André Weil and Algebraic Topology

Armand Borel

André Weil is associated more with number theory or algebraic geometry than with algebraic topology. But the latter was very much on his mind during a substantial part of his career. This led him first to contributions to algebraic topology proper, in a differential geometric setting, and then also to the use in abstract algebraic geometry and several complex variables of ideas borrowed from it.

According to [W3], I, p. 562, his first contacts with algebraic topology took place in Berlin, 1927, in long conversations with, and lectures from, Heinz Hopf. The first publication of H. Hopf on the Lefschetz fixed point formula appeared the following year, so it is rather likely that Weil heard about it at the time. At any rate, his first paper involving algebraic topology is indeed an application of that formula to the proof of a fundamental theorem on compact connected Lie groups (which Weil attributes to E. Cartan, but is in fact due to H. Weyl): Let G be a compact connected Lie group. Then the maximal tori (i.e. maximal connected abelian subgroups) of G are conjugate by inner automorphisms and contain all elements of G ([1935c] in [W3], I, 109–111).

The proof is a repeated application of the Lefschetz fixed point formula to translations by group elements on the homogeneous space G/T, where T is a maximal torus. Note that the isotropy groups on G/T are the conjugates of T, so that an element belongs to a conjugate of T if and only if it fixes some point in G/T. Weil first points out that T is of finite index in its normalizer $N(T)$. If $t \in T$ generates a dense subgroup of T, then its fixed points are the same as those of T, and a local computation shows their indexes to be simultaneously equal to 1 or to -1. The Lefschetz number of t is then $\neq 0$. But since t is connected to the identity, this number is equal to the Euler-Poincaré characteristic $\chi(G/T)$ of G/T, which is therefore $\neq 0$. As a consequence, any element $g \in G$ has a non-zero Lefschetz number, hence a fixed point, and belongs to a conjugate of T. If T' is another torus and t' generates a dense subgroup of T', then any torus containing t' will also contain T', whence the conjugacy statement.

This was the first new proof of that theorem, completely different from the original one, which relied on a study of singular elements (cf. H. Weyl, *Collected Papers II*, 629–633). It was rediscovered independently, about five years later, by H. Hopf and H. Samelson (*Comm. Math. Helv.* **13** (1940–41), 240–251).

For about ten years, from 1942 on, topology was present in several works of Weil, often pursued simultaneously, which I first list briefly:

a) In algebraic geometry: foundations, introduction of fibre bundles, formulation of the Weil conjectures.

b) New proof of the de Rham theorems. Together with Leray's work, this was the launching pad for H. Cartan's work in sheaf theory.

c) Characteristic classes for differentiable bundles: Allendoerfer-Weil generalization of the Gauss-Bonnet theorem, theory of connections, the Chern-Weil homomorphism, the Weil algebra.

d) Joint work with Cartan, Koszul, and Chevalley on cohomology of homogeneous spaces.

e) A letter to H. Cartan (August 1, 1950) on complex manifolds, advocating the use of analytic fibre bundles in the formulation of problems such as those of Cousin.

There is a last item I would like to add, dating from 1961–62:

f) Local rigidity of discrete cocompact subgroups of semisimple Lie groups.

On the face of it, it does not belong to algebraic topology, but can be fitted under my general title when stated as a theorem on group cohomology. This formulation was originally an afterthought, but turned out to be important to suggest further developments.

Algebraic Geometry

The algebraic geometry, as developed mainly by the Italian School, did not offer a secure framework for the proof of the Riemann hypothesis for curves and

---

Armand Borel is professor emeritus of mathematics at the Institute for Advanced Study. His e-mail address is `borel@ias.edu`.

This article is based on a lecture at the Institute for Advanced Study on January 8, 1999, as part of a conference on the Work of André Weil and Its Influence, January 8–9, 1999.

The article is also appearing in the Gazette des Mathématiciens.

other researches of Weil in algebraic geometry. He had to develop new foundations, with as one of its main goals a theory of intersections of subvarieties. It had also to be over any field. This implied a massive recourse to algebra, but Weil still wanted to keep a geometric language and picture. Until then, only projective, affine, or quasi-projective varieties had been considered, i.e. subvarieties of some standard spaces. He wanted a notion of "abstract variety" which would be the analogue of a manifold (albeit with singularities). His first version [W1] is a bit awkward, as acknowledged in the foreword to the second edition, because no topology is introduced. From ([1949c], [W3], I, 411–413) on, however, he uses the language of the Zariski topology (introduced in 1944 by O. Zariski), and I shall do so right away. Fix a "universal field" K, i.e. an algebraically closed field of infinite transcendence degree over its prime field. Let V be an algebraic subset of K^n, i.e. an affine variety. In the Zariski topology, the closed subsets of V are the algebraic subsets. The open sets are, of course, their complements and are quite big. If V is irreducible, any two nonempty ones intersect in a dense open one, so that the topology is decidedly not Hausdorff (unless V is a point), which may explain some reluctance to use it initially. To define an (irreducible) abstract variety V, start from a finite collection $(V_i, f_{ji}), (i, j \in I)$, where V_i is an irreducible affine algebraic set, f_{ji} a birational correspondence from V_i to V_j satisfying certain conditions, so that, in particular: f_{ii} is the identity, $f_{ij} = f_{ji}^{-1}$, there exist open subsets $D_{ji} \subset V_i$ such that f_{ji} is a biregular mapping of D_{ji} onto D_{ij}, and $f_{ji} = f_{jk} \circ f_{ki}$. Two points $P_i \in D_{ji}$ and $P_j \in D_{ij}$ are equivalent if $f_{ji}(P_i) = P_j$. The "abstract variety" V is by definition the quotient of the disjoint union $\tilde{V}$ of the V_i by that equivalence relation.

Note that V is obtained by gluing together disjoint affine sets. For lack of suitable concepts, it was not possible to start from a topological space and require that it be endowed locally with a given structure, as is done for manifolds (as was done later by Serre using the notion of ringed space [S2]). As a result, the V_i and f_{ji} are part of the structure, which is rather unwieldy and requires a somewhat discouraging amount of algebra to be worked with. Nevertheless, Weil develops the theory of such varieties and of the intersection of cycles. For the latter, the analogy with the complex case and the intersection product in the homology of manifolds (on which he had lectured earlier at the Hadamard Seminar ([W3], I, 563)) is always present. In particular, a key property is the analogue of Hopf's inverse homomorphism (see [W1], Introduction, xi–xii). Weil also introduces an analogue of compact manifolds, the complete varieties, which include the projective ones. [W1] supplied the framework for a detailed proof of the Riemann hypothesis for curves and for further work on abelian varieties, and it supplied essentially the only framework for algebraic geometry over any field until Grothendieck's theory of schemes (from about 1960 on).

Photograph courtesy of Sylvie Weil.

Weil (left) with Armand Borel in Chicago about 1955.

Algebraic topology also underlies the formulation of the conjectures in ([1949b], cf. [W3], I, 399–410), soon to be called the Weil conjectures, which suggest looking for a cohomology theory for complete smooth varieties in which a Lefschetz fixed point formula would be valid. This vision, which turned out to be prophetic, was unique at the time.

In ([1949c], cf. [W3], I, 411–12), Weil introduces in algebraic geometry fibre bundles with an algebraic group, say G, as structural group. Given a variety B and a finite open cover $\{V_i\}(i \in I)$ of B, assume one is given regular maps $s_{ij} : V_i \cap V_j \to G$ $(i, j \in I; s_{ii}$ is the constant map to the identity), with the usual transitivity conditions. Let F be a variety on which G operates. Then a fibre bundle E on B, with typical fibre F, is obtained by gluing the products $V_i \times F$ by means of the s_{ij}, as usual. Weil also considers the case of principal bundles ($F = G$, acted upon itself by right translations). In particular, if $G = \mathbb{C}^*$ is the multiplicative group of nonzero complex numbers, the isomorphism classes of such bundles correspond to linear equivalence classes of divisors. It also allowed Weil to interpret in a more conceptual way earlier work on algebraic curves (see [W3], I, 531, 541, 570 for comments). A detailed exposition is given in [W2], where the classification of such bundles is studied in some simple cases.

In view of the big size of the neighborhoods on which such a bundle is trivial, it was not a priori clear this would lead to an interesting theory. That it did is one reason why Weil began to gain confidence in the Zariski topology. Of course, his definition of fibre bundle was greatly generalized later. Already in [1949c], Weil points out it would be desirable to have a notion broad enough so that B could be the set of prime spots of a number field. Later (Séminaire Chevalley 1958, I), J.-P. Serre introduced an important generalization of local triviality: a bundle is locally isotrivial if every point has an open neighborhood admitting an unramified covering on which the lifted bundle is trivial. This notion, which encompasses the fibration of an algebraic group by a closed subgroup, led A. Grothendieck to the definition of étale topology.

The de Rham Theorems

In January 1947 Weil wrote a letter to H. Cartan ([W3], II, 45–47) outlining a new proof of the de Rham theorems, published later in ([1952a], [W3], II, 17–43), the first one since de Rham's thesis. It is limited to compact manifolds, but this restriction is lifted, with very little complication, in the final version.

Given a smooth compact connected manifold M, Weil first shows the existence of a finite open cover $\{U_i\}_{i\in I}$ of M such that any nonempty intersection of some of the U_i's is contractible. Let N be the nerve of this cover, and, for each simplex $\sigma \in N$, let U_σ be the intersection of the U_i represented by the vertices of σ. Given $p, q \in \mathbb{N}$, let $A^{p,q}$ be the function assigning to each p-simplex σ of N the space of differential q-forms on U_σ. The direct sum A of $A^{p,q}$ is endowed with two differentials

$$(1) \quad d : A^{p,q} \to A^{p,q+1}, \qquad \delta : A^{p,q} \to A^{p+1,q},$$

where d stems from exterior differentiation and δ from the coboundary operator on N, followed by restriction of differential forms. Let $F^{p,q}$ (resp. $H^{p,q}$) be the subspace of $A^{p,q}$ of elements annihilated by $d\delta$ (resp. d or δ). Then Weil establishes the isomorphisms

$$(2) \quad F^{0,m}/H^{0,m} = H^m_{DR}(M), \qquad F^{m,0}/H^{m,0} = H^m(N).$$

$$(3) \quad F^{p,q}/H^{p,q} = F^{p+1,q-1}/H^{p+1,q-1} \qquad (0 \le q \le m),$$

where $H^m_{DR}(M)$ refers to real de Rham cohomology and $H^m(N)$ to the real cohomology of N. This proves, by induction, that $H^m_{DR}(M)$ is isomorphic to $H^m(N)$, hence to $H^m(M)$, since the U_σ are contractible. This last fact is taken for granted in the letter, but Weil also shows in [1952a] by similar arguments that $H^m(M)$ is equal to the m-th singular cohomology of M over the reals.

H. Cartan had studied the work of Leray in topology, in particular his wartime paper (*J. Math. Pure Appl.* **29** (1945), 95–248), and he noticed a similarity with Weil's proof. That was tremendously suggestive to him and quickly gave rise to a flurry of letters to Weil, in which Cartan initiated his theory of *faisceaux et carapaces* (sheaves and gratings), of which he gave later three versions (see [B] for references), first following Leray rather closely, arriving eventually at a much greater generality.

What Cartan had noticed is an analogy between the proofs of the isomorphisms in (3) and an argument which occurs repeatedly in Leray's paper, to which Leray himself traced later the origin of the spectral sequence (see [B]). However, Weil's argument was completely independent from it: As stated in a slightly later letter to Cartan, Weil did not know that paper and in fact suspected, on the strength of a report by S. Eilenberg in *Math. Reviews*, that it did not bring much new, if anything. On the other hand, it is quite plausible that the definition of the $A^{p,q}$ was in part inspired by a short conversation Weil had with Leray in summer 1945, in which the latter spoke of a cohomology "with variable coefficients". In fact, an analogue of A in the theory Leray was developing at the time would be a *couverture*, N, with coefficients in the differential graded sheaf associated to differential forms.

Characteristic Classes

In 1941–42 Weil was for some time at Haverford College, Pennsylvania, where he met C. Allendoerfer. This led to their joint work on the generalized Gauss-Bonnet theorem [AW]. Given a smooth compact oriented Riemannian manifold M of even dimension m, it expresses the Euler-Poincaré characteristic $\chi(M)$ of M as the integral over M of a differential m-form built from the components of the curvature tensor. Such a formula had already been proved by Allendoerfer and by Fenchel for submanifolds of euclidean space. At the time, the Allendoerfer-Weil theorem was in principle more general, since it was not known whether a Riemannian manifold was globally isometrically diffeomorphic to a submanifold of euclidean space, though it had been established locally. Because of that, the nature of their proof forced them to prove a more general statement, though I do not know whether the added generality has led to further applications.

Recall the Gauss-Bonnet formula in the most classical case: P is a relatively compact open subspace on a surface in $\mathbb{R}^3$, bounded by a simple closed curve, union of finitely many smooth arcs. Then the integral of the Gaussian curvature K on P, plus the sum of the integrals of the curvature on the boundary arcs and of the outside angles at the meeting points of those, is equal to 2π. The Allendoerfer-Weil formula gives a generalization of such a formula for a Riemannian polyhedron. It is proved first for polyhedra in euclidean space. The general case then follows by using a polyhedral subdivision, small enough so that its building blocks can be isometrically embedded in euclidean space, and by proving a suitable addition formula.

In 1944 S. S. Chern produced a proof of the Allendoerfer-Weil formula for closed manifolds (*Ann.

Math. **45** (1944), 747–52) which was much simpler and a harbinger of further developments on characteristic classes. On M choose a vector field X with only one zero, of order $|\chi(M)|$ at some point x_0, which is always possible. Let E be the unit tangent bundle to M, $p : E \to M$ the canonical projection, and Ω the Gauss-Bonnet form. The key point is that $p^*\Omega = d\Pi$ is the exterior derivative of some explicitly given form Π, the restriction of which to a fibre F represents the fundamental class [F] of the fibre. The vector field X defines a submanifold V in E, a copy of $M - \{x_0\}$, with boundary the unit sphere $F_0 = p^{-1}(x_0)$, with multiplicity $|\chi(M)|$. The Gauss-Bonnet formula then follows from the Stokes theorem, applied to $V \cup F_0$.

The relationship between Ω, Π, and $[F_0]$ is a first example of a notion developed later under the name of *transgression* in a fibre bundle: a cohomology class β of a fibre F is transgressive if there is a cochain (in the cohomology theory used, here a differential form) on the total space E whose restriction to F is closed, represents β, and whose coboundary belongs to the image of a cohomology class η of the base B, under the map induced by the projection $p : E \to B$. The classes β and η will be said to be related by transgression. This notion, and the terminology, were introduced first by J.-L. Koszul in a Lie algebra cohomology setting in his thesis (*Bull. Soc. Math. France* **78** (1950), 65–127).

In *Ann. Math.* **47** (1946), 85–121, Chern gives several definitions of the characteristic classes $c^i(M) \in H^{2i}(M;\mathbb{C})$, since then called the Chern classes ($1 \le i \le m$). In particular, if M is endowed with a hermitian metric, they can be expressed by closed differential forms which are locally defined in terms of the curvature tensor. Again, each one is related by transgression in a suitable bundle to the fundamental class of the fibre.

It is at this point that Weil comes in. He was familiar with the work of Chern, with the theory of fibre bundles, in particular with the classification theorem in terms of universal bundles, having written jointly with S. Eilenberg, with some help from N. Steenrod, a report on fibre bundles for Bourbaki (which, incidentally, provided much background material for the second Cartan seminar [C2]). He was also aware of Ehresmann's publications on fibre bundles and on the formulation of E. Cartan's theory of connections in that framework, as well as of Koszul's work towards his thesis quoted above. All this came together in a series of letters to Cartan, Chevalley, and Koszul, of which the first four were published (almost completely) for the first time thirty years later ([1949e], in [W3], I, 422–36). Some were shown around at the time, however. In particular, the first one is the basis of Chapter III in [C4], and this is how its contents became widely known.

Let G be a compact connected Lie group, ξ a principal G-bundle, E (resp. B) the total space (resp. base) of ξ. A connection on ξ is defined by means of a 1-form on E with values in the Lie algebra $\mathfrak{g}$ of G, satisfying certain conditions. Let I_G be the algebra of polynomials on $\mathfrak{g}$ invariant under the adjoint representation and $P \in I_G$ a homogeneous element of degree q. Replacing the variables in P by the components of the curvature tensor of the connection, Weil associates to P a differential $2q$-form on M, which is proved to be closed, hence to define an element $c_P \in H^{2q}(M;\mathbb{R})$. A fundamental theorem asserts that c_P is independent of the connection. The proof is short but stunning. In the fall of 1949, in Paris, I read this letter and said once to Cartan that this proof seemed to come out of the blue and I could not trace it back to anything. "That's genius. You don't explain genius," was his answer. The image of I_G under this homomorphism, which became known as the Chern-Weil homomorphism, is then the characteristic algebra of ξ.

Photograph courtesy of Armand Borel.

Weil at the Tata Institute of Fundamental Research in Bombay, January 1967.

At the end of the first letter, Weil states a conjecture relating the primitive generators of $H^*(G;\mathbb{R})$ (recall that it is an exterior algebra with a distinguished set of generators, called primitive) to the characteristic algebra by transgression, soon proved by Chevalley. This already provided a generalization of Chern's treatment of characteristic classes of hermitian bundles, modulo some normalization and plausible identifications. In the third letter, which, like the fourth, was addressed to Koszul, Weil makes the analogy closer. Recall that in the classical case the characteristic classes are the images of cohomology classes of a classifying space (a complex Grassmannian for hermitian bundles), under the homomorphism induced by a classifying map (see [C4], for example). Weil proposes an algebraic analogue of that situation. He introduces an algebra which, following Cartan [C3], I shall denote $W(\mathfrak{g})$ and call the Weil algebra of $\mathfrak{g}$. By definition, $W(\mathfrak{g}) = S(\mathfrak{g}^*) \otimes \wedge\mathfrak{g}^*$ is the tensor product of the symmetric algebra $S(\mathfrak{g}^*)$ by the exterior algebra $\wedge\mathfrak{g}^*$ of the dual $\mathfrak{g}^*$ of $\mathfrak{g}$. It is graded, anticommutative, an element $x \in \mathfrak{g}^*$ being given the degree 1 (resp. 2) if it is viewed as belonging to $\wedge\mathfrak{g}^*$ (resp. $S(\mathfrak{g}^*)$). The Weil algebra is further endowed with a specific differential. The latter leaves $S(\mathfrak{g}^*) \otimes \mathbb{R}$ stable, the cohomology of which is isomorphic to I_G. The algebra $\wedge\mathfrak{g}^*$, endowed with the Lie algebra cohomology differential, is a quotient of $W(\mathfrak{g})$. The transgression in $W(\mathfrak{g})$ provides a bijection of the space of primitive generators of $H^*(\mathfrak{g})$ (which is isomorphic to $H^*(G;\mathbb{R})$) onto a space spanned by independent homogeneous generators of I_G (the latter is, by a theorem of Chevalley, a polynomial algebra). A connection on ξ provides a homomorphism of $W(\mathfrak{g})$ onto a subalgebra of differential forms on E which, after having passed to cohomology, yields the Chern-Weil homomorphism. Thus $W(\mathfrak{g})$ plays the role of an algebra of differential forms on a universal G-bundle, an analogy reinforced by the fact, proved by Cartan [C3], that $W(\mathfrak{g})$ is acyclic.

So far, I have focused on characteristic classes. But these letters, combined with Koszul's thesis, led to further correspondence on the cohomology of homogeneous spaces and to more results announced by H. Cartan (*Colloque de Topologie*, C.B.R.M., Bruxelles, 1950, 57–71) and J.-L. Koszul (ibid., 73–81). A full exposition is given in [GHV].

Complex Manifolds and Holomorphic Fibre Bundles

On August 1, 1950, Weil wrote to H. Cartan a letter about global analysis in several complex variables (unpublished). He first claims that it is high time to stop viewing the object of these investigations as a sort of "domain" spread over n-space or complex projective space. One should look at complex manifolds, noting, of course, that not much can be proved without further assumptions such as compact, Kähler, global existence of holomorphic functions with nonzero Jacobians, etc. Then he points out that analytic fibre bundles underlie some classical problems. For instance, the Cousin data for the multiplicative Cousin problem (find a function with a given divisor of zeros and poles) lead to a principal $\mathbb{C}^*$-bundle. For a solution to exist, the bundle should first be topologically trivial. This condition is not always sufficient, but it is on a domain of holomorphy. Pursuing that idea, he conjectures that a complex vector bundle on a polycylinder with structural group a complex Lie group which is topologically trivial should be analytically trivial.

Unfortunately, I could only find the first page of this letter in Weil's papers; the original seems to be lost, or at any rate could not be located. The beginning of the last sentence: *"Once one has taken the habit to look for fibre bundles in these questions, one soon sees them everywhere (or 'almost everywhere') and there is an enormous gain…"*, makes one strongly wish to see the rest.

These remarks were taken into account by H. Cartan (*Proceeding I.C.M.*, Vol. 1, 1950, 152–164), who also pointed out that the first Cousin problem (find a meromorphic function with given polar parts) leads to a principal complex bundle too, but with fibre the additive group of $\mathbb{C}$.

Local Rigidity

It is well known that compact Riemann surfaces of higher genus have moduli (noncompact ones too, but I confine myself to the compact case). Such a surface is a quotient $\Gamma\backslash X$ of the upper half-plane $X = \mathbf{SL}_2(\mathbb{R})/\mathbf{SO}(2)$ by a discrete cocompact subgroup Γ of $\mathbf{SL}_2(\mathbb{R})$. Equivalently, this means that there are small deformations of Γ in $\mathbf{SL}_2(\mathbb{R})$ which are not conjugate to Γ. In the 1950s it began to be suspected that these phenomena were pretty much unique to that case among compact locally symmetric spaces $\Gamma\backslash X$, where $X = G/K$ is the quotient of a noncompact semisimple Lie group with finite center by a maximal compact subgroup K. The question is then to show that the locally symmetric space structure on $\Gamma\backslash X$ is locally rigid (no small deformation which is not an isomorphism) or, equivalently, that Γ is locally rigid (any local deformation of Γ in G is a conjugate of Γ). The first results along those lines were obtained by E. Calabi [C1]; E. Calabi-E. Vesentini [CV], from the geometric point of view; and A. Selberg [S1], for $G = \mathbf{SL}_n(\mathbb{R})$, from the group theoretical point of view.

The paper [S1] and an unpublished sequel to [C1] were the starting point for the three papers of Weil on that topic ([1960c], [1962b], [1964a] in [W3], II, 449–464, 486–510, 517–525). In the first one, Weil proves, for any connected Lie group, a conjecture of Selberg in [S1], to the effect that if Γ is discrete cocompact, any small deformation of Γ is discrete, cocompact, isomorphic to Γ. To formulate the problem, he introduces the variety $R(\Gamma, G)$ of homomorphisms of Γ into G. The group Γ is finitely presented (as fundamental group of the compact smooth manifold G/Γ). Let $(g_1, \dots, g_N)$ be a generating subset. Then $R(\Gamma, G)$ may be viewed as the real analytic subvariety of the product $G^{(N)}$ of N copies of G defined by the relations between these generators. Let $x_o = (g_1, \dots, g_N)$. The theorem is then that x_o has a neighborhood in $R(\Gamma, G)$, all elements of which represent discrete, cocompact subgroups of G isomorphic to Γ.

Assume now that G is semisimple, with finite center (an assumption which is implicit in [1962b], but could be lifted) with no factor which is compact or three dimensional. Then it is shown in [1962b] that Γ is locally rigid, as conjectured in [S1] too, by proving that the orbit

$$G.x_o = \{g.g_1.g^{-1}, \dots, g.g_N.g^{-1}, \ (g \in G)\}$$

contains a neighborhood of x_o in $R(\Gamma, G)$. The theorem is further extended to the case where G has some factors locally isomorphic to $\mathbf{SL}_2(\mathbb{R})$, provided that the projection of Γ on any such factor is not discrete. At the time, it was rumored (and in fact stated in [S1]) that Calabi had proved local rigidity when X is the hyperbolic n-space ($n \geqq 3$), but this was not contained in his only publication on that matter [C1], and Weil kept telling me that an essential idea was still missing. But he found it in notes by Kodaira of some 1958–59 seminar lectures by Calabi, and then proved the above results within a few days.

The paper [CV] considers first of all the case where X is an irreducible bounded symmetric domain and shows that its complex structure is locally rigid, provided X is not isomorphic to the unit ball in $\mathbb{C}^n$ ($n \geq 2$). Both [C1] and [CV] follow the model of the Kodaira-Spencer theory of deformations of complex structures. Local rigidity follows then from the vanishing of a first cohomology group, with coefficients in germs of Killing vector fields in [C1], of holomorphic tangent vector fields in [CV]. In [1964a] Weil provides similarly a cohomological translation of [1962b] by showing that the proof there implies the vanishing of the first group cohomology space $H^1(\Gamma;\mathfrak{g})$ of Γ with coefficients in the Lie algebra $\mathfrak{g}$ of G, acted upon by the adjoint representation.

The proof in [1962b] was already cohomological in spririt and is described so by Weil in his comments. It is first reduced to the case of a one-parameter group of deformations, defined by a vector field ξ. Without changing its class modulo inner automorphisms, he replaces ξ by a "harmonic" one, i.e. by the minimum of a suitable variation problem. It is then shown to be G-invariant and a direct Lie algebra computation shows that it is zero if G has no factor which is either compact or locally isomorphic to $\mathbf{SL}_2(\mathbb{R})$.

Weil was in fact not a newcomer to group cohomology. In 1951, he had asked a student, Arnold Shapiro, to prove a certain lemma on the cohomology of finite groups. The latter complied and the lemma came up later in countless variations, all known as "Shapiro's lemma".

Weil never came back to these questions, but several further developments originated in these papers. If instead of $\mathfrak{g}$ we take $\mathbb{C}$ acted upon trivially, then $H^1(\Gamma;\mathbb{C})$ is trivial if and only if the commutator subgroup of Γ is of finite index in Γ. The vanishing of $H^1(\Gamma;\mathbb{C})$ was proved in many cases, using an approach similar to Weil's, by Matsushima, who extended it further to determine some higher cohomology groups (*Osaka J. Math.* **14** (1962), 1–20). Later I generalized Matsushima's theorem to noncocompact arithmetic groups, which yielded the determination of the rational K-groups of rings of algebraic integers (*Ann. Sci. École Norm. Sup. Paris* (4) **7** (1974), 235–272) and led to the study of higher regulators in algebraic K-theory (*Ann. Sci. École Norm. Sup. Pisa* (4) **4** (1977), 613–656). In another direction, N. Mok, Y.-T. Siu, and S.-K. Yeung used a nonlinear version of Matsushima's approach to establish archimedean superrigidity of cocompact discrete subgroups (*Invent. Math.* **113** (1993), 57–83).

This concludes my survey of algebraic topology in the work of A. Weil. Viewed as part of his overall output, it is quantitatively minor. Still, it reaches out to an impressive amount of mathematics, has been very influential, and testifies to the breadth of his outlook, as well as to his concentration on essential questions.

Weil 1987.

Photo by C. J. Mozzochi, provided couretsy of Armand Borel, with permission of the photographer.

References

[AW] A. Allendoerfer and A. Weil, The Gauss-Bonnet theorem for Riemannian polyhedra, *Trans. Amer. Math. Soc.* (VI) **53** (1943), 101–129; [W3], I, 299–327.

[B1] A. Borel, Jean Leray and algebraic topology, *Leray's Selected Papers I*, Springer and Soc. Math. France, 1997, pp. 1–21.

[C1] E. Calabi, On compact, Riemannian manifolds with constant curvature, I, *Differential Geometry*, Proc. Sympos. Pure Math. vol. III, Amer. Math. Soc., Providence, RI, 1961, pp. 155–180.

[CV] E. Calabi and E. Vesentini, On compact locally symmetric Kählerian manifolds, *Ann. Math.* **71** (1960), 472–507.

[C2] H. Cartan et al., Homotopie et espaces fibrés, Sém. École Norm. Sup., 1949–50.

[C3] H. Cartan, Notions d'algèbre différentielle; application aux groupes de Lie et aux variétés où opère un groupe de Lie, *Colloque de Topologie*, C.B.R.M., Bruxelles, 1950, pp. 15–27; *Collected Papers III*, pp. 1255–67.

[C4] S. S. Chern, Topics in differential geometry, Notes, Institute for Advanced Study, 1951.

[GHV] W. Greub, S. Halperin, and R. Vanstone, *Connections, Curvature and Cohomology*, Vol. 3, Pure Appl. Math. vol. 47, Academic Press, 1976.

[S1] A. Selberg, On discontinuous groups in higher-dimensional symmetric spaces, *Contributions to Function Theory*, Internat. Colloquium TIFR, Bombay, 1960, pp. 147–164.

[S2] J.-P. Serre, Faisceaux algébriques cohérents, *Ann. Math.* **61** (1955), 197–278; *Collected Papers I*, pp. 310–91.

[W1] A. Weil, *Foundations of Algebraic Geometry*, Colloquium Publ., vol. XXIX, Amer. Math. Soc., New York, NY, 1946; second edition, Providence, RI, 1962.

[W2] ____, Fibre spaces in algebraic geometry (Notes by A. Wallace), Notes, University of Chicago, 1952.

[W3] ____, *Collected Papers*, 3 volumes, Springer-Verlag, New York, 1979.