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CORRECTION TO “PRIMARY SOLUTIONS OF GENERAL BELTRAMI EQUATIONS”

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As was noticed by several readers, I was not sufficiently careful in the formulation and the proof of Proposition 2 in [6]. The correct Beltrami equation for the function $G(\zeta)$—in the notation of [6], used throughout in this correction—is more complicated than stated in [6] and the use of its correct, in particular, quasilinear, form in a proof of the Theorem in [6] should be reconsidered. However the main Theorem as stated and the factorisation strategy for the proof proposed is valid.

Since the problem of “primary” solutions for Beltrami equations, started in the present form by Iwaniec et al. [7], aroused a rather long list of publications, extensively referred to, in particular, in the recent monograph [1], it seems proper to reconsider the topic from another perspective. The corresponding paper is in preparation.

Below we sketch another approach to the main theorem in [6]. The theorem reads as follows.

Theorem. *The Beltrami equation*

$$\frac{\partial w}{\partial \bar z} - q(z)\Big(\frac{\partial w}{\partial z} - \overline{\frac{\partial w}{\partial z}}\Big) = 0 \tag{1}$$

with the compactly supported measurable dilatation ($q \equiv 0$ *in the neighbourhood of* ∞) *satisfying the ellipticity condition*

$$|q(z)| < k_0 < \tfrac{1}{2}, \quad 2k_0 = k, \tag{2}$$

admits a global solution $\psi\colon \mathbf{C} \to \mathbf{C}$ *satisfying condition*

$$\operatorname{Im} \frac{\partial \psi}{\partial z} \neq 0 \quad a.e. \tag{3}$$

Then the pair $(z, \psi(z))$ is a primary pair for the Beltrami equation (1).

As is well known, see e.g. [8] and [5], the global homeomorphic solutions of (1), $w = \psi(z)$, $\psi(\infty) = \infty$, admit the (unique) representation

$$w(z) = az - \frac{1}{\pi} \iint_C \frac{\omega(t)\, d\sigma_t}{t - z}, \quad a \neq 0, \tag{4}$$

for some complex a and $\omega \in L^p(\mathbf{C})$, where the density ω satisfies some uniquely solvable integral equation. It is convenient here to expose explicitly the dependence of the solution on the dilatation $q(z)$ and the coefficient a interpreted as $\psi_z(\infty) = a$.

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Thus we write $\psi(z) \equiv \psi(a, q, z)$. If $\operatorname{Im} a = 0$, then the density $\omega \equiv 0$ and the solution (4) reduces to $w(z) \equiv az$. If

$$\operatorname{Im} a \neq 0, \tag{5}$$

the solution $\psi = \psi(a, q, z)$ satisfies the requirements of the theorem.

The proof of the theorem for the "clasical" case when the dilatation $q(z)$ is smooth, say of class C_0^k, $k \geq 2$, follows immediately. Indeed, then the generalized solutions of (1) are smooth, e.g. Vekua [8] or any monographic treatment of elliptic 2D equations.

In particular, by the argument principle any complex (locally) homeomorphic solutions $w = w(z)$ of such equations do not allow any critical points, i.e.,

$$dw(z_0) \neq 0, \quad w_z(z_0) \neq 0, \tag{6}$$

for any finite $z_0 \in \mathbf{C}$. Now by the quoted Vekua's and my works [8, 6] the global homeomorphic solutions of (general) Beltrami equations—the principal solutions—are classified by the coefficient $a \neq 0$ in (4). If, for a smooth dilatation $q(z)$, the solution $\psi \equiv \psi(i, q, z)$ would violate the condition (3) at a finite point $z_0 \in \mathbf{C}$ then obviously some real combination

$$w(z) = \alpha z + \beta\psi(z), \quad \alpha^2 + \beta^2 > 0,$$

for some α, β would produce a global homeomorphic solution of (1) with $a = \alpha + \beta a \neq 0$ and a critical point at $z = z_0$, which is absurd. Thus for q smooth the solution (4) satisfies the stronger inequality

$$\operatorname{Im} a \cdot \operatorname{Im} \frac{\partial\psi}{\partial z}(a, q, z) > 0 \tag{3$'$}$$

everywhere.

For q not smooth the solution $\psi(a, q, z)$ is the strong limit in $W^{1,p}_{\text{loc}}(\mathbf{C})$ (this means that the corresponding densities $\omega_n \to \omega$ in L_p) of smooth quasiconformal homeomorphisms $\psi(a, q_n, z)$, $q_n(z) \to q(z)$ pointwise, by the convergence results for solutions of general Beltrami equations with measurable coefficients, discussed in detail in [6]. In consequence the inequality (3) follows in a weaker form

$$\operatorname{Im} \frac{\partial\psi}{\partial z} \geq 0 \quad (\leq 0) \quad \text{for } \operatorname{Im} a > 0 \ (\operatorname{Im} a < 0). \tag{3$''$}$$

What is left to be shown is to exclude the possibility

$$\operatorname{mes} \Sigma_\psi > 0, \quad \text{where } \Sigma_\psi = \{z\colon \operatorname{Im} \psi_z(a, q, z) = 0\} \tag{7}$$

if (5) is satisfied.

As we know now, there are several ways to achieve this result. They all work by reducing ad absurdum the assumption $\operatorname{mes} \Sigma_\psi > 0$. At a density point of the set Σ_ψ the contradiction is obtained with a Belinskii–Wittich or Sard type theorem for homeomorphic quasiconformal mappings [3], by considering the family of QC-homeomorphisms of $\mathbf{C}$, parametrized as follows:

$$w(z) = \alpha z + \beta\psi(i, q, z) + w_0$$

for α, β real and w_0 complex.

Also integral inverse Hölder inequalities can be used ([1], [2]).

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