ON DIRICHLET PROBLEM FOR BELTRAMI EQUATIONS WITH TWO CHARACTERISTICS

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November 5, 2012 (BGR-DIRICHLET-ARXIV-02-11-2012.tex)

Abstract

We establish a series of criteria on the existence of regular solutions for the Dirichlet problem to general degenerate Beltrami equations $\overline{\partial} f = \mu \partial f + \nu \overline{\partial f}$ in arbitrary Jordan domains in $\mathbb{C}$.

2000 Mathematics Subject Classification: Primary 30C65; Secondary 30C75

1 Introduction

Let D be a domain in the complex plane $\mathbb{C}$. Throughout this paper we use the notations $z = x + iy$, $B(z_0, r) := \{z \in \mathbb{C} : |z - z_0| < r\}$ for $z_0 \in \mathbb{C}$ and $r > 0$, $\mathbb{B}(r) := B(0, r)$, $\mathbb{B} := \mathbb{B}(1)$, and $\overline{\mathbb{C}} := \mathbb{C} \cup \infty$.

The purpose of this paper is to study the Dirichlet problem

$$\left\{ \begin{array}{ll} f_{\overline{z}} = \mu(z) \cdot f_z + \nu(z) \cdot \overline{f_z}, & z \in D, \\ \lim\limits_{z \to \zeta} \operatorname{Re} f(z) = \varphi(\zeta), & \forall\, \zeta \in \partial D, \end{array} \right. \tag{1.1}$$

in a Jordan domain D of the complex plane $\mathbb{C}$ with continuous boundary data $\varphi(\zeta) \not\equiv \text{const}$. Here $\mu(z)$ and $\nu(z)$ stand for measurable coefficients satisfying the inequality $|\mu(z)| + |\nu(z)| < 1$ a.e. in D. The degeneracy of the ellipticity for the Beltrami equations

$$f_{\overline{z}} = \mu(z) \cdot f_z + \nu(z) \cdot \overline{f_z} \tag{1.2}$$

is controlled by the dilatation coefficient

$$K_{\mu,\nu}(z) := \frac{1 + |\mu(z)| + |\nu(z)|}{1 - |\mu(z)| - |\nu(z)|} \in L^1_{\text{loc}}. \tag{1.3}$$

We will look for a solution as a continuous, discrete and open mapping $f : D \to \mathbb{C}$ of the Sobolev class $W^{1,1}_{\text{loc}}$ and such that the Jacobian $J_f(z) \neq 0$ a.e. in D. Such a solution we will call a **regular solution** of the Dirichlet problem (1.1) in a domain D.

Recall that a mapping $f : D \to \mathbb{C}$ is called **discrete** if the preimage $f^{-1}(y)$ consists of isolated points for every $y \in \mathbb{C}$, and **open** if f maps every open set $U \subseteq D$ onto an open set in $\mathbb{C}$.

For the uniformly elliptic case, i.e. when $K_{\mu,\nu}(z) \le K < \infty$ a.e. in D the Dirichlet problem was studied in [2] and [31]. The solvability of the Dirichlet problem in the partial case, when $\nu(z) = 0$ and the degeneracy of the ellipticity for the Beltrami equations

$$f_{\bar{z}} = \mu(z) \cdot f_z \tag{1.4}$$

is controlled by the dilatation coefficient

$$K_\mu(z) = K_{\mu,0}(z) = \frac{1 + |\mu(z)|}{1 - |\mu(z)|} \notin L^\infty, \tag{1.5}$$

is given in [10], [14] and [20].

Recall that the problem on existence of homeomorphic solutions for the equation (1.4) was resolved for the uniformly elliptic case when $\|\mu\|_\infty < 1$ long ago, see e.g. [1], [2], [21]. The existence problem for the degenerate Beltrami equations (1.4) when $K_\mu \notin L^\infty$ is currently an active area of research, see e.g. the monographs [14] and [22] and the surveys [13] and [29] and further references therein. A series of criteria on the existence of regular solutions for the Beltrami equation (1.2) were given in our recent papers [4]–[6]. There we called a homeomorphism $f \in W^{1,1}_{\text{loc}}(D)$ by a **regular solution** of (1.2) if f satisfies (1.2) a.e. in D and $J_f(z) = |f_z|^2 - |f_{\bar{z}}|^2 \neq 0$ a.e. in D.

2 Preliminaries

To derive criteria for existence of regular solutions for the Dirichlet problem (1.1) in a Jordan domain $D \in \mathbb{C}$ we make use of the approximate procedure based on the existence theorems for the case $K_{\mu,\nu} \in L^\infty$ given in [2] and convergence theorems for the Beltrami equations (1.2) when $K_{\mu,\nu} \in L^1_{\text{loc}}$ established in [5]. The Schwarz formula

$$f(z) = i\,\mathrm{Im}\, f(0) + \frac{1}{2\pi i} \int\limits_{|\zeta|=1} \mathrm{Re}\, f(\zeta) \cdot \frac{\zeta + z}{\zeta - z} \frac{d\zeta}{\zeta}\,, \tag{2.1}$$

that allows to recover an analytic function f in the unit disk $\mathbb{B}$ by its real part $\varphi(\zeta) = \mathrm{Re}\, f(\zeta)$ on the boundary of $\mathbb{B}$ up to a purely imaginary additive constant $c = i\mathrm{Im}\, f(0)$, see, e.g., Section 8, Chapter III, Part 3 in [16], as well as the Arzela–Askoli theorem combined with moduli techniques are also used.

The following statement, that is a consequence of Theorems 5.1 and 6.1 and the point 8.1 in [2], is basic for our further considerations. See also Theorem VI.2.2 and the point VI.2.3 in [21], on the regularity of a $W^{1,1}_{\text{loc}}$ solution to the Beltrami equation (1.4) with the bounded dilatation coefficient K_μ.

2.2. Proposition. *Let D, $0 \in D$, be a Jordan domain in the complex plane $\mathbb{C}$ and $\varphi : \partial D \to \mathbb{R}$ be a nonconstant continuous function. If $K_{\mu,\nu} \in L^\infty$, then the Dirichlet problem (1.1) has the unique regular solution f normalized by* $\mathrm{Im} f(0) = 0$. *This solution has the representation*

$$f = \mathcal{A} \circ g \circ \mathcal{R} \tag{2.3}$$

where $\mathcal{R} : D \to \mathbb{B}$, $\mathcal{R}(0) = 0$, *is a conformal mapping and* $g : \overline{\mathbb{B}} \to \overline{\mathbb{B}}$ *stands for a homeomorphic regular solution of the quasilinear equation*

$$g_{\overline{\zeta}} \;=\; \mu^*(\zeta) \cdot g_\zeta + \nu^*(\zeta) \cdot \frac{\overline{\mathcal{A}'(g(\zeta))}}{\mathcal{A}'(g(\zeta))} \cdot \overline{g_\zeta} \tag{2.4}$$

in $\mathbb{B}$ *normalized by* $g(0) = 0$, $g(1) = 1$. *Here* $\mu* = \frac{\overline{\mathcal{R}'}}{\mathcal{R}'} \cdot \mu \circ \mathcal{R}^{-1}$, $\nu* = \nu \circ \mathcal{R}^{-1}$ *and*

$$\mathcal{A}(w) := \frac{1}{2\pi i} \int\limits_{|\omega|=1} \varphi(\mathcal{R}^{-1}(g^{-1}(\omega))) \cdot \frac{\omega + w}{\omega - w} \frac{d\omega}{\omega} \tag{2.5}$$

is an analytic function in the unit disk $\mathbb{B}$.

2.6. Remark. Let $\tilde{\mu} : \mathbb{C} \to \mathbb{C}$ coincide a.e in the domain D with

$$\frac{(g \circ \mathcal{R})_{\overline{z}}}{(g \circ \mathcal{R})_z} = \frac{g_{\overline{\zeta}} \circ \mathcal{R} \cdot \overline{\mathcal{R}'}}{g_\zeta \circ \mathcal{R} \cdot \mathcal{R}'} = \mu + \nu \cdot \frac{\overline{\mathcal{R}'}}{\mathcal{R}'} \cdot \frac{\overline{g_\zeta}}{g_\zeta} \circ \mathcal{R} \cdot \frac{\overline{\mathcal{A}'}}{\mathcal{A}'} \circ g \circ \mathcal{R} \tag{2.7}$$

and equal to 0 outside of D, see e.g. the formulas I.C(1) in [1]. Note that $K_{\tilde{\mu}} \le K_{\mu,\nu}$ a.e. in D and there is a regular solution $G : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ of the equation $G_{\overline{z}} = \tilde{\mu} G_z$ such that $G(0) = 0$, $|G(\mathcal{R}^{-1}(1))| = 1$, $G(\infty) = \infty$ and $G = \mathcal{H} \circ g \circ \mathcal{R}$ in $\overline{D}$. Here $\mathcal{H} : \mathbb{B} \to G(D)$ is a conformal mapping normalized by $\mathcal{H}(0) = 0$, $\mathcal{H}'(0) > 0$. Thus,

$$f = \mathcal{A} \circ h, \tag{2.8}$$

$$\mathcal{A}(w) = \frac{1}{2\pi i} \int\limits_{|\omega|=1} \varphi(h^{-1}(\omega)) \cdot \frac{\omega + w}{\omega - w} \frac{d\omega}{\omega} \tag{2.9}$$

where

$$h = g \circ \mathcal{R} = \mathcal{H}^{-1} \circ G \tag{2.10}$$

stands for a homeomorphism $h : \overline{D} \to \overline{\mathbb{B}}$, $h(0) = 0$, which is a regular solution in D of the quasilinear equation

$$h_{\overline{z}} \;=\; \mu(z) \cdot h_z + \nu(z) \cdot \frac{\overline{\mathcal{A}'(h(z))}}{\mathcal{A}'(h(z))} \cdot \overline{h_z} \tag{2.11}$$

Denote such f, g, $\mathcal{A}$, G, $\mathcal{H}$ and h by $f_{\mu,\nu,\varphi}$, $g_{\mu,\nu,\varphi}$, $\mathcal{A}_{\mu,\nu,\varphi}$ $G_{\mu,\nu,\varphi}$, $\mathcal{H}_{\mu,\nu,\varphi}$ and $h_{\mu,\nu,\varphi}$, respectively.

Recall also that, given a family of paths Γ in $\mathbb{C}$, a Borel function $\rho : \mathbb{C} \to [0, \infty]$ is called **admissible** for Γ, abbr. $\rho \in adm\,\Gamma$, if

$$\int\limits_\gamma \rho(z)\,|dz| \;\ge\; 1 \tag{2.12}$$

for each $\gamma \in \Gamma$. The **modulus** of Γ is defined by

$$M(\Gamma) = \inf_{\rho \in adm\,\Gamma} \int\limits_{\mathbb{C}} \rho^2(z)\; dxdy\ . \tag{2.13}$$

2.14. Remark. Note the following useful fact for a quasiconformal mapping $f : D \to \mathbb{C}$, see e.g. V(6.6) in [21], that

$$M(f(\Gamma)) \leq \int_{\mathbb{C}} K(z) \cdot \rho^2(z)\, dxdy \tag{2.15}$$

for every path family Γ in D and for all $\rho \in adm\,\Gamma$ where

$$K(z) = \frac{|f_z| + |f_{\overline{z}}|}{|f_z| - |f_{\overline{z}}|} \tag{2.16}$$

is the (local) maximal dilatation of the mapping f at a point $z \in D$.

Given a domain D and two sets E and F in $\overline{\mathbb{C}}$, $\Delta(E, F, D)$ denotes the family of all paths $\gamma : [a, b] \to \overline{\mathbb{C}}$ which join E and F in D, i.e., $\gamma(a) \in E$, $\gamma(b) \in F$ and $\gamma(t) \in D$ for $a < t < b$. Recall that a **ring domain**, or shortly a **ring** in $\overline{\mathbb{C}}$ is a domain R whose complement $\overline{\mathbb{C}} \setminus R$ consists of two connected components.

Recall that, for points $z, \zeta \in \overline{\mathbb{C}}$, the **spherical (chordal) distance** $s(z, \zeta)$ between z and ζ is given by

$$s(z,\zeta) = \frac{|z-\zeta|}{(1+|z|^2)^{\frac{1}{2}}(1+|\zeta|^2)^{\frac{1}{2}}} \quad \text{if} \quad z \neq \infty \neq \zeta\,, \tag{2.17}$$

$$s(z,\infty) = \frac{1}{(1+|z|^2)^{\frac{1}{2}}} \quad \text{if} \quad z \neq \infty\,.$$

By $\delta(A)$ we denote the spherical diameter of a set $A \subset \mathbb{C}$, i.e. $\sup\limits_{z,\zeta \in A} s(z, \zeta)$.

The following statement is a direct consequence of the known estimate of the capacity of a ring formulated in terms of moduli, see e.g. Lemma 2.16 in [5].

2.18. Lemma. *Let $f : D \to \mathbb{C}$ be a homeomorphism with $\delta(\overline{\mathbb{C}} \setminus f(D)) \geq \Delta > 0$ and let z_0 be a point in D, $\zeta \in B(z_0, r_0)$, $r_0 < dist\,(z_0, \partial D)$. Then*

$$s(f(\zeta), f(z_0)) \leq \frac{32}{\Delta} \cdot exp\left(-\frac{2\pi}{M(\Delta(fC, fC_0, fA))}\right) \tag{2.19}$$

where $C_0 = \{z \in \mathbb{C} : |z - z_0| = r_0\}$, $C = \{z \in \mathbb{C} : |z - z_0| = |\zeta - z_0|\}$ and $A = \{z \in \mathbb{C} : |\zeta - z_0| < |z - z_0| < r_0\}$.

3 BMO, VMO and FMO functions

Recall that a real-valued function u in a domain D in $\mathbb{C}$ is said to be of **bounded mean oscillation** in D, abbr. $u \in \mathrm{BMO}(D)$, if $u \in L^1_{\mathrm{loc}}(D)$ and

$$\|u\|_* := \sup_B \frac{1}{|B|} \int_B |u(z) - u_B|\, dxdy < \infty\,, \tag{3.1}$$

where the supremum is taken over all discs B in D and

$$u_B = \frac{1}{|B|}\int\limits_B u(z)\,dxdy\,.$$

We write $u \in \mathrm{BMO}_{\mathrm{loc}}(D)$ if $u \in \mathrm{BMO}(U)$ for every relatively compact subdomain U of D (we also write BMO or $\mathrm{BMO}_{\mathrm{loc}}$ if it is clear from the context what D is).

The class BMO was introduced by John and Nirenberg (1961) in the paper [18] and soon became an important concept in harmonic analysis, partial differential equations and related areas, see e.g. [15] and [24].

A function u in BMO is said to have **vanishing mean oscillation**, abbr. $u \in$ VMO, if the supremum in (3.1) taken over all balls B in D with $|B| < \varepsilon$ converges to 0 as $\varepsilon \to 0$. VMO has been introduced by Sarason in [28]. There exists a number of papers devoted to the study of partial differential equations with coefficients of the class VMO.

3.2. Remark. Note that $W^{1,2}(D) \subset VMO\,(D)$, see e.g. [7].

Following [17], we say that a function $u : D \to \mathbb{R}$ has **finite mean oscillation** at a point $z_0 \in D$ if

$$\overline{\lim_{\varepsilon\to 0}} \quad ⨍_{B(z_0,\varepsilon)} |u(z) - \tilde{u}_\varepsilon(z_0)|\; dxdy \;<\; \infty \tag{3.3}$$

where

$$\tilde{u}_\varepsilon(z_0) = ⨍_{B(z_0,\varepsilon)} u(z)\; dxdy$$

is the mean value of the function $u(z)$ over the disk $B(z_0,\varepsilon)$ with small $\varepsilon > 0$. We also say that a function $u : D \to \mathbb{R}$ is of **finite mean oscillation** in D, abbr. $u \in$ FMO(D) or simply $u \in$ **FMO**, if (3.3) holds at every point $z_0 \in D$.

3.4. Remark. Clearly BMO $\subset$ FMO. There exist examples showing that FMO is not $\mathrm{BMO}_{\mathrm{loc}}$, see e.g. [14]. By definition FMO $\subset L^1_{\mathrm{loc}}$ but FMO is not a subset of L^p_{loc} for any $p > 1$ in comparison with $\mathrm{BMO}_{\mathrm{loc}} \subset L^p_{\mathrm{loc}}$ for all $p \in [1,\infty)$.

3.5. Proposition. *If, for some collection of numbers $u_\varepsilon \in \mathbb{R}$, $\varepsilon \in (0,\varepsilon_0]$,*

$$\overline{\lim_{\varepsilon\to 0}} \quad ⨍_{B(z_0,\varepsilon)} |u(z) - u_\varepsilon|\; dxdy \;< \infty\,, \tag{3.6}$$

then u is of finite mean oscillation at z_0.

3.7. Corollary. *If, for a point $z_0 \in D$,*

$$\overline{\lim_{\varepsilon\to 0}} \; ⨍_{B(z_0,\varepsilon)} |u(z)|\; dxdy \;<\; \infty\;, \tag{3.8}$$

then u has finite mean oscillation at z_0.

3.9. Remark. Note that the function $u(z) = \log \frac{1}{|z|}$ belongs to BMO in the unit disk $\mathbb{B}$, see e.g. [24], p. 5, and hence also to FMO. However, $\tilde{u}_\varepsilon(0) \to \infty$ as $\varepsilon \to 0$, showing that the condition (3.8) is only sufficient but not necessary for a function u to be of finite mean oscillation at z_0.

Below we use the notation $A(\varepsilon, \varepsilon_0) = \{z \in \mathbb{C} : \varepsilon < |z| < \varepsilon_0\}$.

3.10. Lemma. *Let $u : D \to \mathbb{R}$ be a nonnegative function with finite mean oscillation at $0 \in D$ and let u be integrable in $B(0, e^{-1}) \subset D$. Then*

$$\int\limits_{A(\varepsilon, e^{-1})} \frac{u(z)\, dxdy}{\left(|z| \log \frac{1}{|z|}\right)^2} \leq C \cdot \log\log \frac{1}{\varepsilon} \qquad \forall\, \varepsilon \in (0, e^{-e}) \tag{3.11}$$

For the proof of this lemma, see [17].

4 The main lemma

The following lemma is the main tool for deriving criteria on the existence of regular solutions for the Dirichlet problem to the Beltrami equations with two characteristics in a Jordan domain in $\mathbb{C}$.

4.1. Lemma. *Let D be a Jordan domain in $\mathbb{C}$ with $0 \in D$ and let μ and $\nu : D \to \mathbb{C}$ be measurable functions with $K_{\mu,\nu} \in L^1(D)$. Suppose that for every $z_0 \in \overline{D}$ there exist $\varepsilon_0 = \varepsilon(z_0) > 0$ and a family of measurable functions $\psi_{z_0,\varepsilon} : (0, \infty) \to (0, \infty)$, $\varepsilon \in (0, \varepsilon_0)$, such that*

$$0 < I_{z_0}(\varepsilon) := \int\limits_{\varepsilon}^{\varepsilon_0} \psi_{z_0,\varepsilon}(t)\, dt < \infty, \tag{4.2}$$

and such that

$$\int\limits_{\varepsilon < |z - z_0| < \varepsilon_0} K_{\mu,\nu}(z) \cdot \psi^2_{z_0,\varepsilon}(|z - z_0|)\, dxdy = o(I^2_{z_0}(\varepsilon)) \tag{4.3}$$

as $\varepsilon \to 0$. Then the the Dirichlet problem (1.1) has a regular solution f with $\mathrm{Im} f(0) = 0$ for each nonconstant continuous function $\varphi : \partial D \to \mathbb{R}$.

Here we assume that μ and ν are extended by zero outside of the domain D.

Proof. Setting

$$\mu_n(z) = \begin{cases} \mu(z), & \text{if } K_{\mu,\nu}(z) \leq n, \\ 0, & \text{otherwise in } \mathbb{C}, \end{cases} \tag{4.4}$$

and

$$\nu_n(z) = \begin{cases} \nu(z), & \text{if } K_{\mu,\nu}(z) \leq n, \\ 0, & \text{otherwise in } \mathbb{C}, \end{cases} \tag{4.5}$$

we have that $K_{\mu_n,\nu_n}(z) \le n$ in $\mathbb{C}$. Denote by f_n, $\mathcal{A}_n$, G_n, $\mathcal{H}_n$ and h_n, the functions $f_{\mu_n,\nu_n,\varphi}$, $\mathcal{A}_{\mu_n,\nu_n,\varphi}$ $G_{\mu_n,\nu_n,\varphi}$, $\mathcal{H}_{\mu_n,\nu_n,\varphi}$ and $h_{\mu_n,\nu_n,\varphi,}$, respectively, from Proposition 2.2 and Remark 2.6.

Let Γ_ε be a family of all paths joining the circles $C_\varepsilon = \{z \in \mathbb{C} : |z - z_0| = \varepsilon\}$ and $C_0 = \{z \in \mathbb{C} : |z - z_0| = \varepsilon_0\}$ in the ring $A_\varepsilon = \{z \in \mathbb{C} : \varepsilon < |z - z_0| < \varepsilon_0\}$. Let also ψ^* be a Borel function such that $\psi^*(t) = \psi(t)$ for a.e. $t \in (0, \infty)$. Such a function ψ^* exists by the Lusin theorem, see e.g. [27], p. 69. Then the function

$$\rho_\varepsilon(z) = \begin{cases} \psi^*(|z-z_0|)/I_{z_0}(\varepsilon), & \text{if } z \in A_\varepsilon, \\ 0, & \text{if } z \in \mathbb{C} \setminus A_\varepsilon, \end{cases}$$

is admissible for Γ_ε. Hence by Remark 2.14 applied to G_n

$$M(G_n\Gamma_\varepsilon) \le \int\limits_{\varepsilon<|z-z_0|<\varepsilon_0} K_{\mu,\nu}(z) \cdot {\rho_\varepsilon}^2(|z-z_0|)\ dxdy\,,$$

and, by the condition (4.3), $M(G_n\Gamma_\varepsilon) \to 0$ as $\varepsilon \to 0$ uniformly with respect to the parameter $n = 1, 2, \ldots$.

Thus, in view of the normalization $G_n(0) = 0$, $|G_n(\mathcal{R}^{-1}(1))| = 1$, $G_n(\infty) = \infty$, the sequence G_n is equicontinuous in $\overline{\mathbb{C}}$ with respect to the spherical distance by Lemma 2.18 with $\Delta = 1/\sqrt{2}$. Consequently, by the Arzela–Ascoli theorem, see e.g. [8], p. 267, and [9], p. 382, it has a subsequence G_{n_l} which converges uniformly in $\overline{\mathbb{C}}$ with respect to the spherical metric to a continuous mapping G in $\overline{\mathbb{C}}$ with the normalization $G(0) = 0$, $|G(\mathcal{R}^{-1}(1))| = 1$, $G(\infty) = \infty$. Note that $G : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is a homeomorphism of the class $W^{1,1}_{\rm loc}(\mathbb{C})$ by Corollary 3.8 in [5].

Hence by the Rado theorem, see e.g. Theorem II.5.2 in [12], $\mathcal{H}_{n_l} \to \mathcal{H}$ as $l \to \infty$ uniformly in $\overline{\mathbb{B}}$ where $\mathcal{H} : \overline{\mathbb{B}} \to G(\overline{D})$ is the conformal mapping of $\mathbb{B}$ onto $G(D)$ with the normalization $\mathcal{H}(0) = 0$ and $\mathcal{H}'(0) > 0$. Moreover, since the locally uniform convergence $G_{n_l} \to G$ and $\mathcal{H}_{n_l} \to \mathcal{H}$ of the sequences G_{n_l} and $\mathcal{H}_{n_l}$ is equivalent to their continuous convergence, i.e., $G_{n_l}(z_l) \to G(z_*)$ if $z_l \to z_*$ and $\mathcal{H}_{n_l}(\zeta_l) \to \mathcal{H}(\zeta_*)$ if $\zeta_l \to \zeta_*$, see [Du], p. 268, and since G and $\mathcal{H}$ are injective, it follows that $G^{-1}_{n_l} \to G^{-1}$ and $\mathcal{H}^{-1}_{n_l} \to \mathcal{H}^{-1}$ continuously, and hence locally uniformly.

Then we have that $\mathcal{A}_{n_l} \to \mathcal{A}$ locally uniformly in $\mathbb{B}$ where

$$\mathcal{A}(w) = \frac{1}{2\pi i} \int\limits_{|\omega|=1} \varphi(h^{-1}(\omega)) \cdot \frac{\omega + w}{\omega - w}\frac{d\omega}{\omega} \tag{4.6}$$

where $h : \overline{D} \to \overline{\mathbb{B}}$, $h(0) = 0$, is a homeomorphism $h = \mathcal{H}^{-1} \circ G$. Note that $\mathcal{A}_{n_l}$ and $\mathcal{A}$ are not constant and hence $\mathcal{A}'_{n_l}$ and $\mathcal{A}'$ have only isolated zeros. The collection of all such zeros is countable. Thus, by Theorem 3.1 and Corollary 3.8 in [5] $h_{n_l} \to h$ locally uniformly in D and h is a homeomorphic $W^{1,1}_{\rm loc}$ solution in D of the quasilinear equation

$$h_{\overline{z}} \;=\; \mu(z) \cdot h_z + \nu(z) \cdot \frac{\overline{\mathcal{A}'(h(z))}}{\mathcal{A}'(h(z))} \cdot \overline{h_z} \tag{4.7}$$

Hence $f_{n_l} \to f$ where $f = \mathcal{A} \circ h$ is a continuous discrete open $W^{1,1}_{\rm loc}$ solution in D of (1.2).

Next, note that $\mathrm{Re}\,\mathcal{A}_{n_l} \to \mathrm{Re}\,\mathcal{A}$ uniformly in $\overline{\mathbb{B}}$ by the maximum principle for harmonic functions and $\mathrm{Re}\,\mathcal{A} = \varphi \circ h^{-1}$ on $\partial\mathbb{B}$ and, consequently, $\mathrm{Re}\, f_{n_l} \to \mathrm{Re}\, f$ uniformly in $\overline{\mathbb{B}}$ and $\mathrm{Re}\, f = \varphi$ on ∂D, i.e., f is a continuous discrete open $W^{1,1}_{\rm loc}$ solution of the Dirichlet problem (1.1) in $\mathbb{B}$ to the equation (1.2). It remains to show that $J_f(z) \neq 0$ a.e. in $\mathbb{B}$.

By a change of variables which is permitted because h_{n_l} and $\tilde{h}_{n_l} = h^{-1}_{n_l}$ belong to the class $W^{1,2}_{\rm loc}$, see e.g. Lemmas III.2.1 and III.3.2 and Theorems III.3.1 and III.6.1 in [21], we obtain that for large enough l

$$\int\limits_{B} |\partial \tilde{h}_{n_l}|^2 \; dudv \;\leq \int\limits_{\tilde{h}_{n_l}(B)} \frac{dxdy}{1-k_l(z)^2} \;\leq \int\limits_{B^*} K_{\mu,\nu}(z)\; dxdy \;<\; \infty \tag{4.8}$$

where $k_l(z) = |\mu_{n_l}(z)| + |\nu_{n_l}(z)|$ and B^* and B are relatively compact domains in D and $\tilde{h}(D)$, respectively, such that $\tilde{h}(\bar{B}) \subset B^*$. The relation (4.8) implies that the sequence $\tilde{h}_{n_l}$ is bounded in $\mathrm{W}^{1,2}(B)$, and hence $h^{-1} \in \mathrm{W}^{1,2}_{\rm loc}$, see e.g. Lemma III.3.5 in [25] or Theorem 4.6.1 in [11]. The latter condition brings in turn that h has $(N^{-1})-$property, see e.g. Theorem III.6.1 in [21], and hence $J_h(z) \neq 0$ a.e., see Theorem 1 in [23]. Thus, $f = \mathcal{A} \circ h$ is a regular solution of the Dirichlet problem (1.1) to the equation (1.2).

4.9. Corollary. *Let D be a Jordan domain in $\mathbb{C}$ with $0 \in D$ and let μ, $\nu : \mathbb{B} \to \mathbb{C}$ be measurable functions with $K_{\mu,\nu} \in L^1(\mathbb{B})$. Suppose that for every $z_0 \in \overline{\mathbb{B}}$ and some $\varepsilon_0 > 0$*

$$\int\limits_{\varepsilon<|z-z_0|<\varepsilon_0} K_{\mu,\nu}(z) \cdot \psi^2(|z-z_0|)\; dxdy \;\leq\; O\left(\int\limits_{\varepsilon}^{\varepsilon_0} \psi(t)\; dt\right) \tag{4.10}$$

as $\varepsilon \to 0$, where $\psi : (0,\infty) \to (0,\infty)$ is a measurable function such that

$$\int\limits_{0}^{\varepsilon_0} \psi(t)\; dt \;=\; \infty\;, \quad 0 \;<\; \int\limits_{\varepsilon}^{\varepsilon_0} \psi(t)\; dt < \;\infty \qquad \forall\, \varepsilon \in (0,\varepsilon_0)\;. \tag{4.11}$$

Then the the Dirichlet problem (1.1) has a regular solution f with $\mathrm{Im} f(0) = 0$ for each nonconstant continuous function $\varphi : \partial D \to \mathbb{R}$.

5 Existence theorems

Everywhere further we assume that the functions μ and $\nu : D \to \mathbb{C}$ are extended by zero outside of the domain D.

5.1. Theorem. *Let D be a Jordan domain in $\mathbb{C}$ with $0 \in D$ and let μ and $\nu : D \to \mathbb{C}$ be measurable functions such that $K_{\mu,\nu}(z) \;\leq\; Q(z) \;\in\; FMO$. Then* *the the Dirichlet problem (1.1) has a regular solution f with $\mathrm{Im} f(0)=0$ for each nonconstant continuous function $\varphi:\partial D\to\mathbb{R}$.*

Proof. Lemma 4.1 yields this conclusion by choosing

$$\psi_{z_0,\varepsilon}(t) \;=\; \frac{1}{t\log\frac{1}{t}}\;, \tag{5.2}$$

see also Lemma 3.10.

5.3. Corollary. *In particular, if*

$$\overline{\lim_{\varepsilon\to 0}}\;\;⨍_{B(z_0,\varepsilon)}\frac{1+|\nu(z)|}{1-|\nu(z)|}\,dxdy\;<\;\infty\qquad\forall\, z_0\in\overline{D}\,, \tag{5.4}$$

Then the the Dirichlet problem

$$\begin{cases} f_{\overline{z}} \,=\, \nu(z)\cdot\overline{f_z}, & z\in D,\\ \lim\limits_{z\to\zeta}\mathrm{Re}\,f(z)=\varphi(\zeta), & \forall\,\zeta\in\partial D, \end{cases} \tag{5.5}$$

in a Jordan domain D, $0\in D$, has a regular solution f with $\mathrm{Im} f(0)=0$ for each nonconstant continuous function $\varphi:\partial D\to\mathbb{R}$.

Similarly, choosing in Lemma 4.1 the function $\psi(t)=1/t$, we come to the following statement.

5.6. Theorem. *Let D be a Jordan domain in $\mathbb{C}$ with $0\in D$ and let μ and $\nu:D\to\mathbb{C}$ be measurable functions such that $K_{\mu,\nu}\in\ L^1_{\mathrm{loc}}(D)$. Suppose that*

$$\int\limits_{\varepsilon<|z-z_0|<\varepsilon_0} K_{\mu,\nu}(z)\,\frac{dm(z)}{|z-z_0|^2}\;=\;o\left(\left[\log\frac{1}{\varepsilon}\right]^2\right)\qquad\forall\; z_0\in\overline{D} \tag{5.7}$$

as $\varepsilon\to 0$ for some $\varepsilon_0=\delta(z_0)$. Then the the Dirichlet problem (1.1) has a regular solution f with $\mathrm{Im} f(0)=0$ for each nonconstant continuous function $\varphi:\partial D\to\mathbb{R}$.

5.8. Remark. Choosing in Lemma 4.1 the function $\psi(t)=1/(t\log 1/t)$ instead of $\psi(t)=1/t$, we are able to replace (5.7) by

$$\int\limits_{\varepsilon<|z-z_0|<\varepsilon_0}\frac{K_{\mu,\nu}(z)\;dm(z)}{\left(|z-z_0|\log\frac{1}{|z-z_0|}\right)^2}=o\left(\left[\log\log\frac{1}{\varepsilon}\right]^2\right) \tag{5.9}$$

In general, we are able to give here the whole scale of the corresponding conditions in log using functions $\psi(t)$ of the form $1/(t\log 1/t\cdot\log\log 1/t\cdot\ldots\cdot\log\ldots\log 1/t)$.

5.10. Theorem. *Let D be a Jordan domain in $\mathbb{C}$ with $0\in D$ and let μ, $\nu:D\to\mathbb{B}$ be measurable functions, $K_{\mu,\nu}\in L^1(D)$ and $k_{z_0}(r)$ be the mean value of $K_{\mu,\nu}(z)$ over the circle $|z-z_0|=r$. Suppose that*

$$\int\limits_0^{\delta(z_0)}\frac{dr}{rk_{z_0}(r)}\;=\;\infty\qquad\forall\; z_0\in\overline{D}\,. \tag{5.11}$$

Then the the Dirichlet problem (1.1) has a regular solution f with $\mathrm{Im} f(0) = 0$ *for each nonconstant continuous function* $\varphi : \partial D \to \mathbb{R}$.

Proof. Theorem 5.10 follows from Lemma 4.1 by special choosing the functional parameter

$$\psi_{z_0,\varepsilon}(t) \;\equiv\; \psi_{z_0}(t) := \begin{cases} 1/[tk_{z_0}(t)]\ , & t \in (0,\varepsilon_0)\ , \\ 0\ , & \text{otherwise} \end{cases} \tag{5.12}$$

where $\varepsilon_0 = \delta(z_0)$.

5.13. Corollary. *In particular, the conclusion of Theorem 5.10 holds if*

$$k_{z_0}(r) \;=\; O\left(\log\frac{1}{r}\right) \quad \text{as} \quad r \to 0 \qquad \forall\ z_0 \in \overline{D}\ . \tag{5.14}$$

In fact, it is clear that the condition (5.11) implies the whole scale of conditions in terms of log with using in the right hand side in (5.14) functions of the form $\log 1/r \cdot \log\log 1/r \cdot \ldots \cdot \log \ldots \log 1/r$.

In the theory of mappings called quasiconformal in the mean, conditions of the type

$$\int\limits_D \Phi(Q(z))\ dxdy \;<\; \infty \tag{5.15}$$

are standard for various characteristics of these mappings. In this connection, in the paper [26], see also the monograph [14], it was established the equivalence of various integral conditions on the function Φ. We give here the conditions for Φ under which (5.15) implies (5.11).

Further we use the following notion of the inverse function for monotone functions. Namely, for every non-decreasing function $\Phi : [0,\infty] \to [0,\infty]$, the **inverse function** $\Phi^{-1} : [0,\infty] \to [0,\infty]$ can be well defined by setting

$$\Phi^{-1}(\tau) \;=\; \inf_{\Phi(t)\geq\tau} t\ . \tag{5.16}$$

As usual, here inf is equal to ∞ if the set of $t \in [0,\infty]$ such that $\Phi(t) \geq \tau$ is empty. Note that the function Φ^{-1} is non-decreasing, too.

5.17. Remark. It is evident immediately by the definition that

$$\Phi^{-1}(\Phi(t)) \;\leq\; t \qquad \forall\ t \in [0,\infty] \tag{5.18}$$

with the equality in (5.18) except intervals of constancy of the function Φ.

Further, in (5.21) and (5.22), we complete the definition of integrals by ∞ if $\Phi(t) = \infty$, correspondingly, $H(t) = \infty$, for all $t \geq T \in [0,\infty)$. The integral in (5.22) is understood as the Lebesgue–Stieltjes integral and the integrals (5.21) and (5.23)–(5.26) as the ordinary Lebesgue integrals.

5.19. Proposition. *Let $\Phi : [0,\infty] \to [0,\infty]$ be a non-decreasing function and set*

$$H(t) \ = \ \log \Phi(t)\ . \tag{5.20}$$

Then the equality

$$\int\limits_{\Delta}^{\infty} H'(t)\ \frac{dt}{t} \ = \ \infty \tag{5.21}$$

implies the equality

$$\int\limits_{\Delta}^{\infty} \frac{dH(t)}{t} \ = \ \infty \tag{5.22}$$

and (5.22) is equivalent to

$$\int\limits_{\Delta}^{\infty} H(t)\ \frac{dt}{t^2} \ = \ \infty \tag{5.23}$$

for some $\Delta > 0$, and (5.23) is equivalent to every of the equalities:

$$\int\limits_{0}^{\delta} H\left(\frac{1}{t}\right)\ dt \ = \ \infty \tag{5.24}$$

for some $\delta > 0$,

$$\int\limits_{\Delta_*}^{\infty} \frac{d\eta}{H^{-1}(\eta)} \ = \ \infty \tag{5.25}$$

for some $\Delta_* > H(+0)$,

$$\int\limits_{\delta_*}^{\infty} \frac{d\tau}{\tau \Phi^{-1}(\tau)} \ = \ \infty \tag{5.26}$$

for some $\delta_* > \Phi(+0)$.

Moreover, (5.21) is equivalent to (5.22) and hence (5.21)–(5.26) are equivalent each to other if Φ is in addition absolutely continuous. In particular, all the conditions (5.21)–(5.26) are equivalent if Φ is convex and non–decreasing.

Finally, we give the connection of the above conditions with the condition of the type (5.11).

Recall that a function $\psi : [0,\infty] \to [0,\infty]$ is called **convex** if $\psi(\lambda t_1 + (1-\lambda)t_2) \leq \lambda\psi(t_1) + (1-\lambda)\psi(t_2)$ for all t_1 and $t_2 \in [0,\infty]$ and $\lambda \in [0,1]$.

5.27. Proposition. *Let $Q : \mathbb{B} \to [0,\infty]$ be a measurable function such that*

$$\int\limits_{\mathbb{B}} \Phi(Q(z))\ dxdy \ < \ \infty \tag{5.28}$$

where $\Phi : [0,\infty] \to [0,\infty]$ *is a non-decreasing convex function such that*

$$\int\limits_{\delta}^{\infty} \frac{d\tau}{\tau\Phi^{-1}(\tau)} \ = \ \infty \tag{5.29}$$

for some $\delta \ > \Phi(0)$. *Then*

$$\int\limits_{0}^{1} \frac{dr}{rq(r)} \ = \ \infty \tag{5.30}$$

where $q(r)$ *is the average of the function* $Q(z)$ *over the circle* $|z| = r$.

Finally, combining Propositions 5.19 and 5.27 we obtain the following conclusion.

5.31. Corollary. *If* $\Phi : [0,\infty] \to [0,\infty]$ *is a non-decreasing convex function and* Q *satisfies the condition (5.28), then every of the conditions (5.21)–(5.26) implies (5.30).*

Immediately on the basis of Theorem 5.10 and Corollary 5.31, we obtain the next significant result.

5.32. Theorem. *Let* D *be a Jordan domain in* $\mathbb{C}$ *with* $0 \in D$ *and let* μ *and* $\nu : D \to \mathbb{C}$ *be measurable functions such that*

$$\int\limits_{D} \Phi(K_{\mu,\nu}(z))\ dxdy \ < \ \infty \tag{5.33}$$

where $\Phi : [0,\infty] \to [0,\infty]$ *is a non-decreasing convex function satisfying at least one of the conditions (5.21)–(5.26). Then the the Dirichlet problem (1.1) has a regular solution* f *with* $\mathrm{Im} f(0) = 0$ *for each nonconstant continuous function* $\varphi : \partial D \to \mathbb{R}$.

On the same basis, we obtain the following consequence.

5.34. Corollary. *In particular, the conclusion of Theorem 5.32 holds if*

$$\int\limits_{D\cap U_{z_0}} e^{\alpha(z_0)K_{\mu,\nu}(z)}\, dxdy \ < \infty \qquad \forall\ z_0 \in \overline{D} \tag{5.35}$$

for some $\alpha(z_0) > 0$ *and a neighborhood* U_{z_0} *of the point* z_0.

5.36. Remark. By the Stoilow theorem, see e.g. [30], every regular solution f to the Dirichlet problem

$$\begin{cases} f_{\overline{z}} \ = \ \mu(z) \cdot f_z, & z \in D, \\ \lim\limits_{z\to\zeta} \mathrm{Re}\, f(z) = \varphi(\zeta), & \forall\ \zeta \in \partial D, \end{cases} \tag{5.37}$$

has the representation $f = h \circ g$ where $g : D \to \mathbb{B}$ stands for a homeomorphic $W^{1,1}_{\mathrm{loc}}$ solution to the Beltrami equation $g_{\overline{z}} = \mu(z) \cdot g_z$, and $h : \mathbb{B} \to \mathbb{C}$ is analytic.

By Theorem 5.50 from [26] the conditions (5.21)–(5.26) are not only sufficient but also necessary to have a homeomorphic $W^{1,1}_{\rm loc}$ solution for all such Beltrami equations with the integral constraint

$$\int\limits_{D} \Phi(K_\mu(z))\ dxdy\ <\ \infty. \tag{5.38}$$

Note also that in the above theorem we may assume that the functions $\Phi_{z_0}(t)$ and $\Phi(t)$ are not convex and non–decreasing on the whole segment $[0,\infty]$ but only on a segment $[T,\infty]$ for some $T\in(1,\infty)$. Indeed, every function $\Phi:[0,\infty]\to[0,\infty]$ which is convex and non-decreasing on a segment $[T,\infty]$, $T\in(0,\infty)$, can be replaced by a non-decreasing convex function $\Phi_T:[0,\infty]\to[0,\infty]$ in the following way. We set $\Phi_T(t)\equiv 0$ for all $t\in[0,T]$, $\Phi(t)=\varphi(t)$, $t\in[T,T_*]$, and $\Phi_T\equiv\Phi(t)$, $t\in[T_*,\infty]$, where $\tau=\varphi(t)$ is the line passing through the point $(0,T)$ and tangent to the graph of the function $\tau=\Phi(t)$ at a point $(T_*,\Phi(T_*))$, $T_*\geq T$. For such a function we have by the construction that $\Phi_T(t)\leq\Phi(t)$ for all $t\in[1,\infty]$ and $\Phi_T(t)=\Phi(t)$ for all $t\geq T_*$.

The equation of the form

$$f_{\overline{z}}\ =\ \lambda(z)\ {\rm Re}\, f_z \tag{5.39}$$

with $|\lambda(z)|<1$ a.e. is called a **reduced Beltrami equation**, considered e.g. in [3] and [32], though the term is not introduced there. The equation (5.39) can be written as the equation (1.2) with

$$\mu(z)\ =\ \nu(z)\ =\ \frac{\lambda(z)}{2} \tag{5.40}$$

and then

$$K_{\mu,\nu}(z)\ =\ K_\lambda(z):=\ \frac{1+|\lambda(z)|}{1-|\lambda(z)|}\ . \tag{5.41}$$

Thus, we obtain from Theorem 5.32 the following consequence for the reduced Beltrami equations (5.39).

5.42. Theorem. *Let D be a Jordan domain in $\mathbb{C}$ with $0\in D$ and let $\lambda:D\to\mathbb{C}$ be a measurable function such that*

$$\int\limits_{D} \Phi(K_\lambda(z))\ dxdy\ <\ \infty \tag{5.43}$$

where $\Phi:[0,\infty]\to[0,\infty]$ is a non-decreasing convex function satisfying at least one of the conditions (5.21)–(5.26). Then the the Dirichlet problem

$$\begin{cases} f_{\overline{z}}\ =\ \lambda(z)\ {\rm Re}\, f_z, & z\in D,\\ \lim\limits_{z\to\zeta} {\rm Re}\, f(z)=\varphi(\zeta), & \forall\ \zeta\in\partial D,\end{cases} \tag{5.44}$$

in a Jordan domain D, $0\in D$, has a regular solution f with ${\rm Im} f(0)=0$ for each nonconstant continuous function $\varphi:\partial D\to\mathbb{R}$.

Finally, on the basis of Corollary 5.34, we obtain the following consequence.

5.45. Corollary. *In particular, the conclusion of Theorem 5.42 holds if*

$$\int\limits_{D\cap U_{z_0}} e^{\alpha(z_0)K_\lambda(z)}\,dxdy\ <\infty \qquad \forall\ z_0\in\overline{D} \tag{5.46}$$

for some $\alpha(z_0)>0$ *and a neighborhood* U_{z_0} *of the point* z_0.

5.47. Remark. Remarks 5.36 are valid for reduced Beltrami equations. Moreover, the above results remain true for the case in (1.2) when

$$\nu(z)\ =\ \mu(z)\ e^{i\theta(z)} \tag{5.48}$$

with an arbitrary measurable function $\theta(z):D\to\mathbb{R}$ and, in particular, for the equations of the form

$$f_{\overline{z}}\ =\ \lambda(z)\ \mathrm{Im}\, f_z \tag{5.49}$$

with a measurable coefficient $\lambda: D\to\mathbb{C}$, $|\lambda(z)|<1$ a.e., see e.g. [3].

Our approach makes possible, under the certain modification, to obtain criteria on the existence of pseudoregular and multi-valued solutions in finitely connected domains that will be published elsewhere.

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