For all definitions covered by the general definition given here, one cannot guarantee the existence of a unique irreducible form if multiple solutions of (1.12) exist. One may find more properties of general multivariate Padé approximants in [3,4,7].

For the sequel we need the standard q-analogues of factorials and binomial coefficients. The *q-factorial* is defined by

$$[n]_q! := [n]! := \frac{(1-q^n)(1-q^{n-1})\cdots(1-q)}{(1-q)^n},$$

where $[0]_q! := 1$. The *q-binomial coefficient* is given by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \begin{bmatrix} n \\ k \end{bmatrix} := \frac{[n]!}{[k]!\cdot[n-k]!}, \quad 0 \leqslant k \leqslant n.$$

Note that for all $0 \leqslant k \leqslant n$,

$$[n]_{q^{-1}}! = q^{-n(n-1)/2}[n]!, \qquad \begin{bmatrix} n \\ k \end{bmatrix}_{q^{-1}} = q^{-k(n-k)}\begin{bmatrix} n \\ k \end{bmatrix},$$

$$\prod_{h=0,\ h\neq k}^{n} \left(q^{-k} - q^{-h}\right) = (-1)^{k+n} q^{-k(k-1)/2-n(n+1)/2}[n-k]!\,[k]!\,(1-q)^n,$$

and for $|t| < q^{-n}$ (see [9]),

$$\frac{1}{\prod_{k=0}^{n}(t - q^{-k})} = (-1)^{n+1} q^{n(n+1)/2} \sum_{l=0}^{\infty} \begin{bmatrix} n+l \\ l \end{bmatrix} t^l. \tag{1.13}$$

We also need the Cauchy binomial theorem

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^{k(k+1)/2} x^k = \prod_{k=1}^{n} \bigl(1 + q^k x\bigr). \tag{1.14}$$

2. Padé approximants to the q analogue of the Appell function $F_1(a, 1, 1; a+1; x, y)$

In this section, we explicitly construct some general Padé approximants to the q analogue of the Appell function $F_1(a, 1, 1; a+1; x, y)$ by using the residue theorem and the functional equation method. The functional equation and the appropriate integral we construct, play a crucial role in finding the explicit formulae for the multivariate Padé approximants for this function. The functional equation used here is simple but the integral is relatively more complicated. We also prove that the rational approximants we obtain are irreducible.

Let $|q| > 1$, $|x|, |y| < |q|$, and let $a \geqslant 1$ be integer, and let

$$L(x, y) := L_q(x, y) := \sum_{i,j=0}^{\infty} \frac{(q^a - 1) x^i y^j}{q^{i+j+a} - 1}. \tag{2.1}$$

As

$$q^{i+j+1} - 1 = (q - 1)\bigl(q^{i+j} + q^{i+j-1} + \cdots + 1\bigr),$$

we find

$$\lim_{q \to 1} L(x, y) = \sum_{i,j=0}^{\infty} \frac{a x^i y^j}{i + j + a} = F_1(a, 1, 1; a+1; x, y).$$

Hence we call $L(x, y)$, as defined above, the q analogue of the Appell function $F_1(a, 1, 1; a+1; x, y)$. Since for $k \geqslant 0$ integer, and $|x|, |y| < |q|$,

$$\begin{aligned} L\bigl(q^{-1}x, q^{-1}y\bigr) &= \sum_{i,j=0}^{\infty} \frac{(q^a - 1) q^{-(i+j)} x^i y^j}{q^{i+j+a} - 1} \\ &= \sum_{i,j=0}^{\infty} \frac{(q^a - 1)(1 - q^{i+j+a} + q^{i+j+a}) x^i y^j}{q^{i+j}(q^{i+j+a} - 1)} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{i,j=0}^{\infty} \frac{q^a(q^a-1)x^i y^j}{q^{i+j+a}-1} - \left(q^a-1\right) \sum_{i,j=0}^{\infty} \frac{x^i y^j}{q^{i+j}} \\
&= q^a L(x,y) - \frac{(q^a-1)}{(1-q^{-1}x)(1-q^{-1}y)},
\end{aligned}
$$

then the functional equation we need is given by

$$
\begin{aligned}
L\left(q^{-k}x, q^{-k}y\right) &= L\left(q^{-1}q^{-k+1}x, q^{-1}q^{-k+1}y\right) \\
&= q^a L\left(q^{-k+1}x, q^{-k+1}y\right) - \frac{(q^a-1)}{(1-q^{-k}x)(1-q^{-k}y)} \\
&\ \ \vdots \\
&= q^{ka} L(x,y) - \sum_{j=1}^{k} \frac{(q^a-1)q^{(k-j)a}}{(1-q^{-j}x)(1-q^{-j}y)} \\
&= q^{ka} L(x,y) - S_k(x,y), \qquad (2.2)
\end{aligned}
$$

where

$$
S_k(x,y) := \sum_{j=1}^{k} \frac{(q^a-1)q^{(k-j)a}}{(1-q^{-j}x)(1-q^{-j}y)} \qquad (2.3)
$$

and

$$
S_0(x,y) := 0.
$$

Theorem 2.1. Let $L(x,y)$ and $S_k(x,y)$ be defined by (2.1) and (2.3), and let

$$
R_n(x,y) := \prod_{j=0}^{n-1} \left(\left(1-q^j x\right)\left(1-q^j y\right)\right). \qquad (2.4)
$$

Let $m, n \in \mathbb{N}$, $m \geqslant n+1 \geqslant 1$, and

$$
\begin{aligned}
W &:= \left\{(i,j):\ 0 \leqslant i, j,\ 0 \leqslant i+j \leqslant m\right\}, \qquad (2.5) \\
N &:= \left\{(i,j):\ 0 \leqslant i, j \leqslant n\right\}, \qquad (2.6) \\
M &:= N \cup W, \qquad (2.7) \\
E &:= \left\{(i,j):\ 0 \leqslant i+j \leqslant m+n,\ i, j \geqslant 0\right\}. \qquad (2.8)
\end{aligned}
$$

Let

$$
I(x,y) := \frac{1}{2\pi \mathrm{i}} \int_{\Gamma} \frac{R_n(tx,ty)L(tx,ty)}{t^{m+1}\prod_{k=0}^{n}(t-q^{-k})}\, \mathrm{d}t, \qquad (2.9)
$$

where Γ is a circular contour containing $0, q^0, q^{-1}, \dots, q^{-n}$, and let

$$Q(x,y) := \frac{(-1)^n q^{n(n+1)/2}}{(1-q)^n[n]!} \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{k(k+1)/2+k(m+a)} R_n\left(q^{-k}x, q^{-k}y\right) \tag{2.10}$$

and

$$P(x,y) := \frac{(-1)^n q^{n(n+1)/2}}{(1-q)^n[n]!} \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{k(k+1)/2+km} R_n\left(q^{-k}x, q^{-k}y\right) S_k(x,y) - \frac{1}{m!}\frac{\mathrm{d}^m}{\mathrm{d}t^m}\left\{\frac{R_n(tx,ty)F(tx,ty)}{\prod_{k=0}^{n}(t-q^{-k})}\right\}_{t=0}. \tag{2.11}$$

Then: (i)

$$I(x,y) = L(x,y)Q(x,y) - P(x,y);$$

(ii)

$$Q(x,y) = \sum_{(i,j)\in N} b_{ij} x^i y^j, \quad b_{ij} \in \mathbb{C}, \tag{2.12}$$

$$P(x,y) = \sum_{(i,j)\in M} a_{ij} x^i y^j, \quad a_{ij} \in \mathbb{C}. \tag{2.13}$$

More precisely,

$$Q(x,y) = \frac{(-1)^n q^{n(n+1)/2}}{(1-q)^n[n]!} \times \sum_{i,j=0}^{n} \left\{ (-1)^{i+j} q^{i(i-1)/2+j(j-1)/2} \left(\prod_{k=1}^{n} \left(1-q^{k+m+a-i-j}\right) \right) \begin{bmatrix} n \\ i \end{bmatrix} \begin{bmatrix} n \\ j \end{bmatrix} x^i y^j \right\}, \tag{2.14}$$

$$P(x,y) = \frac{(-1)^n q^{n(n+1)/2}}{(1-q)^n[n]!} \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{k(k+1)/2+km} R_n\left(q^{-k}x, q^{-k}y\right) S_k(x,y) + (-1)^n\left(q^a-1\right) q^{n(n+1)/2} \times \sum_{\substack{i+j+h+l+k=m \\ 0\leqslant i,j,h\leqslant m,\ 0\leqslant k,l\leqslant n}} \frac{(-1)^{k+l} x^{i+k} y^{j+l}}{(q^{i+j+a}-1)} \begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} n \\ l \end{bmatrix} \begin{bmatrix} n+h \\ h \end{bmatrix} q^{k(k-1)/2+l(l-1)/2}; \tag{2.15}$$

(iii)

$$I(x,y) = \sum_{(i,j)\in \mathbb{N}^2\setminus E} d_{ij} x^i y^j, \quad d_{ij} \in \mathbb{C}, \tag{2.16}$$

with

$$Q(0,0) \neq 0;$$

(iv)

$$M \subseteq E \quad \text{and} \quad \#(E\backslash M) \geqslant \#N - 1.$$

Hence an (M, N) general multivariate Padé approximant to $L(x, y)$ on the lattice E is given by

$$[M/N]_E(x,y) = \frac{P(x,y)}{Q(x,y)}.$$

Proof. (i) We can see that the integrand in (2.9) has simple poles at $t = 1, q^0, q^{-1}, \ldots, q^{-n}$, and a pole of order $m+1$ at $t = 0$, all inside the contour Γ. By the residue theorem, the functional equation (2.2) and (1.13), (1.14), we have

$$\begin{aligned}
I(x,y) &= \frac{1}{2\pi \mathrm{i}} \int_\Gamma \frac{R_n(tx,ty)L(tx,ty)}{t^{m+1}\prod_{k=0}^n (t-q^{-k})}\,\mathrm{d}t \\
&= \sum_{k=0}^n \frac{R_n(q^{-k}x, q^{-k}y)L(q^{-k}x, q^{-k}y)}{\big(\prod_{\substack{h=0\\h\neq k}}^n (q^{-k}-q^{-h})\big)q^{-k(m+1)}} + \frac{1}{m!}\frac{\mathrm{d}^m}{\mathrm{d}t^m}\left\{\frac{R_n(tx,ty)L(tx,ty)}{\prod_{k=0}^n (t-q^{-k})}\right\}_{t=0} \\
&= \frac{(-1)^n q^{n(n+1)/2}}{(1-q)^n [n]!} \sum_{k=0}^n (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{k(k+1)/2+km} R_n\big(q^{-k}x, q^{-k}y\big) \\
&\quad \times \big(q^{ka}L(x,y) - S_k(x,y)\big) + \frac{1}{m!}\frac{\mathrm{d}^m}{\mathrm{d}t^m}\left\{\frac{R_n(tx,ty)L(tx,ty)}{\prod_{k=0}^n (t-q^{-k})}\right\}_{t=0} \\
&= Q(x,y)L(x,y) - P(x,y).
\end{aligned}$$

(ii) It is easy to see from the definition of $Q(x, y)$ and $R_n(x, y)$ that (2.12) holds. Now from the Cauchy binomial theorem (1.15), we have

$$\begin{aligned}
R_n(tx,ty) &= \prod_{j=0}^{n-1}\big(\big(1-q^j tx\big)\big(1-q^j ty\big)\big) \\
&= \prod_{j=1}^{n}\big(\big(1-q^j q^{-1} tx\big)\big(1-q^j q^{-1} ty\big)\big) \\
&= \left(\sum_{i=0}^n (-1)^i \begin{bmatrix} n \\ i \end{bmatrix} q^{i(i+1)/2-i} t^i x^i\right)\left(\sum_{j=0}^n (-1)^j \begin{bmatrix} n \\ j \end{bmatrix} q^{j(j+1)/2-j} t^j y^j\right) \\
&= \sum_{i,j=0}^n (-1)^{i+j} \begin{bmatrix} n \\ i \end{bmatrix}\begin{bmatrix} n \\ j \end{bmatrix} q^{i(i-1)/2+j(j-1)/2} x^i y^j t^{i+j},
\end{aligned} \tag{2.17}$$

and then

$$R_n\big(q^{-k}x, q^{-k}y\big) = \sum_{i,j=0}^{n} (-1)^{i+j} \begin{bmatrix} n \\ i \end{bmatrix} \begin{bmatrix} n \\ j \end{bmatrix} q^{i(i-1)/2+j(j-1)/2-k(i+j)} x^i y^j.$$

Putting this into (2.10), we have, by using (1.15) again,

$$\begin{aligned} Q(x,y) &= \frac{(-1)^n q^{n(n+1)/2}}{(1-q)^n [n]!} \sum_{k=0}^{n} \Bigg\{ (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{k(k+1)/2+k(m+a)} \\ &\quad \times \sum_{i,j=0}^{n} (-1)^{i+j} \begin{bmatrix} n \\ i \end{bmatrix} \begin{bmatrix} n \\ j \end{bmatrix} q^{i(i-1)/2+j(j-1)/2-k(i+j)} x^i y^j \Bigg\} \\ &= \frac{(-1)^n q^{n(n+1)/2}}{(1-q)^n [n]!} \sum_{i,j=0}^{n} \Bigg\{ (-1)^{i+j} \begin{bmatrix} n \\ i \end{bmatrix} \begin{bmatrix} n \\ j \end{bmatrix} q^{i(i-1)/2+j(j-1)/2} x^i y^j \\ &\quad \times \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{k(k+1)/2+k(m+a-i-j)} \Bigg\} \\ &= \frac{(-1)^n q^{n(n+1)/2}}{(1-q)^n [n]!} \sum_{i,j=0}^{n} (-1)^{i+j} \left(\prod_{k=1}^{n} \big(1 - q^{k+m+a-i-j}\big) \right) \begin{bmatrix} n \\ i \end{bmatrix} \begin{bmatrix} n \\ j \end{bmatrix} \\ &\quad \times q^{i(i-1)/2+j(j-1)/2} x^i y^j. \end{aligned}$$

This proves (2.14). Now for $0 \leqslant k \leqslant n$,

$$\begin{aligned} R_n\big(q^{-k}x, q^{-k}y\big) &= \prod_{j=0}^{n-1} \big(1 - q^{j-k}x\big)\big(1 - q^{j-k}y\big) \\ &= \left(\prod_{j=1}^{k} \big(1 - q^{-j}x\big)\big(1 - q^{-j}y\big) \right) \left(\prod_{j=0}^{n-k-1} \big(1 - q^{j}x\big)\big(1 - q^{j}y\big) \right), \end{aligned}$$

which implies that

$$\begin{aligned} R_n\big(q^{-k}x, q^{-k}y\big) S_k(x,y) &= \big(q^a - 1\big) \left(\prod_{j=0}^{n-k-1} \big(1 - q^{j}x\big)\big(1 - q^{j}y\big) \right) \\ &\quad \times \sum_{h=1}^{k} q^{(k-h)a} \prod_{j=1,\, j\neq h}^{k} \big(1 - q^{-j}x\big)\big(1 - q^{-j}y\big), \end{aligned}$$

and hence

$$R_n\big(q^{-k}x, q^{-k}y\big) S_k(x,y) = \sum_{(i,j)\in N} s_{ij} x^i y^j, \quad s_{ij} \in \mathbb{C}. \tag{2.18}$$

Also

$$L(tx,ty)=\sum_{i,j=0}^{\infty}\frac{(q^a-1)x^iy^jt^{i+j}}{q^{i+j+a}-1}. \tag{2.19}$$

Then from (1.14), (2.17) and (2.19), for $|t|\leqslant q^{-n}$,

$$\frac{R_n(tx,ty)L(tx,ty)}{\prod_{k=0}^{n}(t-q^{-k})}=(-1)^{n+1}q^{n(n+1)/2}\sum_{i,j,h=0}^{\infty}\sum_{k,l=0}^{n}\left\{(-1)^{k+l}\begin{bmatrix}n\\k\end{bmatrix}\begin{bmatrix}n\\l\end{bmatrix}\begin{bmatrix}n+h\\h\end{bmatrix}\right.$$
$$\left.\times q^{k(k-1)/2+l(l-1)/2}\frac{(q^a-1)x^{i+k}y^{j+l}t^{i+j+h+k+l}}{(q^{i+j+a}-1)}\right\}.$$

So

$$\frac{1}{m!}\frac{\mathrm{d}^m}{\mathrm{d}t^m}\left\{\frac{R_n(tx,ty)L(tx,ty)}{\prod_{k=0}^{n}(t-q^{-k})}\right\}_{t=0}$$
$$=(-1)^{n+1}\big(q^a-1\big)q^{n(n+1)/2}\sum_{\substack{i+j+h+l+k=m\\0\leqslant i,j,h,\ 0\leqslant k,l\leqslant n}}\left\{\frac{(-1)^{k+l}x^{i+k}y^{j+l}}{(q^{i+j+a}-1)}\right.$$
$$\left.\times\begin{bmatrix}n\\k\end{bmatrix}\begin{bmatrix}n\\l\end{bmatrix}\begin{bmatrix}n+h\\h\end{bmatrix}q^{k(k-1)/2+l(l-1)/2}\right\}, \tag{2.20}$$

and hence

$$\frac{1}{m!}\frac{\mathrm{d}^m}{\mathrm{d}t^m}\left\{\frac{R_n(tx,ty)L(tx,ty)}{\prod_{k=1}^{n}(t-q^{k})}\right\}_{t=0}=\sum_{(i,j)\in W}r_{ij}x^iy^j,\quad r_{ij}\in\mathbb{C}. \tag{2.21}$$

Thus (2.13) follows from (2.18) and (2.21), and (2.15) follows from (2.11) and (2.20).

(iii) From (2.9), (2.17) and (2.19),

$$I(x,y)=\frac{1}{2\pi\mathrm{i}}\int_\Gamma\frac{R_n(tx,ty)L(tx,ty)}{t^{m+n+2}\prod_{k=0}^{n}(1-1/(q^kt))}\,\mathrm{d}t$$
$$=\frac{1}{2\pi\mathrm{i}}\int_\Gamma\frac{R_n(tx,ty)L(tx,ty)}{t^{m+n+2}}\left(\sum_{j_0,\dots,j_n\geqslant0}\prod_{k=0}^{n}\left(\frac{1}{q^kt}\right)^{j_k}\right)\mathrm{d}t$$
$$=\sum_{j_0,\dots,j_n\geqslant0}q^{-\sum_{k=0}^{n}kj_k}\times\frac{1}{2\pi\mathrm{i}}\int_\Gamma\left\{\frac{1}{t^{m+n+2+(j_0+\cdots+j_n)}}\right.$$
$$\left.\times\sum_{i,j=0}^{\infty}\sum_{k,l=0}^{n}(-1)^{k+l}\begin{bmatrix}n\\k\end{bmatrix}\begin{bmatrix}n\\l\end{bmatrix}q^{k(k-1)/2+l(l-1)/2}\frac{(q^a-1)x^{i+k}y^{j+l}t^{i+j+k+l}}{(q^{i+j+a}-1)}\right\}\mathrm{d}t$$

$$= \sum_{j_0,\dots,j_n \geqslant 0} q^{-\sum_{k=0}^n kj_k} \sum_{\substack{i+j+l+k-(m+n+j_0+\cdots+j_n+2)=-1 \\ 0\leqslant i,j<\infty,\ 0\leqslant l,k\leqslant n}} (-1)^{k+l} \begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} n \\ l \end{bmatrix}$$

$$\times q^{k(k-1)/2+l(l-1)/2} \frac{(q^a-1)x^{i+k}y^{j+l}}{(q^{i+j+a}-1)}$$

$$= \sum_{\substack{i+j+l+k=m+n+j_0+\cdots+j_n+1 \\ 0\leqslant i,j<\infty,\ 0\leqslant l,k\leqslant n \\ 0\leqslant j_0,\dots,j_n}} q^{-\sum_{k=0}^n kj_k} (-1)^{k+l} \begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} n \\ l \end{bmatrix}$$

$$\times q^{k(k-1)/2+l(l-1)/2} \frac{(q^a-1)x^{i+k}y^{j+l}}{(q^{i+j+a}-1)}.$$

So (2.16) holds. Now from (2.10) and (1.15), for $|q| > 1$,

$$Q(0,0) = \frac{(-1)^n q^{n(n+1)/2}}{(1-q)^n[n]!} \sum_{k=0}^n (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{k(k+1)/2+k(m+a)}$$

$$= \frac{(-1)^n q^{n(n+1)/2}}{(1-q)^n[n]!} \prod_{k=1}^n \bigl(1-q^{k+m+a}\bigr) \neq 0.$$

(iv) $M \subseteq E$ is obvious, and

$$\#W = \#\bigl\{(i,j):\ 0 \leqslant i+j \leqslant m,\ i,j \geqslant 0\bigr\} = \frac{(m+1)(m+2)}{2}.$$

For $n < m < 2n$, i.e. $n+1 \leqslant m \leqslant 2n-1$,

$$m-n \geqslant 1, \qquad 2n-m \geqslant 1,$$

and

$$\begin{aligned} \#M &= \#N + 2 \times \frac{(m-n)(m-n+1)}{2} \\ &= (n+1)^2 + (m-n)(m-n+1) = m^2+2n^2-2mn+m+n+1, \\ \#E &= \frac{(m+n+1)(m+n+2)}{2}, \end{aligned}$$

so

$$\begin{aligned} \#(E\backslash M) &= \frac{1}{2}(m+n+1)(m+n+2) - \bigl(m^2+2n^2-2mn+m+n+1\bigr) \\ &= 3mn - \frac{1}{2}m^2 - \frac{3}{2}n^2 + \frac{1}{2}m + \frac{1}{2}n \\ &= mn - \frac{1}{2}m(m-n) + \frac{3}{2}n(m-n) + \frac{1}{2}(m+n) \end{aligned}$$

$$
\begin{aligned}
&= mn + \frac{1}{2}(m-n)(3n-m) + \frac{1}{2}(m+n) \\
&\geqslant mn + \frac{1}{2}(n+1) + \frac{1}{2}(m+n) \\
&\geqslant (n+1)n + \frac{1}{2}(n+1) + \frac{1}{2}(2n+1) \\
&\geqslant n^2 + 2n = \#N - 1.
\end{aligned}
$$

For $m \geqslant 2n$, we have $N \subseteq W$, and hence $M = W$ and

$$E\backslash M = \{(i,j)\colon\ m+1 \leqslant i+j \leqslant m+n,\ i,j \geqslant 0\}.$$

Then

$$
\begin{aligned}
\#(E\backslash M) &= \frac{(m+n+1)(m+n+2)}{2} - \frac{(m+1)(m+2)}{2} \\
&= \frac{n(2m+n+3)}{2} \\
&\geqslant \frac{n(5n+3)}{2} \quad (\text{as } m \geqslant 2n) \\
&\geqslant n^2 + 2n = \#N - 1.
\end{aligned}
$$

Then for all $m \geqslant n+1$,

$$\#(E\backslash M) \geqslant \#N - 1.$$

Combining (i)–(iv), we have

$$[M/N]_E(x,y) = \frac{P(x,y)}{Q(x,y)}.$$

This completes the proof of theorem 2.1. □

Theorem 2.2. Let M, N, E and $L(x,y)$, $P(x,y)$, $Q(x,y)$ be defined in theorem 2.1 and let $m \geqslant n+1$. Then the coefficient matrix of the homogeneous linear system (1.12)

$$\sum_{\mu=0}^{i}\sum_{\nu=0}^{j} c_{\mu\nu} b_{i-\mu,j-\nu} = 0, \quad (i,j) \in E\backslash M,$$

has rank $\#N - 1$, where $b_{kl} = 0$ for $(k,l) \notin N$.

Proof. From part (iv) of theorem 2.1, $\#(E\backslash M) \geqslant \#N - 1$, so the number of variables in the homogeneous linear system is less than or equal to the number of equations. Since we have obtained a nontrivial solution $Q(x,y)$ in theorem 2.1, the rank r of the coefficient matrix of the homogeneous linear system (1.12) is at most $\#N - 1$, i.e.

$$r \leqslant \#N - 1. \tag{2.22}$$

To prove that also $r \geqslant \#N - 1$, we consider the following points in the set $E \backslash M$,

$$\begin{array}{llll} & (0, m+1) & \ldots & (0, m+n-1) \quad (0, m+n) \\ (1, m) & (1, m+1) & \ldots & (1, m+n-1) \\ (2, m) & (2, m+1) & \ldots & \\ \vdots & \vdots & & \\ (n, m) & & & \end{array} \tag{2.23}$$

and

$$\begin{array}{llllll} (m+1, 0) & (m+2, 0) & (m+3, 0) & \cdots & (m+n-1, 0) & (m+n, 0) \\ (m+1, 1) & (m+2, 1) & (m+3, 1) & \cdots & (m+n-1, 1) & \\ (m+1, 2) & (m+2, 2) & (m+3, 2) & \cdots & & \\ \vdots & \vdots & \vdots & & & \\ (m+1, n-2) & (m+2, n-2) & & & & \\ (m+1, n-1) & & & & & \end{array} \tag{2.24}$$

These $n + 2n(n+1)/2 = n(n+2) = (n+1)^2 - 1 = \#N - 1$ points of $E \backslash M$ represent $(\#N - 1)$ homogeneous linear equations of the linear system (1.12). The first $n + n(n+1)/2$ equations corresponding to the index points given in (2.23) are

$$\left.\begin{aligned} c_{0,m+1}b_{0,0} + c_{0,m}b_{0,1} + \cdots + c_{0,m-n+1}b_{0,n} &= 0, \\ c_{0,m+2}b_{0,0} + c_{0,m+1}b_{0,1} + \cdots + c_{0,m-n+2}b_{0,n} &= 0, \\ &\vdots \\ c_{0,m+n}b_{0,0} + c_{0,m+n-1}b_{0,1} + \cdots + c_{0,m}b_{0,n} &= 0; \end{aligned}\right\} \quad n \text{ equations (1st row in (2.23))}$$

$$\left.\begin{aligned} c_{1,m}b_{0,0} + c_{1,m-1}b_{0,1} + \cdots + c_{1,m-n}b_{0,n} & \\ + c_{0,m}b_{1,0} + c_{0,m-1}b_{1,1} + \cdots + c_{0,m-n}b_{1,n} &= 0, \\ c_{1,m+1}b_{0,0} + c_{1,m}b_{0,1} + \cdots + c_{1,m-n+1}b_{0,n} & \\ + c_{0,m+1}b_{1,0} + c_{0,m}b_{1,1} + \cdots + c_{0,m-n+1}b_{1,n} &= 0, \\ &\vdots \\ c_{1,m+n-1}b_{0,0} + c_{1,m+n-2}b_{0,1} + \cdots + c_{1,m-1}b_{0,n} & \\ + c_{0,m+n-1}b_{1,0} + c_{0,m+n-2}b_{1,1} + \cdots + c_{0,m-1}b_{1,n} &= 0; \end{aligned}\right\} \quad \begin{array}{l} n \text{ equations} \\ \text{(2nd row in (2.23))} \end{array}$$

$$\vdots$$

$$\left.\begin{aligned} c_{n,m}b_{0,0} + c_{n,m-1}b_{0,1} + \cdots + c_{n,m-n}b_{0,n} \\ + c_{n-1,m}b_{1,0} + c_{n-1,m-1}b_{1,1} + \cdots + c_{n-1,m-n}b_{1,n} \\ + \cdots + c_{0,m}b_{n,0} + c_{0,m-1}b_{n,1} + \cdots + c_{0,m-n}b_{n,n} = 0 \end{aligned}\right\} \quad \text{1 equation}$$

and the following $n(n+1)/2$ equations corresponding to the index points listed in (2.24) are

$$\left.\begin{aligned} c_{m+1,0}b_{0,0} + c_{m,0}b_{1,0} + \cdots + c_{m-n+1,0}b_{n,0} = 0, \\ c_{m+2,0}b_{0,0} + c_{m+1,0}b_{1,0} + \cdots + c_{m-n+2,0}b_{n,0} = 0, \\ \vdots \\ c_{m+n,0}b_{0,0} + c_{m+n-1,0}b_{1,0} + \cdots + c_{m,0}b_{n,0} = 0; \end{aligned}\right\} \quad n \text{ equations}$$

$$\left.\begin{aligned} c_{m+1,1}b_{0,0} + c_{m+1,0}b_{0,1} + c_{m,1}b_{1,0} + c_{m,0}b_{1,1} \\ + \cdots + c_{m-n+1,1}b_{n,0} + c_{m-n+1,0}b_{n,1} = 0, \\ c_{m+2,1}b_{0,0} + c_{m+2,0}b_{0,1} + c_{m+1,1}b_{1,0} + c_{m+1,0}b_{1,1} \\ + \cdots + c_{m-n+2,1}b_{n,0} + c_{m-n+2,0}b_{n,1} = 0, \\ \vdots \\ c_{m+n-1,1}b_{0,0} + c_{m+n-1,0}b_{0,1} + c_{m+n-2,1}b_{1,0} + c_{m+n-2,0}b_{1,1} \\ + \cdots + c_{m-1,1}b_{n,0} + c_{m-1,0}b_{n,1} = 0; \end{aligned}\right\} \quad (n-1) \text{ equations}$$

$$\vdots$$

$$\left.\begin{aligned} c_{m+1,n-2}b_{0,0} + c_{m+1,n-3}b_{0,1} + \cdots + c_{m+1,0}b_{0,n-2} \\ + c_{m,n-2}b_{1,0} + c_{m,n-3}b_{1,1} + \cdots + c_{m,0}b_{1,n-2} \\ + \cdots + c_{m-n+1,n-2}b_{n,0} + c_{m-n+1,n-3}b_{n,1} + \cdots + c_{m-n+1,0}b_{n,n-2} = 0, \\ c_{m+2,n-2}b_{0,0} + c_{m+2,n-3}b_{0,1} + \cdots + c_{m+2,0}b_{0,n-2} \\ + c_{m+1,n-2}b_{1,0} + c_{m+1,n-3}b_{1,1} + \cdots + c_{m+1,0}b_{1,n-2} \\ + \cdots + c_{m-n+2,n-2}b_{n,0} + c_{m-n+2,n-3}b_{n,1} + \cdots + c_{m-n+2,0}b_{n,n-2} = 0; \end{aligned}\right\} \quad \text{2 equations}$$

$$\left.\begin{aligned} c_{m+1,n-1}b_{0,0} + c_{m+1,n-2}b_{0,1} + \cdots + c_{m+1,0}b_{0,n-1} \\ + c_{m,n-1}b_{1,0} + c_{m,n-2}b_{1,1} + \cdots + c_{m,0}b_{1,n-1} \\ + \cdots + c_{m-n+1,n-1}b_{n,0} + c_{m-n+1,n-2}b_{n,1} + \cdots + c_{m-n+1,0}b_{n,n-1} = 0 \end{aligned}\right\} \quad \text{1 equation.}$$

The coefficient matrix of this subsystem of (1.12) equals

$$D := \begin{bmatrix} A \\ B \end{bmatrix},$$

where A has $n + n(n+1)/2$ rows and $(n+1)^2$ columns, B has $n(n+1)/2$ rows and $(n+1)^2$ columns, and A and B are respectively given by

$$A := \begin{bmatrix}
c_{0,m+1} & \cdots & c_{0,m-n+1} & & & & & & & \\
\vdots & \vdots & \vdots & & & & & & & \\
c_{0,m+n} & \cdots & c_{0,m} & & & & & & & \\
c_{1,m} & \cdots & c_{1,m-n} & c_{0,m} & \cdots & c_{0,m-n} & & & & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & & & \\
c_{1,m+n-1} & \cdots & c_{1,m-1} & c_{0,m+n-1} & \cdots & c_{0,m-1} & & & & \\
\vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & & & \\
c_{n,m} & \cdots & c_{n,m-n} & c_{n-1,m} & \cdots & c_{n-1,m-n} & \cdots & c_{0,m} & \cdots & c_{0,m-n}
\end{bmatrix}$$

$$B := $$

$$\begin{bmatrix}
c_{m+1,0} & & & & \cdots & c_{m-n+1,0} & & & \\
\vdots & & & & \vdots & \vdots & & & \\
c_{m+n,0} & & & & \cdots & c_{m,0} & & & \\
c_{m+1,1} & c_{m+1,0} & & & \cdots & c_{m-n+1,1} & c_{m-n+1,0} & & \\
\vdots & \vdots & & & \vdots & \vdots & \vdots & & \\
c_{m+n-1,1} & c_{m+n-1,0} & & & \cdots & c_{m-1,1} & c_{m-1,0} & & \\
\vdots & \vdots & \vdots & & & \vdots & \vdots & \vdots & \\
c_{m+1,n-1} & c_{m+1,n-2} & \cdots & c_{m+1,0} & \cdots & c_{m-n+1,n-1} & c_{m-n+1,n-2} & \cdots & c_{m-n+1,0}
\end{bmatrix}.$$

We observe that the coefficients c_{ij} of $L(x, y)$ defined in (2.1), satisfy the property that for $i + j = k + l$, then

$$c_{ij} = c_{kl}.$$

Let us now perform some elementary row operations on the matrices A and B. We start with the last row and subtract the one but last row from the last one, the 4th last from the 2nd last, the 5th last from the 3rd last, $\ldots$, then the nth from the $2n$th, $\ldots$, and finally the first from the $(n+1)$th. In this way A is transformed to

$$A \to \begin{bmatrix} c_{0,m+1} & \cdots & c_{0,m-n+1} & & & & & & & \\ \vdots & \vdots & \vdots & & & & & & & \\ c_{0,m+n} & \cdots & c_{0,m} & & & & & & & \\ & & & c_{0,m} & \cdots & c_{0,m-n} & & & & \\ & & & \vdots & \vdots & \vdots & & & & \\ & & & c_{0,m+n-1} & \cdots & c_{0,m-1} & & & & \\ & & & & & & \ddots & & & \\ & & & & & & & c_{0,m} & \cdots & c_{0,m-n} \end{bmatrix}.$$

We now perform similar row operations on the matrix B, starting with the last row, subtracting the 2nd last from the last one, the 4th last from the 2nd last, …, then the nth from the $(2n-1)$th, the $(n-1)$th from the $(2n-2)$th,…, and finally the 2nd from the $(n+1)$th. Then B is transformed into

$$B \to \begin{bmatrix} c_{m+1,0} & & & & \cdots & c_{m-n+1,0} & & & \\ \vdots & & & & \vdots & \vdots & & & \\ c_{m+n,0} & & & & \cdots & c_{m,0} & & & \\ & c_{m+1,0} & & & \cdots & & c_{m-n+1,0} & & \\ & \vdots & & & \vdots & & \vdots & & \\ & c_{m+n-1,0} & & & \cdots & & c_{m-1,0} & & \\ & & \ddots & & & & & \ddots & \\ & & & c_{m+1,0} & \cdots & & & & c_{m-n+1,0} \end{bmatrix}.$$

Hence

$$D \to [D_1 \quad D_2 \quad \ldots \quad D_{n+1}],$$

where each D_j for $j = 1, 2, \ldots, n+1$, has $n(n+2)$ rows and $(n+1)$ columns, and the matrices D_j are given by

$$D_1 := \begin{bmatrix} c_{0,m+1} & \cdots & c_{0,m-n+2} & c_{0,m-n+1} \\ \vdots & \vdots & \vdots & \vdots \\ c_{0,m+n} & \cdots & c_{0,m+1} & c_{0,m} \\ 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \\ c_{m+1,0} & & & \\ \vdots & & & \\ c_{m+n,0} & & & \\ & \ddots & & \\ & & c_{m+1,0} & 0 \end{bmatrix},$$

$$D_2 := \begin{bmatrix} 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \\ c_{0,m} & \cdots & c_{0,m-n+1} & c_{0,m-n} \\ \vdots & \vdots & \vdots & \vdots \\ c_{0,m+n-1} & \cdots & c_{0,m} & c_{0,m-1} \\ 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \\ c_{m,0} & & & \\ \cdots & & & \\ c_{m+n-1,0} & & & \\ & \ddots & & \\ & & c_{m,0} & 0 \end{bmatrix},$$

$$\vdots$$

$$D_{n+1} := \begin{bmatrix} 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & 0 \\ c_{0,m} & \dots & c_{0,m-n+1} & c_{0,m-n} \\ c_{m-n+1,0} & & & \\ \vdots & & & \\ c_{m,0} & & & \\ & \ddots & & \\ & & c_{m-n+1,0} & 0 \end{bmatrix}.$$

By exchanging columns and rows in D, namely moving the $(k(n+1)+1)$th column to the $(n+k+1)$th column for $k = 2, 3, \dots, n$, and moving the $(n(n+1)/2+n+k)$th row up to the $(n+k)$th row for $k = 1, \dots, n$, and by pulling the nonzero entries between the rows together and moving them up, we have transformed D into

$$D = \begin{bmatrix} A_{n+1} & & & & & & & \\ & B_n & & & & & & \\ & & A_n & & & O & & \\ & & & & \ddots & & & \\ & & & D^* & & B_2 & & \\ & & & & & & A_2 & & \\ & & & & & & & B_1 & \\ & & & & & & & & A_1 \end{bmatrix},$$

where

$$A_{n+1} := \begin{bmatrix} c_{0,m+1} & c_{0,m} & \dots & c_{0,m-n+1} \\ \vdots & \vdots & \vdots & \vdots \\ c_{0,m+n} & c_{0,m+n-1} & \dots & c_{0,m} \end{bmatrix}_{n\times(n+1)},$$

$$A_n := \begin{bmatrix} c_{0,m-1} & \dots & c_{0,m-n} \\ \vdots & \vdots & \vdots \\ c_{0,m+n} & \dots & c_{0,m-1} \end{bmatrix}_{n\times n},$$

$$\vdots$$

$$A_1 := [c_{0,m-n}]_{1\times 1}$$

and

$$B_n := \begin{bmatrix} c_{m,0} & \cdots & c_{m-n+1,0} \\ \vdots & \vdots & \vdots \\ c_{m+n-1,0} & \cdots & c_{m,0} \end{bmatrix}_{n\times n},$$

$$B_{n-1} := \begin{bmatrix} c_{m-1,0} & \cdots & c_{m-n+1,0} \\ \vdots & \vdots & \vdots \\ c_{m+n-3,0} & \cdots & c_{m-1,0} \end{bmatrix}_{(n-1)\times(n-1)},$$

$$\vdots$$

$$B_2 := \begin{bmatrix} c_{m-n+2,0} & c_{m-n+1,0} \\ c_{m-n+3,0} & c_{m-n+2,0} \end{bmatrix}_{2\times 2},$$

$$B_1 := [c_{m-n+1,0}]_{1\times 1},$$

and O contains only zero entries. Observe that all the square matrices A_j and B_j for $j = 1, \dots, n$, are encountered in the computation of Padé approximants to the univariate function ${}_2F_1(a, 1; a+1; z)$. Since the Padé table for ${}_2F_1(a, 1; a+1; z)$ is normal, these matrices are all regular (see [8,13] for details) and then the rank of A_j and the rank of B_j are both j. Now write

$$A_{n+1} = [CA^*],$$

where

$$C := \begin{bmatrix} c_{0,m+1} \\ \vdots \\ c_{0,m+n} \end{bmatrix}_{n\times 1}$$

and

$$A^* := \begin{bmatrix} c_{0,m} & \cdots & c_{0,m-n+1} \\ \vdots & \vdots & \vdots \\ c_{0,m+n-1} & \cdots & c_{0,m} \end{bmatrix}_{n\times n}.$$

Since the rank of A^* is n, so is the rank of A_{n+1}. Therefore the rank of D is the sum of the ranks of A_j where $j = 1, \dots, n+1$, and B_j where $j = 1, \dots, n$, i.e. the rank of D equals $n + 2n(n+1)/2 = n(n+2) = \#N - 1$. Since D is the coefficient matrix of a subsystem of the linear system (1.12), we find that the rank of the coefficient matrix of (1.12)

$$r \geqslant \#N - 1.$$

Combined with (2.22), we have

$$r = \#N - 1. \tag{2.25}$$

□

Theorem 2.3. Let M, N, E and $L(x, y)$ be defined in theorem 2.1 and let $m \geqslant n + 1$ and $m + a \geqslant 2n$. Then the (M, N) general multivariate Padé approximant to $L(x, y)$ on the set E

$$[M/N]_E(x, y) = \frac{P(x, y)}{Q(x, y)}$$

is irreducible.

Proof. From part (iii) of theorem 2.1, $Q(0, 0) \neq 0$. This implies that a common factor of $P(x, y)$ and $Q(x, y)$ needs to have a nonzero constant term. Suppose that $t(x, y)$ is a true common factor, not only a constant, then $t(x, y)$ has to contain a nonzero constant term. From (2.14) and $m + a - 2n \geqslant 0$ we know

$$b_{nn} = \frac{(-1)^n q^{n(3n-1)/2}}{(1-q)^n [n]!} \prod_{k=1}^{n} \left(1 - q^{k+m+a-2n}\right) \neq 0.$$

Hence $p(x, y)$ in $P(x, y) = p(x, y)t(x, y)$ and $q(x, y)$ in $Q(x, y) = q(x, y)t(x, y)$ must be indexed by some index sets strictly smaller than and contained in M and N, respectively. As $t(0, 0) \neq 0$, then $1/t(x, y)$ can be expanded around the origin and then

$$(Fq - p)(x, y) = \frac{1}{t(x, y)}(FQ - P)(x, y) = \sum_{(i,j)\in\mathbb{N}^2\setminus E} e_{ij} x^i y^j, \quad e_{ij} \in \mathbb{C}.$$

This implies that $p(x, y)/q(x, y)$ is another solution to the (M, N) general multivariate Padé approximant to $L(x, y)$ on the set E. It is impossible because of theorem 2.2. Then $t(x, y)$ must be a constant. This completes the proof of theorem 2.3. □

3. Padé approximants to the Appell function $F_1(a, 1, 1; a + 1; x, y)$

Now we can obtain the Padé approximant $[M/N]_E = p(x, y)/q(x, y)$ to the Appell function $F_1(a, 1, 1; a + 1; x, y)$ by taking the limits

$$\lim_{q\to 1} L(x, y) = F_1(a, 1, 1; a + 1; x, y),$$
$$\lim_{q\to 1} Q(x, y) = q(x, y),$$
$$\lim_{q\to 1} P(x, y) = p(x, y).$$

This is guaranteed by [5, theorem 3]. It states that the general multivariate Padé operator, which maps a power series to its general multivariate Padé approximant, is continuous, if two conditions are satisfied. First of all the system (1.12) must have maximal rank. The second condition, in the particular case of the sets M, N and E defined by (2.6), (2.7) and (2.8), translates to $b_{nn} \neq 0$ with b_{nn} defined by (2.12). The former condition was proved in our theorem 2.2. The latter is satisfied when $m + a \geqslant 2n$ as in theorem 2.3:

$$b_{nn} = \frac{(-1)^n q^{n(3n-1)/2}}{(1-q)^n [n]!} \prod_{k=1}^{n} \bigl(1 - q^{k+m+a-2n}\bigr) \neq 0.$$

Moreover,

$$\lim_{q \to 1} b_{nn} = (-1)^n \binom{a+m-n}{n} \neq 0.$$

In this section, we derive an explicit formula for the general multivariate Padé approximants $[M/N]_E$ to the Appell function $F_1(a, 1, 1; a+1; x, y)$ in theorem 3.1, and prove the normality of the so-called contracted table of multivariate Padé approximants for the Appell function $F_1(a, 1, 1; a+1; x, y)$ in theorem 3.2.

Theorem 3.1. Let m and n be integers such that $m \geqslant n+1$, $m+a \geqslant 2n$ and let N, M and E be defined by (2.6), (2.7) and (2.8), respectively. Then the general multivariate Padé approximants $[M/N]_E$ to the Appell function $F_1(a, 1, 1; a+1; x, y)$, where $a \geqslant 1$ is an integer, are given by

$$[M/N]_E = \frac{p(x, y)}{q(x, y)},$$

where

$$q(x, y) = (-1)^n \binom{m+n+a}{n} F_1(-m-a, -n, -n; -m-n-a; x, y) \tag{3.1}$$

and

$$\begin{aligned} p(x, y) = (-1)^n & \sum_{\substack{0 \leqslant i+h_1 \leqslant n, \\ 0 \leqslant j+h_2 \leqslant n, \\ 0 \leqslant i, j, h_1, h_2 \leqslant n}} \left\{ \frac{(-1)^{i+j} a}{h_1+h_2+a} \binom{n}{i} \binom{n}{j} x^{i+h_1} y^{j+h_2} \right. \\ & \times \left. \left(\binom{m+n+a-i-j}{n} - \binom{m+n-i-j-h_1-h_2}{n} \right) \right\} \\ & + (-1)^n \sum_{\substack{0 \leqslant i+j+l+k \leqslant m \\ 0 \leqslant i, j, h \leqslant m, 0 \leqslant k, l \leqslant n}} \frac{(-1)^{k+l} a x^{i+k} y^{j+l}}{i+j+a} \binom{n}{k} \binom{n}{l} \binom{m+n-i-j-k-l}{n}. \end{aligned} \tag{3.2}$$

Proof. From the discussion above and (2.14), we have

$$\begin{aligned}
q(x, y) &= \lim_{q\to 1} Q(x, y) \\
&= (-1)^n \lim_{q\to 1} \frac{q^{n(n+1)/2}}{(1-q)^n[n]!} \sum_{i,j=0}^{n} \left\{ (-1)^{i+j} \begin{bmatrix} n \\ i \end{bmatrix} \begin{bmatrix} n \\ j \end{bmatrix} q^{i(i-1)/2+j(j-1)/2} x^i y^j \right. \\
&\quad \left. \times \prod_{k=1}^{n} \left(1 - q^{k+m+a-i-j}\right) \right\} \\
&= (-1)^n \lim_{q\to 1} \sum_{i,j=0}^{n} (-1)^{i+j} \begin{bmatrix} n \\ i \end{bmatrix} \begin{bmatrix} n \\ j \end{bmatrix} q^{i(i-1)/2+j(j-1)/2} x^i y^j \frac{\prod_{k=1}^{n}(1-q^{k+m+a-i-j})}{\prod_{k=1}^{n}(1-q^k)} \\
&= (-1)^n \lim_{q\to 1} \sum_{i,j=0}^{n} (-1)^{i+j} \begin{bmatrix} n \\ i \end{bmatrix} \begin{bmatrix} n \\ j \end{bmatrix} q^{i(i-1)/2+j(j-1)/2} x^i y^j \begin{bmatrix} m+n+a-i-j \\ n \end{bmatrix} \\
&= (-1)^n \sum_{i,j=0}^{n} (-1)^{i+j} \binom{m+n+a-i-j}{n} \binom{n}{i} \binom{n}{j} x^i y^j \\
&= (-1)^n \sum_{i,j=0}^{n} (-1)^{i+j} \frac{(m+n+a-i-j)!}{n!(m+a-i-j)!} \cdot \frac{n!}{i!(n-i)!} \cdot \frac{n!}{j!(n-j)!} x^i y^j \\
&= (-1)^n \frac{1}{n!} \sum_{i,j=0}^{n} \frac{(-n)_i(-n)_j}{i!j!} \frac{(m+a)!(m+n+a)!/(m+a-i-j)!}{(m+a)!(m+n+a)!/(m+n+a-i-j)!} x^i y^j \\
&= (-1)^n \frac{(m+n+a)!}{n!(m+a)!} \sum_{i,j=0}^{n} \frac{(-n)_i(-n)_j}{i!j!} \\
&\quad \times \frac{(-1)^{i+j}(m+a)!/(m+a-i-j)!x^i y^j}{(-1)^{i+j}(m+n+a)!/(m+n+a-i-j)!} \\
&= (-1)^n \binom{m+n+a}{n} \sum_{i,j=0}^{n} \frac{(-n)_i(-n)_j}{i!j!} \cdot \frac{(-m-a)_{i+j}}{(-m-n-a)_{i+j}} x^i y^j \\
&= (-1)^n \binom{m+n+a}{n} \sum_{i,j=0}^{\infty} \frac{(-m-a)_{i+j}(-n)_i(-n)_j x^i y^j}{(-m-n-a)_{i+j} i! j!} \\
&= (-1)^n \binom{m+n+a}{n} F_1(-m-a, -n, -n; -m-n-a; x, y).
\end{aligned}$$

Here we used the fact that

$$\lim_{q\to 1} \begin{bmatrix} n \\ k \end{bmatrix}_q = \binom{n}{k}.$$

So (3.1) holds. We calculate the limits of the two parts of $P(x, y)$ in (2.15) separately to prove (3.2). First,

$$\lim_{q\to 1}\frac{(-1)^n q^{n(n+1)/2}}{(1-q)^n[n]!}\sum_{k=0}^{n}(-1)^k\begin{bmatrix}n\\k\end{bmatrix}q^{k(k+1)/2+km}R_n\big(q^{-k}x, q^{-k}y\big)S_k(x, y)$$

$$= (-1)^n\lim_{q\to 1}\big(q^a-1\big)\sum_{\substack{0\leqslant i+h_1\leqslant n,\\ 0\leqslant j+h_2\leqslant n,\\ 0\leqslant i,j,h_1,h_2\leqslant n}}\sum_{k=0}^{n}\bigg\{(-1)^{i+j+k}\frac{q^{n(n+1)/2}}{(1-q)^n[n]!}\begin{bmatrix}n\\k\end{bmatrix}\begin{bmatrix}n\\i\end{bmatrix}\begin{bmatrix}n\\j\end{bmatrix}$$

$$\times q^{k(k+1)/2+k(m-i-j)}\big(q^{ka}-q^{-k(h_1+h_2)}\big)q^{i(i-1)/2+j(j-1)/2}\frac{x^{i+h_1}y^{j+h_2}}{q^{h_1+h_2+a}-1}\bigg\}$$

$$= (-1)^n\lim_{q\to 1}\big(q^a-1\big)\sum_{\substack{0\leqslant i+h_1\leqslant n,\\ 0\leqslant j+h_2\leqslant n,\\ 0\leqslant i,j,h_1,h_2\leqslant n}}\bigg\{(-1)^{i+j}\begin{bmatrix}n\\i\end{bmatrix}\begin{bmatrix}n\\j\end{bmatrix}q^{i(i-1)/2+j(j-1)/2}\frac{x^{i+h_1}y^{j+h_2}}{q^{h_1+h_2+a}-1}$$

$$\times\left(\begin{bmatrix}m+n+a-i-j\\n\end{bmatrix}-\begin{bmatrix}m+n-i-j-h_1-h_2\\n\end{bmatrix}\right)\bigg\}$$

$$= (-1)^n\sum_{\substack{0\leqslant i+h_1\leqslant n,\\ 0\leqslant j+h_2\leqslant n,\\ 0\leqslant i,j,h_1,h_2\leqslant n}}\bigg\{\frac{(-1)^{i+j}a}{h_1+h_2+a}\binom{n}{i}\binom{n}{j}x^{i+h_1}y^{j+h_2}$$

$$\times\left(\binom{m+n+a-i-j}{n}-\binom{m+n-i-j-h_1-h_2}{n}\right)\bigg\},$$

and

$$\lim_{q\to 1}(-1)^n\big(q^a-1\big)q^{n(n+1)/2}\sum_{\substack{i+j+h+l+k=m\\ 0\leqslant i,j,h\leqslant m,\,0\leqslant k,l\leqslant n}}\frac{(-1)^{k+l}q^{k(k-1)/2+l(l-1)/2}}{(q^{i+j+a}-1)}$$

$$\times\begin{bmatrix}n\\k\end{bmatrix}\begin{bmatrix}n\\l\end{bmatrix}\begin{bmatrix}n+h\\h\end{bmatrix}x^{i+k}y^{j+l}$$

$$= (-1)^n\sum_{\substack{i+j+h+l+k=m\\ 0\leqslant i,j,h\leqslant m,\,0\leqslant k,l\leqslant n}}\frac{(-1)^{k+l}a}{i+j+a}\binom{n}{k}\binom{n}{l}\binom{n+h}{n}x^{i+k}y^{j+l}$$

$$= (-1)^n\sum_{\substack{0\leqslant i+j+l+k\leqslant m\\ 0\leqslant i,j,h\leqslant m,\ 0\leqslant k,l\leqslant n}}\frac{(-1)^{k+l}a}{i+j+a}\binom{n}{k}\binom{n}{l}\binom{m+n-i-j-k-l}{n}x^{i+k}y^{j+l}.$$

So (3.2) holds and this completes the proof of theorem 3.1. □

Now let us consider the table of Padé approximants $[M/N]_E$ for the Appell function $F_1(a, 1, 1; a+1; x, y)$ for increasing $m \geqslant 0$ and $n \geqslant 0$. Then we have to define the sets M, N and E for all m and n, also when $m < n + 1$ and we cannot get an explicit formula for $p(x, y)$ and $q(x, y)$. Let $m, n \in \mathbb{N}$, and

$$W := \{(i, j):\ 0 \leqslant i, j,\ 0 \leqslant i + j \leqslant m\}, \tag{3.3}$$

$$N := \{(i, j):\ 0 \leqslant i, j \leqslant n\}, \tag{3.4}$$

$$M := (N \cup W) \setminus \{(i, 0), (0, i):\ m + 1 \leqslant i \leqslant n\}, \tag{3.5}$$

$$E :\supset \{(i, j):\ 0 \leqslant i + j \leqslant m + n,\ i, j \geqslant 0\}. \tag{3.6}$$

Since the index set M is mainly determined by m and N solely depends on n, we can also denote

$$[m/n]_{m+n} := [M/N]_E$$

Then the Padé table looks like

$$\begin{array}{ccccc} [0/0]_0 & [0/1]_1 & [0/2]_2 & [0/3]_3 & \dots \\ [1/0]_1 & [1/1]_2 & [1/2]_3 & [1/3]_4 & \dots \\ [2/0]_2 & [2/1]_3 & [2/2]_4 & [2/3]_5 & \dots \\ [3/0]_3 & [3/1]_4 & [3/2]_5 & [3/3]_6 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array}.$$

We know that the univariate Padé table for the Gauss function ${}_2F_1(a, 1; a + 1; z)$ is normal, which means that for each m and n the Padé approximant of degree m in the numerator and n in the denominator occurs only once in the table. It was shown in [6] that the table of general multivariate Padé approximants for the Appell function $F_1(a, 1, 1; a + 1; x, y)$ is highly non-normal if one considers less specific index sets M, N and E than the ones used in this paper. Compared to the table discussed in [6], the above table of functions $[m/n]_{m+n}(x, y)$ should actually be called a *contracted multivariate Padé table*.

Theorem 3.2. The contracted table of multivariate Padé approximants for the Appell function $F_1(a, 1, 1; a + 1; x, y)$ is normal.

Proof. The proof heavily relies on the univariate results obtained for $F_1(a, 1, 1; a + 1, x, 0) =_2 F_1(a, 1; a + 1; x)$. From the definitions for M, N and E it is easy to see that for each m and n, the projected function $[m/n]_{m+n}(x, 0)$ equals the univariate Padé approximant for ${}_2F_1(a, 1; a+1; x)$ of degree n in the numerator and m in the denominator. This is by the construction of the sets M, N and E and not because of the explicit form for $q(x, y)$ which was only obtained under the conditions $m \geqslant n + 1$ and $m + a \geqslant 2n$. Then the proof goes by contradiction. Suppose that for some specific integers m_1, m_2 and n_1, n_2 it holds that

$$[m_1/n_1]_{m_1+n_1}(x, y) = [m_2/n_2]_{m_2+n_2}(x, y)$$

with $m_1 \neq m_2$ or $n_1 \neq n_2$. Then

$$[m_1/n_1]_{m_1+n_1}(x, 0) = [m_2/n_2]_{m_2+n_2}(x, 0)$$

which contradicts the normality of the Padé table for ${}_2F_1(a, 1; a+1; x)$. □

From theorem 3.2 we can also conclude that in the explicit formula (3.1) for $p(x, y)$ the coefficients a_{m0} and a_{0m} are nonzero. These coefficients are the highest degree coefficients in the numerators of degree m of the Padé approximants to the Gauss function ${}_2F_1(a, 1; a+1; z)$. This nicely complements the result that the coefficients b_{00} and b_{nn} in $q(x, y)$, as given in (3.2), are nonzero, as we already pointed out at the beginning of this section.

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