Aequ. math. 54 (1997) 74–86
0001-9054/97/010074-13 $ 1.50 + 0.20/0

Aequationes Mathematicae

On Hyers–Ulam stability for a class of functional equations

COSTANZA BORELLI

Summary. In this paper we prove some stability theorems for functional equations of the form $g[F(x, y)] = H[g(x), g(y), x, y]$. As special cases we obtain well known results for Cauchy and Jensen equations and for functional equations in a single variable.

1. Introduction

In the last years many papers appeared dealing with the stability of functional equations in the sense of Hyers–Ulam. Most of them concern special functional equations, e.g., Cauchy equation, quadratic equation, d'Alembert equation, Fréchet equation, Hosszú equation, etc. Few papers present results unifying many of the particular cases (see [3], [4]).

The aim of the present note is to give stability theorems for a class of functional equations of the form

$$f[F(x, y)] = H[g(x), g(y), x, y] \tag{1}$$

under suitable hypotheses on the given functions F and H. What is new in respect of other papers is the explicit dependence of the function H on the variables x and y.

As a first step we prove a stability result for a class of functional equations in a single variable; this is not only an intermediate step to get a theorem about equation (1), but it seems interesting in itself and generalizes a result due to J. A. Baker ([2]).

AMS (1991) subject classification. 39B52, 39B72, 47H15.

Work supported by M.U.R.S.T. Research funds (60%).

Manuscript received March 21, 1995 and, in final form, February 1, 1996.

2. A stability result for a functional equation in a single variable

We consider the functional equation

$$g(x) = S[g(B(x)), x], \tag{2}$$

where x is an element of a set X, B is a given function from X to X, $S\colon Y \times X \to Y$, where (Y, d) is a complete metric space and the unknown function g maps X into Y. For every non-negative integer n, B^n denotes the n-th iterate of the function B ($B^0(x) = x$).

Given functions $f\colon X \to Y$, $C\colon X \to X$ and $A\colon Y \times X \to Y$, we define a sequence of functions $\{A_n[f, C]\}$ from X into Y as follows:

$$\begin{aligned} &A_1[f, C](x) = A[f(C(x)), x] \\ &A_n[f, C](x) = A[A_{n-1}[f, C](C(x)), x], \qquad n \in \mathbb{N}. \end{aligned} \tag{3}$$

THEOREM 1. *Assume that S satisfies the Lipschitz condition*

$$d(S(u, x), S(v, x)) \le \lambda d(u, v), \qquad x \in X,\ u, v \in Y. \tag{4}$$

Let $f\colon X \to Y$ be a given function and let $\sigma\colon X \to \mathbb{R}^+$ fulfil the following conditions:

$$d(f(x), S[f(B(x)), x]) \le \sigma(x), \qquad x \in X \tag{5}$$

$$\text{the series } \sum_{i=0}^{\infty} \lambda^i \sigma(B^i(x)) \text{ converges for every } x \in X. \tag{6}$$

Then the sequence $\{S_n[f, B](x)\}$ defined as in (3) *converges for every $x \in X$, its limit function g is a solution of equation* (2) *and*

$$d(f(x), g(x)) \le \sum_{i=0}^{\infty} \lambda^i \sigma(B^i(x)).$$

Moreover, g is the unique solution of (2) *satisfying the above inequality.*

Proof. We have

$$\begin{aligned} d(S_2[f, B](x), S_1[f, B](x)) &= d(S[S_1[f, B](B(x)), x], S[f(B(x)), x]) \\ &\le \lambda d(S_1[f, B](B(x)), f(B(x))) \\ &= \lambda d(S[f(B^2(x)), B(x)], f(B(x))) \\ &\le \lambda \sigma(B(x)). \end{aligned}$$

By induction it is easy to prove the inequality

$$d(S_{n+1}[f, B](x), S_n[f, B](x)) \leq \lambda^n \sigma(B^n(x)). \tag{7}$$

Define now

$$q_n(x) = S_n[f, B](x), \qquad x \in X;$$

we prove that $\{q_n(x)\}$ is a Cauchy sequence for every $x \in X$. Let $n > m$, by (7) we have

$$\begin{aligned} d(q_n(x), q_m(x)) &\leq d(q_n(x), q_{n-1}(x)) + \cdots + d(q_{m+1}(x), q_m(x)) \\ &\leq \lambda^{n-1}\sigma(B^{n-1}(x)) + \cdots + \lambda^m \sigma(B^m(x)) \\ &= \sum_{i=m}^{n-1} \lambda^i \sigma(B^i(x)), \end{aligned}$$

and the last term goes to 0 as $m \to \infty$ by the hypotheses (6). The completeness of Y assures the existence of the limit function g.

Since S is continuous in the first variable, we obtain

$$d(S[S_n[f, B](B(x)), x], S_n[f, B](x)) \to d(S[g(B(x)), x], g(x)) \qquad \text{as } n \to \infty;$$

on the other hand, by (7) we get

$$\begin{aligned} d(S[S_n[f, B](B(x)), x], S_n[f, B](x)) &= d(S_{n+1}[f, B](x), S_n[f, B](x)) \\ &\leq \lambda^n \sigma(B^n(x)) \to 0 \end{aligned}$$

as $n \to \infty$; thus g is a solution of equation (2).

By taking the limit in the following inequality

$$\begin{aligned} d(S_n[f, B](x), f(x)) &\leq d(S_n[f, B](x), S_{n-1}[f, B](x)) + \cdots + d(S_1[f, B](x), f(x)) \\ &\leq \lambda^{n-1}\sigma(B^{n-1}(x)) + \cdots + \sigma(x) \\ &= \sum_{i=0}^{n-1} \lambda^i \sigma(B^i(x)), \end{aligned}$$

we have

$$d(f(x), g(x)) \leq \sum_{i=0}^{\infty} \lambda^i \sigma(B^i(x)).$$

Now we prove the uniqueness. Assume h is a solution of (2) such that

$$d(f(x), h(x)) \leq \sum_{i=0}^{\infty} \lambda^i \sigma(B^i(x)).$$

By equation (2) we have

$$h(x) = S[h(B(x)), x] = S[S[h(B^2(x)), B(x)], x] = S_2[h, B](x)$$
$$= \cdots = S_n[h, B](x) = \cdots.$$

By induction we obtain the following inequality:

$$d(h(x), S_n[f, B](x)) = d(S_n[h, B](x), S_n[f, B](x)) \leq \lambda^n d(h(B^n(x)), f(B^n(x)))$$

and so by the hypotheses we get

$$d(h(x), S_n[f, B](x)) \leq \lambda^n d(h(B^n(x)), f(B^n(x))) \leq \lambda^n \sum_{i=0}^{\infty} \lambda^i \sigma(B^{i+n}(x))$$
$$= \sum_{i=0}^{\infty} \lambda^{i+n} \sigma(B^{i+n}(x)) \to 0 \qquad \text{as } n \to \infty.$$

This implies $h = g$. □

As a corollary we obtain Baker's result.

COROLLARY 2. *In the hypothesis of Theorem* 1 *assume that the function* σ *is bounded by a constant* M. *If the Lipschitz constant* λ *is less than* 1 *then there exists a unique solution* g *of equation* (2) *such that*

$$d(f(x), g(x)) \leq \frac{M}{1-\lambda}, \qquad x \in X.$$

Now we consider the functional equation

$$g(G(x)) = J[g(x), x], \qquad x \in X, \tag{8}$$

where G is a given function from X to X and J: $Y \times X \to Y$. Clearly, if the function G is invertible we come back to equation (2), where $B = G^{-1}$ and $S[u, x] = J[u, G^{-1}(x)]$ and so we can use Theorem 1.

If G is not invertible, we assume on J the following hypotheses:

$$\begin{aligned}&\forall x \in X \quad \exists L(.,x)\colon Y \to Y \qquad \text{such that } \forall u \in Y\\ &J[L(u,x),x] = u \quad \text{and} \quad L[J(u,x),x] = u.\end{aligned} \tag{9}$$

Thus we can prove the following.

THEOREM 3. *Assume that J satisfies condition* (9) *and that the function L satisfies the Lipschitz condition*

$$d(L(u,x), L(v,x)) \le \lambda d(u,v), \qquad x \in X,\ u, v \in Y. \tag{10}$$

Let $f\colon X \to Y$ be a given function and let $\delta\colon X \to \mathbb{R}^+$ fulfil the following conditions:

$$d(f(G(x)), J[f(x), x]) \le \delta(x), \qquad x \in X \tag{11}$$

$$\text{the series } \sum_{i=1}^{\infty} \lambda^i \delta(G^{i-1}(x)) \qquad \text{converges for every } x \in X. \tag{12}$$

Then there exists a unique solution g of equation (8) *such that*

$$d(f(x), g(x)) \le \sum_{i=1}^{\infty} \lambda^i \delta(G^{i-1}(x)), \qquad x \in X.$$

Proof. By conditions (9), (10) and (11) we have

$$\begin{aligned}d(f(x), L[f(G(x)), x]) &= d(L[J(f(x), x), x], L[f(G(x)), x])\\ &\le \lambda d(J[f(x), x], f(G(x))) \le \lambda\delta(x).\end{aligned}$$

Thus we can apply Theorem 1 with G instead of B, L instead of S and $\lambda\delta(x)$ instead of $\sigma(x)$. □

3. The main results

In this section we prove the main results of the paper, that is two stability theorems for the functional equation (1)

$$g[F(x,y)] = H[g(x), g(y), x, y],$$

where $F\colon X \times X \to X$ and $H\colon Y \times Y \times X \times X \to Y$ and $g\colon X \to Y$.

In order to cover two different situations with the same statement, in the next two theorems we denote by $G(x)$ either the function $F(x, x)$ or the function $F(x, a)$, where a is a fixed element in X; by $J(v, x)$ either $H[u, u, x, x]$ or $H[u, f(a), x, a]$ and by $\delta(x)$ either $\Delta(x, x)$ or $\Delta(x, a)$ respectively, where $f\colon X \to Y$ is a given function.

THEOREM. 4 *Let $f\colon X \to Y$ be a given function and let $\Delta\colon X \times X \to \mathbb{R}^+$ be a function such that*

$$d(f[F(x, y)], H[f(x), f(y), x, y]) \le \Delta(x, y), \qquad x, y \in X.$$

Assume the following hypotheses:
- *the function J satisfies condition* (9) *and the function L condition* (10);
- *for every $x, y \in X$ it is $F[G(x), G(y)] = G[F(x, y)]$;*
- *for every $x, y \in X$ and $u, v \in Y$, it is*

$$H[L[u, x], L[v, y], x, y] = L[H[u, v, G(x), G(y)], F(x, y)];$$

- *the function H is continuous in its first two variables.*
- *the function $\delta(x)$ satisfies conditions* (11) *and* (12) *and moreover $\lambda^n \Delta(G^n(x), G^n(y)) \to 0$ as $n \to \infty$, for every $x, y \in X$.*

Then the function g obtained in Theorem 3 *is a solution of equation* (1).

Proof. The function g is defined as the pointwise limit of the sequence $\{L_n[f, G](x)\}$ constructed as in (3).

By the continuity of H for every $x, y \in X$ we have

$$d(L_n[f, G](F(x, y)), H[L_n[f, G](x), L_n[f, G](y), x, y])$$
$$\to d(g[F(x, y)], H[g(x), g(y), x, y]).$$

On the other hand, for every $x, y \in X$, we get

$$d(L_n[f, G](F(x, y)), H[L_n[f, G](x), L_n[f, G](y), x, y])$$
$$= d(L[L_{n-1}[f, G](G[F(x, y)]), F(x, y)],$$
$$L[H[L_{n-1}[f, G](G(x)), L_{n-1}[f, G](G(y)), G(x), G(y)], F(x, y)])$$
$$\le \lambda d(L_{n-1}[f, g](G[F(x, y)]),$$
$$H[L_{n-1}[f, G](G(x)), L_{n-1}[f, G](G(y)), G(x), G(y)])$$
$$\le \lambda^2 d(L_{n-2}[f, G](G(F[G(x), G(y)])),$$

$$H[L_{n-2}[f, G](G^2(x)), L_{n-2}[f, G](G^2(y)), G^2(x), G^2(y)])$$
$$\leq \cdots \leq \lambda^n d(f(G^n[F(x, y)]), H[f(G^n(x)), f(G^n(y)), G^n(x), G^n(y)])$$
$$\leq \lambda^n \Delta(G^n(x), G^n(y)) \to 0.$$

Thus $g[F(x, y)] = H[g(x), g(y), x, y]$, $x, y \in X$. □

The second stability result for equation (1) is obtained under the condition that the function G is invertible.

THEOREM. 5 *Let $f: X \to Y$ be a given function and let $\Delta: X \times X \to \mathbb{R}^+$ be a function such that*

$$d(f[F(x, y)], H[f(x), f(y), x, y]) \leq \Delta(x, y), \qquad x, y \in X. \tag{13}$$

Assume the following hypotheses:
- *the function G is invertible and denote by $B: X \to X$ its inverse;*
- *the function J satisfies the Lipschitz condition*

$$d(J(u, x), J(v, x)) \leq \lambda d(u, v), \qquad x \in X, u, v \in Y.$$

- *for every $x, y \in X$ it is $F[G(x), G(y)] = G[F(x, y)]$;*
- *for every $x, y \in X$ and $u, v \in Y$, it is*

$$H[J[u, x], J[v, y], G(x), G(y)] = J[H[u, v, x, y], F(x, y)];$$

- *the function H is continuous in its first two variables.*
- *the function $\delta(x) := \Delta(x, x)$ is such that the series*

$$\sum_{i=0}^{\infty} \lambda^i \delta(B^{i+1}(x))$$

converges for every $x \in X$ and moreover $\lambda^n \Delta(B^n(x), B^n(y)) \to 0$ as $n \to \infty$, for every $x, y \in X$.

Then there exists a unique solution g of equation (1) such that

$$d(f(x), g(x)) \leq \sum_{i=0}^{\infty} \lambda^i \delta(B^{i+1}(x)), \qquad x \in X. \tag{14}$$

Proof. From (13) for every $x \in X$ we obtain the inequality

$$d(f(G(x)), J[f(x), x]) \leq \delta(x)$$

which is equivalent to the following

$$d(f(x), J[f(B(x)), B(x)]) \leq \delta(B(x)).$$

Thus we can apply Theorem 1 with $S(u, x) = J[u, B(x)]$ and $\sigma(x) = \delta(B(x))$. So the function g, pointwise limit of $\{S_n[f, B]\}$, is a solution of the equation

$$g(x) = J[g(B(x)), B(x)], \qquad x \in X,$$

and satisfies (14).

Arguing in analogy with the proof of Theorem 4 we obtain that the function g is a solution of the equation (1). □

4. Some special cases

In this section we present some examples which can be treated as special cases of the functional equation (1). Naturally, it is possible to treat them without using the full power of the previous results but proving a simpler specific theorem for each of them.

First we assume that the function H doesn't depend explicitly on the variables x and y. So we are considering the equation

$$g[F(x, y)] = H[g(x), g(y)]. \tag{15}$$

This equation is a common generalization of the Cauchy and Jensen equations.

If we set $G(x) = F(x, x)$, the hypotheses of Theorems 4 and 5 coincide with those of the stability theorem proved in [4] and of a special case of Theorem 2 proved in [3], except for the Lipschitz conditions. In the previously quoted papers a different assumption is made. Thus, our theorems give stability results for a different class of functional equations. However, in the case of the Cauchy equation all hypotheses coincide.

Now we apply the previous results to a functional equation which is a generalization of the classic Jensen equation.

We suppose that X is a vector space and Y is a Banach space and consider the functional equation (15) where the functions F and H are defined as follows:

$$F(x, y) = h^{-1}(\alpha h(x) + \beta h(y)), \qquad H(u, v) = s^{-1}(\gamma s(u) + \eta s(v)), \tag{16}$$

where $\alpha, \beta > 0$ and $\alpha + \beta = 1$, $\gamma, \eta > 0$ and $\gamma + \eta = 1$, the function $h: X \to X$ is invertible, the function $s: Y \to Y$ is continuous, invertible with continuous inverse and satisfies the additional condition

$$\mu_1 \|u - v\| \leq \|s(u) - s(v)\| \leq \mu_2 \|u - v\| \qquad \text{for some } \mu_1, \mu_2 > 0.$$

THEOREM. 6 *Let $f: X \to Y$ be a given function and let $\Delta: X \times X \to \mathbb{R}^+$ be a function such that*

$$\|f[F(x, y)] - H[f(x), f(y)]\| \leq \Delta(x, y),$$

where F and H are defined as in (16) *and fix $a \in X$ such that $h(a) = 0$. Define $\delta(x) = \Delta(x, x)$.*

Assume that one of the following conditions holds:

(i) the series

$$\sum_{i=1}^{\infty} \left(\frac{\mu_2}{\mu_1 \gamma}\right)^i \delta(h^{-1}(\alpha^{i-1} h(x)))$$

converges for every $x \in X$ and

$$\left(\frac{\mu_2}{\mu_1 \gamma}\right)^n \Delta(h^{-1}(\alpha^n h(x)), h^{-1}(\alpha^n h(y))) \to 0 \qquad \text{as } n \to \infty \text{ for every } x, y \in X;$$

(ii) the series

$$\sum_{i=0}^{\infty} \left(\frac{\mu_2 \gamma}{\mu_1}\right)^i \delta(h^{-1}(\alpha^{-i-1} h(x)))$$

converges for every $x \in X$ and

$$\left(\frac{\mu_2 \gamma}{\mu_1}\right)^n \Delta(h^{-1}(\alpha^{-n} h(x)), h^{-1}(\alpha^{-n} h(y))) \to 0 \qquad \text{as } n \to \infty \text{ for every } x, y \in X.$$

Then there exists a unique solution g of the equation (15) *such that*

$$\|f(x) - g(x)\| \leq \sum_{i=1}^{\infty} \left(\frac{\mu_2}{\mu_1 \gamma}\right)^i \delta(h^{-1}(\alpha^{i-1} h(x)))$$

in case (i), or

$$\|f(x) - g(x)\| \leq \sum_{i=0}^{\infty} \left(\frac{\mu_2 \gamma}{\mu_1}\right)^i \delta(h^{-1}(\alpha^{-i-1} h(x)))$$

in case (ii).

Proof. We set $G(x) = F(x, a)$ and put $f(a) = b$. Clearly, the function F satisfies the assumptions of Theorems 4 and 5. Moreover we have

$$J(u) = s^{-1}[\gamma s(u) + \eta s(b)],$$

$$L(u) = s^{-1}\left[\frac{s(u) - \eta s(b)}{\gamma}\right],$$

$$\|J(u_1) - J(u_2)\| \leq \frac{\mu_2 \gamma}{\mu_1} \|u_1 - u_2\|,$$

$$\|L(u_1) - L(u_2)\| \leq \frac{\mu_2}{\mu_1 \gamma} \|u_1 - u_2\|,$$

and all other conditions on the function H are fulfilled. Furthermore we get

$$G^n(x) = h^{-1}[\alpha^n h(x)], \qquad B^n(x) = h^{-1}[\alpha^{-n} h(x)].$$

Thus in case (i) we apply Theorem 4 while in case (ii) we apply Theorem 5. □

As a very special case we assume that both F and H are weighted arithmetic means, thus obtaining the following.

COROLLARY. 7 *Let f: $X \to Y$ be a given function and assume that the inequality*

$$\|f[\alpha x + (1-\alpha)y] - [\gamma f(x) + (1-\gamma)f(y)]\| \leq \theta(\|x\|^p + \|y\|^p),$$
$$\alpha, \gamma \in (0, 1), \quad \theta \geq 0, \quad p \in \mathbb{R},$$

is satisfied for all $x, y \in X$ (x and y different from 0 if $p < 0$). If $\alpha^p \neq \gamma$, then there exists a unique function g such that

$$g[\alpha x + (1-\alpha)y] = \gamma g(x) + (1-\gamma)g(y)$$

and

$$\|f(x) - g(x)\| \leq \frac{2\theta}{|\gamma - \alpha^p|} \|x\|^p$$

for every $x \in X$ ($x \neq 0$ if $p < 0$).

Note that in the case $\alpha^p < \gamma$, i.e. when we apply Theorem 4, it is possible to assume that x and y belong to a convex subset of X containing the origin.

We remark that if $\alpha^p = \gamma$ then in general we have not stability. To show this consider the Jensen equation, i.e. $\alpha = \gamma = 1/2$, and take $p = 1$; the function $f(x) = x \log |x|$ for $x \neq 0$ and $f(0) = 0$ provides an example (see [5]).

Now we consider two functional equations where the variables x and y appear explicitly.

The first is the functional equation of the multiplicative derivation

$$g(xy) = xg(y) + yg(x) \tag{17}$$

and we assume $X \subset \mathbb{R}$ and $Y = \mathbb{R}$.

As an application of Theorem 4 we get

THEOREM. 8 *Let* $f: [1, \infty) \to \mathbb{R}$ *be a given function and let* $\Delta: [1, \infty)^2 \to \mathbb{R}^+$ *be a function such that*

$$|f(xy) - xf(y) - yf(x)| \leq \Delta(x, y).$$

Set $\delta(x) = \Delta(x, x)$. If the series

$$\sum_{i=1}^{\infty} 2^{-i}\delta(x^{2^{i-1}})$$

converges for every $x \in [1, \infty)$ *and* $2^{-n}\Delta(x^{2^n}, y^{2^n}) \to 0$ *as* $n \to \infty$ *for every* $x, y \in [1, \infty)$, *then there exists a unique solution* $g: [1, \infty) \to \mathbb{R}$ *of equation* (17) *such that*

$$|f(x) - g(x)| \leq \sum_{i=1}^{\infty} 2^{-i}\delta(x^{2^{i-1}}).$$

Note that the above convergence conditions are satisfied if, for instance,

$$\Delta(x, y) = \theta(\|x\|^p + \|y\|^p) \qquad \text{with } p \leq 0.$$

Still for equation (17) we get the following result by using Theorem 5.

THEOREM. 9 *Let* $f: [0, 1] \to \mathbb{R}$ *be a given function and let* $\Delta: [0, 1]^2 \to \mathbb{R}^+$ *be a function such that*

$$|f(xy) - xf(y) - yf(x)| \leq \Delta(x, y).$$

Set $\delta(x) = \Delta(x, x)$. *If the series*

$$\sum_{i=0}^{\infty} 2^i \delta(x^{2^{-i-1}})$$

converges for every $x \in [0, 1]$ *and* $2^n \Delta(x^{2^{-n}}, y^{2^{-n}}) \to 0$ *as* $n \to \infty$ *for every* $x, y \in [0, 1]$, *then there exists a unique solution* $g: [0, 1] \to \mathbb{R}$ *of equation* (17) *such that*

$$|f(x) - g(x)| \le \sum_{i=0}^{\infty} 2^i \delta(x^{2^{-i-1}}).$$

The second example is given by the equation of Sincov (see [1])

$$g(x + y) = a^{xy} g(x) g(y), \qquad a > 1, \quad x, y \in \mathbb{R}, \quad g: \mathbb{R} \to \mathbb{R}^+. \tag{18}$$

By using Theorem 4 we get

THEOREM. 10 *Let* $f: \mathbb{R} \to \mathbb{R}^+$ *be a given function and let* $\phi, \psi: \mathbb{R}^2 \to \mathbb{R}^+$ *be functions such that* $\phi(x, y) \ge 1$ *and*

$$\psi(x, y) \le \frac{f(x + y)}{a^{xy} f(x) f(y)} \le \phi(x, y), \qquad x, y \in \mathbb{R}, \quad a > 1. \tag{19}$$

Set $\Delta(x, y) = \max\{\log_a \phi(x, y), -\log_a \psi(x, y)\}$ *and* $\delta(x) = \Delta(x, x)$. *If the series*

$$\sum_{i=1}^{\infty} 2^{-i} \delta(2^{i-1} x)$$

converges for every $x \in \mathbb{R}$ *and* $2^{-n} \Delta(2^n x, 2^n y) \to 0$ *as* $n \to \infty$ *for every* $x, y \in \mathbb{R}$, *then there exists a unique solution* $g: \mathbb{R} \to \mathbb{R}^+$ *of equation* (18) *such that*

$$a^{\{-\sum_{i=1}^{\infty} 2^{-i} \delta(2^{i-1} x)\}} \le \frac{f(x)}{g(x)} \le a^{\{\sum_{i=1}^{\infty} 2^{-i} \delta(2^{i-1} x)\}}, \qquad x \in \mathbb{R}.$$

Proof. By taking the logarithm in (19), we obtain the inequality

$$\log_a \psi(x, y) \le \log_a f(x + y) - xy - \log_a f(x) - \log_a f(y) \le \log_a \phi(x, y),$$

i.e.

$$|h(x + y) - xy - h(x) - h(y)| \le \Delta(x, y),$$

where $h = \log_a \circ f$. Now we apply Theorem 4. □

REFERENCES

[1] Aczél, J., *Lectures on functional equations and their applications*. Academic Press, New York–London, 1966.

[2] Baker, J. A., *The stability of certain functional equations*. Proc. Amer. Math. Soc. *112* (1991), 729–732.

[3] Borelli, C. and Forti, G. L., *On a general Hyers-Ulam stability result*. Internat. J. Math. Math. Sci. *18* (1995), 229–236.

[4] Forti, G. L., *An existence and stability theorem for a class of functional equations*. Stochastica *4* (1980), 23–30.

[5] Johnson, B. E., *Approximately multiplicative maps between Banach algebras*. J. London Math. Soc. (2) *37* (1988), 294–316.

[6] Kuczma, M., *Functional equations in a single variable*, Monografie Mat. 46, Polish Scientific Publishers (PWN), Warszawa, 1968.

C. Borelli
Università degli Studi di Milano
Dipartimento di Matematica
Via C. Saldini 50
I-20133 Milano, Italy