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On Integral Conditions for the General Beltrami Equations

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Abstract Under integral restrictions on dilatations, it is proved existence theorems for the degenerate Beltrami equations with two characteristics $\overline{\partial} f = \mu \partial f + \nu \overline{\partial f}$ and, in particular, to the Beltrami equations of the second type $\overline{\partial} f = \nu \overline{\partial f}$ that play a great role in many problems of mathematical physics and to the so-called reduced Beltrami equations $\overline{\partial} f = \lambda \operatorname{Re} \partial f$ that also have significant applications.

Keywords Beltrami equations · Integral conditions · Two characteristics

Mathematics Subject Classification (2000) Primary 30C65; Secondary 30C75

1 Introduction

The existence problem for the Beltrami equations with two characteristics

$$f_{\overline{z}} = \mu(z) \cdot f_z + \nu(z) \cdot \overline{f_z} \tag{1.1}$$

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where $|\mu(z)| + |\nu(z)| < 1$ a.e. was solved first in the case of the bounded dilatations

$$K_{\mu,\nu}(z) := \frac{1 + |\mu(z)| + |\nu(z)|}{1 - |\mu(z)| - |\nu(z)|} \tag{1.2}$$

in [8, Theorem 5.1].

Recently in [11], the existence of homeomophic solutions in $W^{1,s}_{loc}$ for all $s \in [1, 2)$ to the equation (1.1) was stated in the case when $K_{\mu,\nu}$ had a majorant Q in the class BMO, bounded mean oscillation by John–Nirenberg, see [22]. Note that $L^\infty \subset$ BMO $\subset L^p_{loc}$ for all $p \in [1, \infty)$. Our last paper [12] was devoted to the study of more general cases when $K_{\mu,\nu} \in L^1_{loc}$ and when $K_{\mu,\nu}$ had a majorant Q in the class FMO, finite mean oscillation by Ignat'ev–Ryazanov, see [20]. Moreover, the paper [12] contained one new criterion of the Lehto type that is the base for the further development of the theory of the degenerate Beltrami equations (1.1) with integral constraints on the dilatation $K_{\mu,\nu}$ in the present paper, see the next section.

In the theory of quasiconformal mappings, it is well-known the role of the Beltrami equations of the first type

$$f_{\overline{z}} = \mu(z) \cdot f_z \tag{1.3}$$

where $f_{\overline{z}} = \overline{\partial} f = (f_x + i f_y)/2$, $f_z = \partial f = (f_x - i f_y)/2$, $z = x + iy$, and f_x and f_y are partial derivatives of $f = u + iv$ in the variables x and y, respectively, and $\mu : D \to \mathbb{C}$ is a measurable function with $|\mu(z)| < 1$ a.e., see e.g. [1,4,8] and [28] where the existence problem was resolved for the uniformly elliptic case when $\|\mu\|_\infty < 1$. The existence problem for degenerate Beltrami equations (1.3) with unbounded dilatations

$$K_\mu(z) := \frac{1 + |\mu(z)|}{1 - |\mu(z)|} \tag{1.4}$$

is currently an active area of research, see e.g. [3,6,7,13,14,17,21,29–32,34,38–41, 43,45,50]. The study of such homeomorphisms was started in the frames of the theory of the so–called mean quasiconformal mappings, see e.g. [2,5,15,16,18,23,24,26,27, 35–37,44,46].

On the other hand, the Beltrami equations of the second type

$$f_{\overline{z}} = \nu(z) \cdot \overline{f_z} \tag{1.5}$$

play a great role in many problems of mathematical physics, see e.g. [25]. Hence the research of Eqs. (1.1) is so actual.

Recall that a function $f : D \to \mathbb{C}$ is *absolutely continuous on lines*, abbr. $f \in$ **ACL**, if, for every closed rectangle R in D whose sides are parallel to the coordinate axes, $f|R$ is absolutely continuous on almost all line segments in R which are parallel to the sides of R. In particular, f is ACL (possibly modified on a set of Lebesgue measure zero) if it belongs to the Sobolev class $W^{1,1}_{loc}$ of locally integrable functions with locally integrable first generalized derivatives and, conversely, if $f \in$ ACL has locally integrable first partial derivatives, then $f \in W^{1,1}_{loc}$, see e.g. 1.2.4 in [33].

If $f : D \to \mathbb{C}$ is a homeomorphic ACL solution of the Beltrami equation (1.1) with $K_{\mu,\nu} \in L^1_{loc}(D)$, then $f \in W^{1,1}_{loc}(D)$, furthermore, if $K_{\mu,\nu} \in L^p_{loc}(D)$, $p \in [1, \infty]$, then $f \in W^{1,s}_{loc}(D)$ where $s = 2p/(p+1)$. Indeed, if $f \in$ ACL, then f has partial derivatives f_x and f_y a.e. and, for a sense-preserving ACL homeomorphism $f : D \to \mathbb{C}$, the Jacobian $J_f(z) = |f_z|^2 - |f_{\bar{z}}|^2$ is nonnegative a.e. and, moreover,

$$|\bar{\partial} f| \leq |\partial f| \leq |\partial f| + |\bar{\partial} f| \leq \ Q^{1/2}(z) \cdot J_f^{1/2}(z) \text{ a.e.} \tag{1.6}$$

Recall that if a homeomorphism $f : D \to \mathbb{C}$ has finite partial derivatives a.e., then

$$\int_B J_f(z)\, dxdy \leq |f(B)| \tag{1.7}$$

for every Borel set $B \subseteq D$, see e.g. Lemma III.3.3 in [28]. Consequently, applying successively the Hölder inequality and the inequality (1.7) to (1.6), we get that

$$\|\partial f\|_s \leq \|K_{\mu,\nu}\|_p^{1/2} \cdot |f(C)|^{1/2} \tag{1.8}$$

where $\|\cdot\|_s$ and $\|\cdot\|_p$ denote the L^s- and L^p-norm in a compact set $C \subset D$, respectively.

In the classical case when $\|\mu\|_\infty < 1$, equivalently, when $K_\mu \in L^\infty(D)$, every ACL homeomorphic solution f of the Beltrami equation (1.3) is in the class $W^{1,2}_{loc}(D)$ with $f^{-1} \in W^{1,2}_{loc}(f(D))$. In the case $\|\mu\|_\infty = 1$ with $K_\mu \leq Q \in$ BMO, again $f^{-1} \in W^{1,2}_{loc}(f(D))$ and f belongs to $W^{1,s}_{loc}(D)$ for all $1 \leq s < 2$ but already not necessarily to $W^{1,2}_{loc}(D)$. However, there is a number of degenerate Beltrami equations (1.3) for which there exist homeomorphic solutions f of the class $W^{1,1}_{loc}(D)$ with $f^{-1} \in W^{1,2}_{loc}(f(D))$.

Following [12], we call a homeomorphism $f \in W^{1,1}_{loc}(D)$ a *regular solution* of (1.1) if f satisfies (1.1) and $J_f(z) \neq 0$ a.e. Note that by [19] $f^{-1} \in W^{1,2}_{loc}(f(D))$ for such solutions if $K_{\mu,\nu} \in L^1_{loc}(D)$.

2 Preliminaries

The following theorem was recently established in the work [12].

Theorem 2.1 *Let D be a domain in $\mathbb{C}$ and let μ and $\nu : D \to \mathbb{C}$ be measurable functions with $|\mu(z)| + |\nu(z)| < 1$ a.e. and $K_{\mu,\nu} \in L^1_{loc}(D)$. Suppose that*

$$\int_0^{\delta(z_0)} \frac{dr}{r k_{z_0}(r)} = \infty \qquad \forall z_0 \in D \tag{2.2}$$

where $\delta(z_0) < dist\,(z_0, \partial D)$ *and* $k_{z_0}(r)$ *is the average of* $K_{\mu,\nu}(z)$ *over the circle* $|z - z_0| = r$. *Then the Beltrami equation* (1.1) *has a regular solution.*

In general, in the Beltrami equation theory in the plane as well as in the theory of space mappings, the integral conditions of the Lehto type

$$\int\limits_0^1 \frac{dr}{rq(r)} = \infty \tag{2.3}$$

are often met where the function Q is given say in the unit ball $\mathbb{B}^n = \{x \in \mathbb{R}^n : |x| < 1\}$ and $q(r)$ is the average of the function $Q(z)$ over the sphere $|x| = r$, see e.g. [3,12,13,17,29,32,34,35,38–41,48,49].

On the other hand, in the theory of mappings called quasiconformal in the mean, conditions of the type

$$\int\limits_{\mathbb{B}^n} \Phi(Q(x))\, dx < \infty \tag{2.4}$$

are standard for various characteristics Q of these mappings, see e.g. [2,5,15,18,23–27,35,37,44].

In this connection, in the paper [41] it was established interconnections between a series of integral conditions on the function Φ and between (2.3) and (2.4), cf. also [7] and [17]. We give here these conditions for Φ under which (2.4) implies (2.3).

Further we use the following notion of the inverse function for monotone functions. For every non-decreasing function $\Phi : [0, \infty] \to [0, \infty]$, the *inverse function* $\Phi^{-1} : [0, \infty] \to [0, \infty]$ can be well defined by setting

$$\Phi^{-1}(\tau) = \inf_{\Phi(t) \geq \tau} t. \tag{2.5}$$

As usual, here inf is equal to ∞ if the set of $t \in [0, \infty]$ such that $\Phi(t) \geq \tau$ is empty. Note that the function Φ^{-1} is non-decreasing, too.

Remark 2.6 It is evident immediately by the definition that

$$\Phi^{-1}(\Phi(t)) \leq t \quad \forall\, t \in [0, \infty] \tag{2.7}$$

with the equality in (2.7) except intervals of constancy of the function $\Phi(t)$.

Further, the integral in (2.11) is understood as the Lebesgue–Stieltjes integral and the integrals in (2.10) and (2.12)–(2.15) as the ordinary Lebesgue integrals. In (2.10) and (2.11) we complete the definition of integrals by ∞ if $\Phi(t) = \infty$, correspondingly, $H(t) = \infty$, for all $t \geq T \in [0, \infty)$.

Theorem 2.8 *Let* $\Phi : [0, \infty] \to [0, \infty]$ *be a non-decreasing function and set*

$$H(t) = \log \Phi(t). \tag{2.9}$$

Then the equality

$$\int_{\Delta}^{\infty} H'(t)\, \frac{dt}{t} = \infty \tag{2.10}$$

implies the equality

$$\int_{\Delta}^{\infty} \frac{dH(t)}{t} = \infty \tag{2.11}$$

and (2.11) *is equivalent to*

$$\int_{\Delta}^{\infty} H(t)\, \frac{dt}{t^2} = \infty \tag{2.12}$$

for some $\Delta > 0$, *and* (2.12) *is equivalent to every of the equalities:*

$$\int_{0}^{\delta} H\left(\frac{1}{t}\right)\, dt = \infty \tag{2.13}$$

for some $\delta > 0$,

$$\int_{\Delta_*}^{\infty} \frac{d\eta}{H^{-1}(\eta)} = \infty \tag{2.14}$$

for some $\Delta_* > H(+0)$,

$$\int_{\delta_*}^{\infty} \frac{d\tau}{\tau \Phi^{-1}(\tau)} = \infty \tag{2.15}$$

for some $\delta_* > \Phi(+0)$.

Moreover, (2.10) *is equivalent to* (2.11) *and hence* (2.10)–(2.15) *are equivalent each to other if* Φ *is in addition absolutely continuous. In particular, all the conditions* (2.10)–(2.15) *are equivalent if* Φ *is convex and non-decreasing.*

It is necessary here to give one more explanation. From the right hand sides in the conditions (2.10)–(2.15) we have in mind $+\infty$. If $\Phi(t) = 0$ for $t \in [0, t_*]$, then $H(t) = -\infty$ for $t \in [0, t_*]$ and we complete the definition $H'(t) = 0$ for $t \in [0, t_*]$. Note, the conditions (2.11) and (2.12) exclude that t_* belongs to the interval of integrability because in the contrary case the left hand sides in (2.11) and (2.12) are either equal to $-\infty$ or indeterminate. Hence we may assume in (2.10)–(2.13) that $\Delta > t_0$ where $t_0 := \sup_{\Phi(t)=0} t$, $t_0 = 0$ if $\Phi(0) > 0$, and $\delta < 1/t_0$, correspondingly.

Finally, we give the connection of the above conditions with the condition of the Lehto type (2.2).

Recall that a function $\psi : [0, \infty] \to [0, \infty]$ is called *convex* if $\psi(\lambda t_1 + (1-\lambda)t_2) \le \lambda\psi(t_1) + (1-\lambda)\psi(t_2)$ for all t_1 and $t_2 \in [0, \infty]$ and $\lambda \in [0, 1]$.

In what follows, $\mathbb{D}$ denotes the unit disk in the complex plane $\mathbb{C}$,

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}. \tag{2.16}$$

Theorem 2.17 *Let $Q : \mathbb{D} \to [0, \infty]$ be a measurable function such that*

$$\int_{\mathbb{D}} \Phi(Q(z))\, dxdy < \infty \tag{2.18}$$

where $\Phi : [0, \infty] \to [0, \infty]$ is a non-decreasing convex function such that

$$\int_{\delta_0}^{\infty} \frac{d\tau}{\tau\Phi^{-1}(\tau)} = \infty \tag{2.19}$$

for some $\delta_0 > \tau_0 := \Phi(0)$. Then

$$\int_0^1 \frac{dr}{rq(r)} = \infty \tag{2.20}$$

where $q(r)$ is the average of the function $Q(z)$ over the circle $|z| = r$.

Finally, combining Theorems 2.8 and 2.17 we obtain the following conclusion.

Corollary 2.21 *If $\Phi : [0, \infty] \to [0, \infty]$ is a non-decreasing convex function and Q satisfies the condition* (2.18), *then every of the conditions* (2.10)–(2.15) *implies* (2.20).

3 Existence Theorems

Immediately on the base of Theorem 2.1 and Corollary 2.21, we obtain the next significant result.

Theorem 3.1 *Let D be a domain in $\mathbb{C}$ and let μ and $\nu : D \to \mathbb{C}$ be measurable functions with $|\mu(z)| + |\nu(z)| < 1$ a.e. such that*

$$\int\limits_D \Phi(K_{\mu,\nu}(z))\, dxdy < \infty \tag{3.2}$$

where $\Phi : [0,\infty] \to [0,\infty]$ is a non-decreasing convex function. If Φ satisfies at least one of the conditions (2.10)–(2.15), then the Beltrami equation (1.1) has a regular solution.

Remark 3.3 The condition (3.2) can be also localized to neighborhoods U_{z_0} of points $z_0 \in D$ with $\Phi = \Phi_{z_0}$ under the same conditions on the functions Φ_{z_0}. If $\infty \in D$, then the condition (3.2) for $K_{\mu,\nu}(z)$ at $\infty \in D$ should be understood as the corresponding condition for $K_{\mu,\nu}(1/\overline{z})$ at 0. The latter condition can also be rewritten explicitly in terms of $K_{\mu,\nu}(z)$ itself after the inverse change of variables $z \longmapsto 1/\overline{z}$ in the form

$$\int\limits_{U_\infty} \Phi_\infty(K_{\mu,\nu}(z))\frac{dxdy}{|z|^4} < \infty. \tag{3.4}$$

If the domain D is unbounded, then it is better to use the global condition

$$\int\limits_D \Phi(K_{\mu,\nu}(z))\, \frac{dxdy}{(1+|z|^2)^2} < \infty \tag{3.5}$$

instead of the condition (3.2). The latter means the integration of the function $\Phi \circ K_{\mu,\nu}$ in the spherical area.

We may assume in the above theorem that the functions $\Phi_{z_0}(t)$ and $\Phi(t)$ are not convex and non-decreasing on the whole segment $[0,\infty]$ but only on a segment $[T,\infty]$ for some $T \in (1,\infty)$. Indeed, every function $\Phi : [0,\infty] \to [0,\infty]$ which is convex and non-decreasing on a segment $[T,\infty]$, $T \in (0,\infty)$, can be replaced by a non-decreasing convex function $\Phi_T : [0,\infty] \to [0,\infty]$ in the following way. We set $\Phi_T(t) \equiv 0$ for all $t \in [0,T]$, $\Phi(t) = \varphi(t)$, $t \in [T,T_*]$, and $\Phi_T \equiv \Phi(t)$, $t \in [T_*,\infty]$, where $\tau = \varphi(t)$ is the line passing through the point $(0,T)$ and touching upon the graph of the function $\tau = \Phi(t)$ at a point $(T_*, \Phi(T_*))$, $T_* \geq T$. For such a function we have by the construction that $\Phi_T(t) \leq \Phi(t)$ for all $t \in [1,\infty]$ and $\Phi_T(t) = \Phi(t)$ for all $t \geq T_*$.

The equation of the form

$$f_{\overline{z}} = \lambda(z)\,\mathrm{Re}\, f_z \tag{3.6}$$

with $|\lambda(z)| < 1$ a.e. is called the *reduced Beltrami equation*, see e.g. [3,8–10,47]. Equation (3.6) can be rewritten as Eq. (1.1) with

$$\mu(z) = \nu(z) = \frac{\lambda(z)}{2} \tag{3.7}$$

and then

$$K_{\mu,\nu}(z) = K_\lambda(z) := \frac{1 + |\lambda(z)|}{1 - |\lambda(z)|}. \tag{3.8}$$

Thus, we obtain from Theorem 3.1 the following consequence for the reduced Beltrami equations (3.6).

Theorem 3.9 *Let D be a domain in* $\mathbb{C}$ *and let* λ *be a measurable function with* $|\lambda(z)| < 1$ *a.e. such that*

$$\int_D \Phi(K_\lambda(z))\, dxdy < \infty \tag{3.10}$$

where $\Phi : [0, \infty] \to [0, \infty]$ *is a non-decreasing convex function. If* Φ *satisfies at least one of the conditions* (2.10)–(2.15), *then the reduced Beltrami equation* (3.6) *has a regular solution.*

Remark 3.11 Remarks 3.3 are valid for the reduced Beltrami equation. Moreover, the above results remain true for the case in (1.1) when

$$\nu(z) = \mu(z)\, e^{i\theta(z)} \tag{3.12}$$

with an arbitrary measurable function $\theta(z) : D \to \mathbb{R}$ and, in particular, for the equations of the form

$$f_{\overline{z}} = \lambda(z)\ \mathrm{Im}\, f_z \tag{3.13}$$

with a measurable coefficient $\lambda : D \to \mathbb{C}$, $|\lambda(z)| < 1$ a.e., see e.g. [8–10].

Next, note that Theorem 5.1 from the work [42] for the Beltrami equations of the first type (1.3) shows that the conditions (2.10)–(2.15) are not only sufficient but also necessary for the general Beltrami equations (1.1) to have regular solutions.

Finally, the same is valid for the reduced Beltrami equations (3.6) because the examples in the mentioned theorem had the form

$$f(z) = \frac{z}{|z|}\, \rho(|z|)$$

where $\rho(t) = e^{I(t)}$ and

$$I(t) := \int_0^t \frac{dr}{rK(r)}.$$

Indeed, setting $z = re^{i\vartheta}$ we have that

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial r} + \frac{\partial f}{\partial \overline{z}} \cdot \frac{\partial \overline{z}}{\partial r} = e^{i\vartheta} \cdot \frac{\partial f}{\partial z} + e^{-i\vartheta} \cdot \frac{\partial f}{\partial \overline{z}}$$

and

$$\frac{\partial f}{\partial \vartheta} = \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial \vartheta} + \frac{\partial f}{\partial \overline{z}} \cdot \frac{\partial \overline{z}}{\partial \vartheta} = ire^{i\vartheta} \cdot \frac{\partial f}{\partial z} - ire^{-i\vartheta} \cdot \frac{\partial f}{\partial \overline{z}}$$

and hence

$$\frac{\partial f}{\partial z} = \frac{e^{-i\vartheta}}{2} \left(\frac{\partial f}{\partial r} + \frac{1}{ir} \cdot \frac{\partial f}{\partial \vartheta} \right) = \frac{1}{2} \left(\frac{\rho(r)}{rK(r)} + \frac{\rho(r)}{r} \right) = \frac{\rho(r)}{2r} \cdot \frac{1 + K(r)}{K(r)} > 0$$

and

$$\begin{aligned} \frac{\partial f}{\partial \overline{z}} &= \frac{e^{i\vartheta}}{2} \left(\frac{\partial f}{\partial r} - \frac{1}{ir} \cdot \frac{\partial f}{\partial \vartheta} \right) \\ &= \frac{e^{2i\vartheta}}{2} \left(\frac{\rho(r)}{rK(r)} - \frac{\rho(r)}{r} \right) = e^{2i\vartheta} \cdot \frac{\rho(r)}{2r} \cdot \frac{1 - K(r)}{K(r)}, \end{aligned}$$

i.e.

$$\lambda(z) = e^{2i\vartheta} \cdot \frac{1 - K(r)}{1 + K(r)} = -\frac{z}{\overline{z}} \cdot \frac{K(|z|) - 1}{K(|z|) + 1}$$

and, consequently, $K_\lambda(z) = K(|z|)$.

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