

# Sobolev spaces and averaging I\*

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## Abstract

An apparently new concept of maximal mean difference quotient is defined for functions in the Lebesgue space  $L_{\text{loc}}(\mathbb{R}^n)$ . Our definitions are meaningful for vector valued functions [53] on general measure metric spaces as well and seem to lead to the most natural class of metric Sobolev spaces [35], [42], [49]. The discussion of higher order Sobolev spaces and higher order mean difference quotients on regular subsets of Euclidean spaces is also possible [17] in the context of the generalized Taylor–Whitney jets, [12]–[20], [26], [40], [84]. This paper is a direct sequel to [15], [16].

## Introduction

Averaging and integration, especially integration by parts formulae and their multidimensional analogues (Stokes–Green theorems etc), lie at the heart of Sobolev’s theory and his concepts of duality in linear function spaces.

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The Hardy–Littlewood idea [46] of maximal function gives the natural and most direct way to introduce and characterize the Sobolev spaces  $W^{m,p}(\mathbb{R}^n)$  and  $W_{\text{loc}}^{m,p}(\mathbb{R}^n)$  and their far going generalizations to measure metric spaces [42], [49], [52] etc.

We illustrate this fact and the corresponding ideas on the simplest case of Sobolev spaces  $W^{m,p}(\mathbb{R})$ ,  $m \geq 1$ ,  $p \geq 1$  ( $m$ —an integer) on the real line, starting with the case  $m = 1$ ,  $p = 1$ . Since a function  $f \in W^{1,1}(\mathbb{R})$  is absolutely continuous, by the fundamental theorem of the calculus for  $x, y$  real,  $x < y$ , we have

$$f(y) - f(x) = \int_x^y f'(t) dt. \quad (0.1)$$

Hence for  $x < \xi < y$

$$f(y) - f(x) = (y - x) \left[ \frac{\xi - x}{y - x} \int_x^\xi f'(t) dt + \frac{y - \xi}{y - x} \int_\xi^y f'(t) dt \right] \quad (0.2)$$

and, by the definition<sup>1)</sup> of the Hardy–Littlewood maximal function [46], [41]

$$g_f(x) \equiv M(|f'|)(x) \stackrel{2)}{=} \sup_{\xi > x} \int_x^\xi |f'(x)| dt \quad (0.3)$$

we get the pointwise inequality

$$|f(y) - f(x)| \leq |y - x|[g_f(x) + g_f(y)]. \quad (0.4)$$

Now for  $f \in W_{\text{loc}}^{1,p}(\mathbb{R})$ ,  $p > 1$ ,  $f'(t) \equiv \nabla f(t)$ , [46], [41], [79]

$$\|g_f\|_p \leq C_p \|\nabla f\|_p \quad (0.5)$$

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<sup>1)</sup>We skip here over the precise definitions, distinctions of left  $M_L$  and right  $M_R$  maximal functions on the line etc., for details see [22], [41], [42].

<sup>2)</sup>Here, and below:  $f_V g dx \equiv \frac{1}{|V|} \int_V g dx$ ,  $|V| > 0$ .

and we conclude [12], [18], [42] that, for  $p > 1$ , the pointwise inequality (0.4) characterizes the Sobolev class  $W^{1,p}(\mathbb{R}^n)$  [18], [42], [43].

It is a classical fact of analysis that, for  $f \in W^{1,p}(\mathbb{R})$ ,  $p > 1$ , (0.1) implies, by Hölder inequality, the following estimate

$$|f(y) - f(x)| \leq |y - x|^\alpha \|f'\|_p \quad \alpha = 1 - \frac{1}{p} > 0. \quad (0.6)$$

This describes, when generalized to  $\mathbb{R}^n$ ,  $n > 1$ , the simplest of the famous Sobolev imbeddings

$$W_{\text{loc}}^{1,p}(\mathbb{R}^n) \rightarrow C_{\text{loc}}^\alpha(\mathbb{R}^n), \quad \alpha = 1 - \frac{n}{p}, \quad p > n, \quad n \geq 1. \quad (0.7)$$

For  $p \leq n$  the Sobolev functions admit discontinuities at, possibly everywhere dense, sets of points and may be even locally unbounded [5].

The inequalities (0.6) and (0.7) imply the uniform continuity of bounded families of functions in  $W^{1,p}(\mathbb{R}^n)$ ,  $p > n$ , and play a crucial role for various compactness arguments in the calculus of variations, PDE, and many other applications of Sobolev spaces in analysis and geometry.

However (0.6) is very far away from characterizing the Sobolev functions. In particular the Weierstrass nowhere differentiable function [32], [41], [35] is Hölder continuous, for some  $\alpha > 0$ . In fact a much deeper statement holds: in general an  $\alpha$ -Hölder continuous function, with  $\alpha < 1$ , cannot be “corrected” on any subset of positive measure of its domain of definition to become Lipschitz (smooth)!

On the contrary, any function  $f$  in the Sobolev class  $W^{1,p}(\mathbb{R}^n)$ , or any function just satisfying the inequality (0.4), can be modified on a subset of arbitrary small positive Lebesgue measure in its domain of definition to become Lipschitz or even  $C^1$  [32]. This is the important Luzin’s property (or Luzin approximation

property) [14], [32], [48], [20], of function classes, satisfying inequalities of the type (0.4) and their numerous generalizations.

The essence of the argument runs as follows: For  $L > 0$  consider the set  $E_L = \{x : g_f(x) \leq L\}$ . On  $E_L$  the function  $f$  is (uniformly) Lipschitz. By Tschebyscheff inequality the Lebesgue measure of the complement  $CE_L$  of  $E_L$

$$|CE_L| \rightarrow 0 \quad \text{for } L \rightarrow \infty \quad (0.8)$$

Invoking the Tietze–McShane–Kirschbraun–Whitney extension theorems [32], [48], [49] we come to the required conclusions.

For some Sobolev type function spaces the details of this procedure are given in the quoted references. Some others will be mentioned in Section 3 below.

Let us remark at this point that the remarkable Luzin’s property has essentially a semi global character: it refers to the behaviour of the function (mapping!) on the *complement* of a set *very small in measure*! The deep structure theorems of N. N. Luzin for measurable functions on general metric spaces allow us to understand the duality between measurability and continuity: Luzin’s continuity of a mapping is an infinitesimal expression of the global property of measurability of the mapping!! In a still more refined form these questions led A. Denjoy and A. Khintchine, [33], to the discovery of the concept of the approximate continuity almost everywhere as Luzin’s dual to measurability. See e.g. references in [14]. This point of view is presented in extenso in [14].

When applied to the Sobolev type function spaces it allowed Hassler Whitney, [83], [87], to extend the Luzin’s duality to continuously differentiable functions in the class  $C^1(\mathbb{R})$ , on the real line. In [87] H. Whitney showed that the semiglobal Luzin’s property of order 1, see [14], has an infinitesimal description in terms of the almost everywhere approximate Peano differentiability [33]. In this connection let us also state that the long standing Federer–Whitney con-

jecture, (3.1.17) on pp. 228–229 of [33], was solved in [14] thanks to the work of F. Ch. Liu and the author described in [14]<sup>3)</sup>. These works allowed us also to understand the case of Luzin’s property for Sobolev spaces  $W^{m,p}(\mathbb{R}^n)$  of higher order  $m \geq 1$ . The detailed discussion is postponed to [17].

Another important property of the pointwise inequalities (0.4)–(1.9) (below, (1.9) for the euclidean spaces  $\mathbb{R}^n$ ) is that they are directly related, in fact equivalent, with the classical Riesz potential estimates: the starting point of Sobolev’s theory [76], [77], [78] in  $\mathbb{R}^n$  is, essentially, again the fundamental theorem of the calculus (0.1) written in the form

$$f(x) - f(y) = - \int_0^{|x-y|} D_t f(x + t\omega) dt \quad (0.9)$$

where  $x, y$  in  $\mathbb{R}^n$  are in the ball  $B_r \equiv B(x, r) = \{y, |x - y| < r\}$  and  $\omega = \frac{y-x}{|y-x|}$  is the unit vector in  $\mathbb{R}^n$ . Averaging (0.9) with respect to  $y$  over the ball  $B_r$ , after introducing polar coordinates and a change of variables, Sobolev comes to the pointwise estimates

$$|f(x) - f_B| \leq C(n) I_1(|\nabla f|)(x) \quad (0.10)$$

where  $I_\alpha(g)(x) \equiv (|y|^{\alpha-n} * g)(x)$  is the Riesz potential of order  $\alpha > 0$ . In (0.10)  $\alpha = 1$ .

Written in a symmetric way (0.10) takes the form

$$|f(x) - f(y)| \leq C(n) (I_1(|\nabla f|)(x) + I_1(|\nabla f|)(y)), \quad (0.11)$$

which is a special case of (0.4) in the classical Euclidean case of  $\mathbb{R}^n$ .

As a matter of fact, the inequality (0.10) was the starting point of the systematic approach to pointwise Sobolev inequalities initiated in [18]. For the

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<sup>3)</sup>At this point it is necessary to say that the Russian edition translator’s remark on p. 247 of the Federer’s monograph [33] (Nauka, Moscow, 1987) and the referred theorem of E. E. Movskovitz is erroneous.

inverse way: from (0.4) to the pointwise Poincaré type inequality (0.10) see [42], [43], [45], [61] and the inequalities (1.17), (1.18) below.

S. L. Sobolev introduced his spaces  $W^{m,p}(\mathbb{R}^n)$ ,  $m \geq 1$ ,  $p \geq 1$  ( $W^{m,p}(\Omega)$ ,  $\Omega$  a domain in  $\mathbb{R}^n$ ) as closed subspaces of the Lebesgue spaces  $L^p(\mathbb{R}^n)$ , ( $L^p(\Omega)$ ) defined by some integral identities resulting from the classical integration by parts formulae [75], [76], [77], [5], [35].

These identities also imply the recurrence relations (under some obvious conditions on  $f$ )

$$f \in W^{m,p}(\mathbb{R}^n) \iff \nabla f \in W^{m-1,p}(\mathbb{R}^n), \quad (m \geq 1), \quad (W^{0,p}(\mathbb{R}^n) \equiv L^p(\mathbb{R}^n)) \quad (0.12)$$

where  $\nabla f$  is the generalized (weak) Sobolev gradient of  $f$ . They open the way for the inductive, with respect to  $m$ , treatment of the Sobolev spaces of order  $m \geq 2$  [90].

However, for various reasons, the direct approach to  $W^{m,p}(\mathbb{R}^n)$  (for  $m \geq 2$ ) has been, and continues to be, widely used as well [5], [80], [82]. The gradient operators  $\nabla^k$  ( $k \geq 1$ ) combined with the finite difference and shift operators

$$\Delta f = \Delta_y f = f(y) - f(x), \quad T_\tau f(x) \equiv f_\tau(x) = f(x + \tau), \quad y, x, \tau \in \mathbb{R}^n \quad (0.13)$$

and their iterates create the general framework for the discussion of Sobolev type functions spaces [5], [15], [78], [79] in general.

In the immense and permanently growing literature of the subject there are thousands of works describing Sobolev spaces as Banach space closures of smooth functions in the corresponding Sobolev integral norms.

In various constructive computational treatments of PDE and their applications in sciences many new, ingenious methods were invented [30], [74], [79] etc.

Sobolev functions on the real line  $\mathbb{R}^1$  occupy a special place in the general theory of Sobolev spaces  $W^{m,p}(\mathbb{R}^n)$ : elements  $f$  in  $W^{m,p}(\mathbb{R}^n)$  (or on manifolds) have representatives  $\bar{f}$ ,  $f \equiv \bar{f}$  a.e., [90], [5], which, restricted to almost all lines  $l \subset \mathbb{R}^n$  parallel to the coordinate axes, may be regarded as generalizations of Sobolev functions in  $W^{m,p}(\mathbb{R}^1)$ ,  $f_l = \bar{f}|_l$  is in  $W^{m,p}(l)$ .

For a short review of the theory of Sobolev spaces on the real line, with special emphasis on the use of the concepts of Hardy–Littlewood maximal functions, pointwise inequalities,  $B$ -Splines, numerical analysis and approximation theory [30], [32], [34], [73], see [22] where the presentation is somehow adapted to the needs and concepts of this paper (see also [16]).

For  $m = 1$ ,  $p = 1$  the space  $W^{1,1}(\Omega)$ ,  $\Omega \in \mathbb{R}^n$  is identified with the famous class  $ACL(\Omega)$ . The class  $ACL$  is widely used in the analytic theory of quasiconformal mappings, e.g. [21], [49], [50], of real valued functions absolutely continuous on almost all lines (line segments) in  $\Omega$ , parallel to the coordinate axes, and whose partial derivatives  $f_{x_i}$ , defined a.e. in  $\Omega$ , belong to  $L^p(\Omega)$ .

This remarkable property, the directional regularity of Sobolev functions is analogous to the famous Hartogs theorem, see [55], (and references therein) for separately holomorphic functions in  $\mathbb{C}^n$ . It is a kind of Fubini theorem for Sobolev functions. The general case of the theorem, for the product decomposition  $\mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k}$ ,  $k \geq 1$  and for manifolds, is due to S. M. Nikolskii in his early paper [68], see also [5], [11], [90].

The pointwise inequalities characterizing the elements  $f \in W_{\text{loc}}^{m,p}(\mathbb{R}^n)$ ,  $m \geq 2$ , are formulated, for a pair  $(x, y)$ ,  $x \neq y$ , of points in  $\mathbb{R}^n$  on an affine line  $l = l(x, y)$  in  $\mathbb{R}^n$ , as estimates of the type (0.4) for the “errors” or differences

$$R^{m-1}f(x, y) \equiv R_Z^{m-1}f(x, y) = f(y) - L(y; f, x_0, \dots, x_{m-1}), \quad x_0 = x \quad (0.14)$$

between the value  $f(y)$  and the value at  $t = y$  of the Newton–Lagrange interpo-

lation polynomial  $p_n(t) = L(t; f, x_0, \dots, x_{m-1})$  [16], [25], [30], [37], [73] of order  $n = m - 1$ , interpolating the function  $f(t)$  at the nodes  $x_i$

$$p_n(x_i) = f(x_i) \quad i = 0, \dots, n. \quad (0.15)$$

The nodes  $x_i$  in (0.14) and (0.15) are assumed to belong to the affine line  $l = l(x, y)$ , or rather to the affine segment  $I(x, y) \subset l$ ;  $t$  is the natural parameter on  $l$ .

The letter  $Z$  in (0.14) stands for the general term of “interpolation scheme” [4], [41] i.e. the set of general conditions of the type of Hermite–Birkhoff–Lagrange conditions: see [4], (or their equivalence classes under affine transformations of  $\mathbb{R}^n$ ) used to define the interpolating polynomial  $p_n(t)$ .

Though the classical interpolation theory [4], [25], [30], [32] is considered for complex valued functions on the real line  $\mathbb{R}^1$  only, the term Newton–Lagrange interpolation polynomial  $L(t; f, x_0, \dots, x_{m-1})$  is meaningful for affine lines in  $\mathbb{R}^n$ . See the comments in this respect in [15], [16].

For simplicity, in this paper, we assume that all nodes  $x_i$  are simple and ordered

$$x = x_0 < x_1 < \dots < x_n \leq y \quad (0.16)$$

except the case  $x = x_0 = \dots = x_n < y$  when the Lagrange interpolation reduces to the Taylor formula.

For many reasons it is convenient to represent the interpolating polynomial in the Newton form [23], [24], [79]

$$p_n(t) = \sum_{i=0}^n c_i q_i(t), \quad q_0(t) \equiv 1 \quad (0.17)$$

where the polynomials in  $t$ ,  $q_i(t) = \sum_{k=0}^{i-1} (t - x_k)$ , are the Newton polynomials

for the assigned nodes  $x_i, i = 0, \dots, n - 1$  of the interpolation scheme  $Z$ . The coefficients  $c_i$  in (0.17) are the divided differences of the function  $f$  for the scheme  $Z$ . In the classical notation [4], [36], [31], [72],

$$c_i = f[x_0, \dots, x_i] \quad (0.18)$$

and the remainder  $R_Z^n f(t, x)$  takes the form

$$R_Z^n f(t, x) = f[x_0, \dots, x_n, t]q_{n+1}(t) \quad x_0 = x. \quad (0.19)$$

For Sobolev spaces  $W^{m,1}(\mathbb{R}^1)$ ,  $m = n + 1$ , the normalized remainders have the beautiful form of the general formula

$$\frac{R_Z^{m-1} f(t, x)}{q_m(t)} = f[x_0, \dots, x_n, t] \quad (0.20)$$

i.e. they are represented as divided differences themselves.

For suitable choices of the Lagrange–Newton interpolation scheme, the formula (0.19) comprises all classical formulas for the Newton–Taylor–Lagrange remainders used in the literature [72]. Thus, for  $m = 1$  and  $f \in C^1(R)$ ,  $t = y$ , we get the identity

$$Rf(y, x) = R^0 f(y, x) = (y - x)f[y, x] = (y - x) \int_0^1 f'(x + \tau(y - x)) d\tau \quad (0.21)$$

used in (0.2) and (0.4).

For  $m \geq 2$  and the equidistant nodes  $x_i = x + ih$ ,  $h = \frac{y-x}{m}$ ,  $n = m - 1$ , we get the classical finite difference remainders [15], [16], [82]

$$R^{m-1} f(y, x) = \Delta^m f(y, x) \equiv \sum_{i=0}^m (-1)^{m-i} \binom{m}{i} f(x_i) \quad (0.22)$$

leading to the pointwise inequalities

$$|\Delta^m f(y, x)| \leq |y - x|^m [g_f(x) + g_f(y)] \quad (0.23)$$

characterizing the Sobolev spaces  $W^{m,p}(\mathbb{R}^n)$ ,  $p > 1$  by the condition  $g_f \in L^p(\mathbb{R}^n)$ . Thus (0.23) is a direct generalization of (0.4) and (1.9) below.

If the nodes  $x_i$ ,  $i = 1, \dots, n$  shrink to the center  $x_0 = x$ ,

$$\lim_{l \rightarrow \infty} x_i^l \rightarrow x \quad i = 1, \dots, m \text{ or } \|y - x\| \rightarrow 0 \quad (0.24)$$

for a family  $Z^l$  of interpolation schemes, the remainders  $R_{Z^l}^{m-1} f(y; x)$  reduce to the classical Taylor or, Taylor–Whitney [83] remainders for  $f$  differentiable (of class  $C^m(\mathbb{R}^n)$ ). We recall that for  $f \in C^m(\mathbb{R}^n)$  ( $x_m = y$ )

$$f[x, x_1, \dots, x_m] \rightarrow \frac{f^{(m)}(x)}{m!} \quad (0.25)$$

when  $|x_i - x| \rightarrow 0$ ,  $i = 1, \dots, m$  [36]. In this case the Newton representation formula (0.17) reduces to the classical Taylor formula [22], [36].

The general form of the pointwise inequality for functions  $f$  in  $W^{m,p}(\mathbb{R}^m)$  has the form

$$|f[x, x_1, \dots, x_{m-1}, y]| \leq g_f(x) + g_f(y), \quad (0.26)$$

for some  $g_f \in L^p(\mathbb{R}^n)$ .

The function  $g_f$  is majorized from above by the Hardy–Littlewood maximal function of the  $m$ -th Sobolev gradient  $\nabla^m f$ . For  $m = 1$ , it is shown below that  $g_f$  can be taken as  $g_f = MQf$ , the mean maximal quotient, see (1.9). For the (harmonic) interpolation scheme with equidistant nodes (0.23) the proof of the asserted estimate is given in [16].

The general case, for functions on the real-line [31], [36], [72], is deduced from

Genocchi–Hermite formula, [22], [72], in analogy with the direct proof of (0.4) above. For Taylor–Whitney interpolation scheme (0.24)–(0.25) the inequality (0.26) reduces to the case considered in [18].

The discussion of the simplest, now available, proof of (0.26) for the general interpolation schemes with arbitrary nodes  $\{x_i\}$  will be given in [17].

Since the Newton polynomial  $q_m(y)$  for the interpolation scheme  $Z : \{x_i \leq y, i = 1, \dots, m-1, x_0 = x\}$  is majorized by  $|y-x|^m$  we can also write (0.26) in the more familiar form

$$|R^{m-1}f(y, x)| \equiv |R_Z^{m-1}f(y, x)| \leq |y-x|^m(g_f(x) + g_f(y)). \quad (0.27)$$

For  $m = 1$  (0.27) reduces to the inequality

$$|Rf(y, x)| = |f(y) - f(x)| \leq |y-x|[g_f(x) + g_f(y)] \quad (0.28)$$

which also makes sense for measure-metric spaces  $(X, d, \mu)$ ,  $x, y \in X$  with distance  $d(x, y) \doteq |y-x|$  and measure  $\mu$ .

Of course the coefficients  $g_f$  in the formulas (0.23)–(0.28) vary from case to case and they appear in the literature on Sobolev spaces and their applications, over the last 25 years, at various places with various names used: variable Lipschitz coefficient, Sobolev metric gradient etc. Perhaps an appropriate term for  $g_f$  could be *Sobolev smoothness* or *Sobolev smoothness density*, which occasionally is used in this and related papers, of the author and his students and colleagues [12], [18], [19], [42], [43], [48], [52], etc.

In the works involved, the right hand side coefficients  $g_f(x)$ , in general, are estimated from above in terms of the maximal functions of the  $m$ -th gradient  $|\nabla^m f|$  evaluated at  $x$  and some geometric constants (universal, depending on  $n, p$  and  $Z$  only) in analogy with (1.8) below.

Even for functions on the real line the inequalities (0.23) and (0.26) seem to be absent in the literature.

This last point has been confronted with the expertise at some leading mathematical centers in the area (Sobolev Institute at Novosibirsk, Courant Institute at NYU, some other universities in USA and Europe, especially Finland, and a number of leading experts in Sobolev space theory.

The author will be grateful for any additional related references.

A natural and important problem is the understanding of the dependence of the pointwise estimate (0.26) on the interpolation scheme  $Z$  [4], [40]. It turns out that the divided difference  $f[Z]$  (0.26), and (0.23), (0.27) as well, are estimated from above by the values of the Sobolev smoothness  $g_f$  at the extremal points  $x$  and  $y$  of  $Z$ , and the upper bound does not depend on the intermediate nodes  $x_i$  in (0.16).

Even in the intensively studied classical case of the strictly one dimensional interpolation and approximation theory on the real line  $\mathbb{R}$  [25], [30], [32], [38], [72]... this problem seems not to have been satisfactorily clarified. Also in the rather recent papers of H. Triebel [82] and his school [47] (and some others following, available on the net) the independence of the right hand side in (0.23) from the intermediate nodes is overlooked.

Notice, in this connection, that in [30] a somehow different concept of A. P. Calderon's maximal function is used to characterize the higher order Sobolev type smoothness of measurable functions. Here again the values at all intermediate nodes of the corresponding maximal functions are used in the estimates of the expressions (0.22). Apparently, in view of (0.23) and [16], this fact deserves to be reconsidered.

The pointwise inequalities (0.23) and (0.28) are intimately related with S. M. Nikolskii's trace theorem quoted above: all quantities in the left hand

sides of (0.25), (0.26), (0.27) are determined by the fact that the restriction  $f_I$  of the function  $f$  to the segment  $I = I(x, y)$  belongs to the Sobolev space  $W^{m,p}[I]$  on the segment  $I$ . The evaluation of the terms  $g_f(x)$  and  $g_f(y)$  in the right hand sides of (0.23)–(0.27) is determined by averaging of the gradients, difference quotients, and other quantities describing the “generalized” Sobolev regularity of  $f$ , over open neighbourhoods of the segment  $I(x, y)$  in the enhancing space  $\mathbb{R}^n$  see (1.2) and (1.22), (2.7) below. The quantities  $g_f(x)$  and  $g_f(y)$  reflect the fact that the function  $f \in W^{m,p}(\mathbb{R}^n)$ .

All in all, the pointwise inequalities appear as primary facts of the Sobolev space theory:

they are formulated and proved at the very initial stages of the theory;

in a properly understood conceptual context they have simple and natural proofs <sup>4)</sup>;

in a rather direct way they lead to the central results of the theory: the famous Sobolev embedding theorems and delicate, far from obvious, continuity, differentiability, Peano approximate differentiability, Luzin’s property and other crucial properties of Sobolev space theory and allow us to see them in their natural general settings.

For the Sobolev space  $W^{m,p}$  these are: for  $m = 1$  measure, metric spaces [42] and for  $m \geq 2$  measurable subsets or submanifolds of domains in  $\mathbb{R}^n$ .

The text below and [17] is a continuation of a series of papers published over the last two decades, partially quoted in our references, illustrating some of the above statements.

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<sup>4)</sup>in mathematics simple ideas usually come last (J. Hadamard).

# 1 Maximal mean difference quotients

Let  $f$  be a real valued locally integrable function  $f \in L_{\text{loc}}(\mathbb{R}^n)$ . The maximal mean difference quotient at the point  $x \in \mathbb{R}^n$ , denoted by  $MQf(x)$ , is defined as the following l.u.b.

$$MQf(x) = \sup_{r>0} M_r Qf(x) \tag{1.1}$$

of the averaged values of the difference quotients

$$M_r Qf(x) = \int_{B(x,r)} \frac{|f(z) - f(x)|}{|z - x|} dz. \tag{1.2}$$

(Here  $B(x, r)$  is the open ball  $\{y : |y - x| < r\}$ .)

Our general assumption is that for bounded subsets  $\Omega \subset \mathbb{R}^n$  the integral

$$\int_{\Omega} \int_{\Omega} \frac{|f(z) - f(x)|}{|z - x|} dz dx < \infty \tag{1.3}$$

is finite. Condition (1.3) implies that the averaged values (1.2) are well defined for  $x$  a.e.

**Definition 1.1.** For  $p \geq 1$  the class  $MQ^{1,p}(\mathbb{R}^n)$  is defined as the set of all  $f$  in  $L^p(\mathbb{R}^n)$  for which  $MQf$  is in  $L^p(\mathbb{R}^n)$ .

If  $MQf(x)$  is finite a.e. the following proposition holds

**Proposition 1.1.** *With an absolute constant  $c = c(n)$  the pointwise inequality*

$$|f(x) - f(y)| \leq c|x - y|(MQf(x) + MQf(y)) \tag{1.4}$$

*holds a.e.; specifically for all  $x, y$  for which all terms in (1.4) are meaningful and finite, e.g. for all Lebesgue points of  $f$  (the right-hand side of (1.4) is defined, possibly  $\infty$ , for all  $x, y, x \neq y$ ).*

*Proof.* For  $x, y$  fixed let  $\Sigma_r = \Sigma_r(x, y)$  be the spherical segment

$$\Sigma_r = B(x, r) \cap B(y, r), \quad r = |x - y|. \quad (1.5)$$

For all  $z \in \Sigma_r$   $z \neq x \neq y$  we have

$$|f(x) - f(y)| \leq |f(x) - f(z)| \frac{|z - x|}{|z - x|} + |f(z) - f(y)| \frac{|z - y|}{|z - y|} \quad (1.6)$$

thus

$$\frac{|f(x) - f(y)|}{|x - y|} \leq \frac{|f(x) - f(z)|}{|x - z|} + \frac{|f(z) - f(y)|}{|z - y|}, \quad (1.7)$$

since  $|x - z| \leq |x - y|$ ,  $|y - z| \leq |x - y|$ .

Averaging (1.7) with respect to  $z$  over  $\Sigma_r$  we get

$$\begin{aligned} & \frac{|f(x) - f(y)|}{|x - y|} \\ & \leq \frac{|B(x, r)|}{|\Sigma_r|} \left( \int_{B(x, r)} \frac{|f(x) - f(z)|}{|x - z|} dz + \int_{B(y, r)} \frac{|f(z) - f(y)|}{|z - y|} dz \right). \end{aligned}$$

By the geometry of  $\mathbb{R}^n$

$$\frac{|B(x, r)|}{|\Sigma_r|} = \frac{|B(y, r)|}{|\Sigma_r|} = c(n) \quad (1.8)$$

and (1.4) follows with  $c = c(n)$ .

Written in the form

$$|f(x) - f(y)| \leq |x - y|(g(x) + g(y)), \quad g(x) = g_f(x) = MQf(x), \quad (1.9)$$

(1.4) is the first and, apparently, most natural example of the pointwise inequalities characterizing the general classes of function spaces of Sobolev type, see [18], [42].

We shall call the inequality (1.9) the *Sobolev pointwise inequality* of order 1.

Inequalities (1.9) (and some their modifications) have been widely used in the last two decades to define various generalizations of classical Sobolev spaces  $W^{1,p}(\mathbb{R}^n)$  [15], [42], [44], [60], [49]. We shall use for these spaces the general term *metric Sobolev spaces* [49], [52] and, for a given  $f \in L^p(\mathbb{R}^n)$ , any admissible function  $g$  in the right-hand side of (1.9) will be termed a Sobolev metric gradient of  $f$ . The collection  $SMG(f)$  of all Sobolev metric gradients for  $f \in L^p(\mathbb{R}^n)$  is a convex (closed) subset of nonnegative measurable functions in  $\mathbb{R}^n$ . In fact it is a lattice (i.e.,  $g_1$  and  $g_2 \in SMG(f)$  implies  $\min(g_1, g_2) \in SMG(f)$ ). Also if  $g \in SMG(f)$  any  $g_1 \geq g$  is in  $SMG(f)$  as well.

In particular, the following definition [42] is broadly used.

**Definition 1.2.**  $M^{1,p}(\mathbb{R}^n)$  is the class of all  $f \in L^p_{\text{loc}}(\mathbb{R}^n)$  which admit Sobolev metric gradients in  $L^p(\mathbb{R}^n)$ .

Our Proposition 1.1 implies

**Corollary 1.1.**  $MQ^{1,p}(\mathbb{R}^n)$  ( $p \geq 1$ ) is a Sobolev metric space:  $MQ^{1,p} \subset M^{1,p}(\mathbb{R}^n)$ .

Notice that each  $f \in MQ^{1,p}$  has a well specified Sobolev metric gradient  $g = MQ(f)$  defined by the formulas (1.1) and (1.2).

In fact the above definitions and the Hardy–Littlewood theorem (recalled below) imply also

**Proposition 1.2.** *If  $f \in M^{1,p}(\mathbb{R}^n)$ ,  $p \geq 1$ , then any  $g \in SMG(f)$  can be estimated from below pointwise by the mean maximal quotient*

$$MQf(x) \leq g(x) + Mg(x) \leq 2Mg(x) \quad \text{for a.e. } x, \quad (1.10)$$

where  $Mg(x)$  is the Hardy–Littlewood maximal function for  $g \in L^p_{\text{loc}}(\mathbb{R}^n)$ ,  $g \geq 0$ .

This is a direct consequence of the definitions and the inequality (1.9).

In particular, the next corollary follows.

**Corollary 1.2.** *For  $p > 1$ ,  $M^{1,p}(\mathbb{R}^n) \subset MQ^{1,p}(\mathbb{R}^n)$  (with equivalent seminorms).*

In the sense described by Proposition 1.2, the maximal mean quotient takes the role of a “pointwise minimal” element in  $SMG(f)$ .

$MQ^{1,p}(\mathbb{R}^n)$  is a linear space with the seminorm  $\|f\|_{1,p} = \|MQf\|_{L_p}$ .

In Proposition 1.1 the Sobolev pointwise inequality (1.9) was obtained by averaging the difference quotient (1.7). If, instead, we average the oscillation  $|f(z) - f(x)|$  on open non empty convex subsets  $\Sigma$  of  $\mathbb{R}^n$  we come to the Poincaré inequalities as we now recall.

For the pair of points  $x, z$  of  $\Sigma$ ,  $x \neq z$  we have

$$|f(z) - f(x)| \leq |z - x| \frac{|f(z) - f(x)|}{|z - x|} \leq \text{diam } \Sigma \frac{|f(z) - f(x)|}{|z - x|}. \quad (1.11)$$

If  $\Sigma$ , rather a family  $\Sigma(r)$ ,  $\Sigma = \Sigma(r)$ , is a subset family of  $B(x, r)$  such that

$$\frac{|B|}{|\Sigma|} \leq C, \quad \text{independent of } r, \quad (1.12)$$

e.g.,  $\Sigma = B(x, r)$  or  $\Sigma = B(x, r) \cap B(y, r)$ , we get

$$|f_\Sigma - f(x)| \leq \text{diam } \Sigma \frac{|B|}{|\Sigma|} \int_B \frac{|f(z) - f(x)|}{|z - x|} dz \leq C(\text{diam } \Sigma) M_r Qf(x)$$

consequently

$$|f_\Sigma - f(x)| \leq C(\text{diam } \Sigma) MQf(x). \quad (1.13)$$

In particular, for  $\Sigma = B(x, r)$

$$|f(x) - f_{B(x,r)}| \leq CrMQf(x) \doteq rg(x) \quad (1.14)$$

for almost all  $x$ , e.g. for all Lebesgue points of  $f \in L^p(\mathbb{R}^n)$  and, analogously, for  $\Sigma_r = B(x, r) \cap B(y, r)$ ,  $c_n$  as in (1.8),

$$|f(x) - f_{\Sigma_r}| \leq rc_n g(x), \quad r = |x - y|. \quad (1.15)$$

(1.14) and (1.15) can be viewed as the pointwise form of the Poincaré type inequalities for the function space  $MQ^{1,p}$ .

Raised to the exponent  $p \geq 1$  and integrated, (1.14) and (1.15) lead to the familiar integral form of the Poincaré inequalities. They are recalled below for the case of smooth functions  $f \in C^1(\mathbb{R}^n)$  or  $f \in W^{1,p}(\mathbb{R}^n)$ .

For functions in the class  $C^1(\mathbb{R}^n)$  the fundamental theorem of calculus

$$f(z) - f(x) = \int_0^1 \frac{d}{dt} f(x + t(z-x)) dt = \int_0^1 \langle \nabla f(x + t(z-x)), z-x \rangle dt \quad (1.16)$$

implies the inequality

$$|f(z) - f(x)| \leq |z-x| \int_0^1 |\nabla f|(x + t(z-x)) dt. \quad (1.17)$$

Let  $\Sigma$  be a convex set (in  $\mathbb{R}^n$ ) and consider pairs  $x, z \in \Sigma$ , then  $|x-z| \leq \text{diam } \Sigma$ . Averaging (1.17) over  $\Sigma$  with respect to  $x$  and  $z$  we get

$$\begin{aligned} |f(x) - f_\Sigma| &\leq \text{diam } \Sigma \int_0^1 \int_\Sigma |\nabla f|(x + t(z-x)) dt dz \\ &\leq c_n \text{diam } \Sigma \int_\Sigma |\nabla f| dz \end{aligned} \quad (1.18)$$

$$\int_{\Sigma} |f(x) - f_{\Sigma}| dx \leq \text{diam } \Sigma \int_0^1 \int_{\Sigma} |\nabla f|(x + t(z - x)) dt dz dx. \quad (1.19)$$

After standard computation using the change of variables  $\zeta = x + t(z - x)$ ,  $d\zeta = t^n dz$  and the Hölder inequality  $\int_{\Sigma} g d\mu \leq (\int_{\Sigma} g^p d\mu)^{1/p}$ ,  $p \geq 1$ ,  $g \in L^p(\Sigma, \mu)$ , we come [39], [60] to the classical Poincaré inequalities

$$\int_{\Sigma} |f - f_{\Sigma}| dx \leq c_{n,p} \text{diam } \Sigma \int_{\Sigma} |\nabla f| dx \quad (1.20)$$

and

$$\left( \int_{\Sigma} |f - f_{\Sigma}|^p dx \right)^{1/p} \leq c_{n,p} \text{diam } \Sigma \left( \int_{\Sigma} |\nabla f|^p dx \right)^{1/p}, \quad p \geq 1. \quad (1.21)$$

The Poincaré integral inequalities (1.21), (1.20), (1.19) are the only ones above<sup>5)</sup> which are meaningful and valid for arbitrary functions  $f$  in the classes  $W_{\text{loc}}^{1,p}(\mathbb{R}^n)$ ,  $p \geq 1$ . They are generally considered to be much weaker than the more subtle Sobolev integral inequalities discussed later. Sobolev integral inequalities have an essentially different form for the values of the parameter  $p$ :  $1 \leq p < n$ ,  $p = n$  and  $p > n$ .

The pointwise form (1.13), (1.14) of the Poincaré inequalities can be given sense for a.e. point of  $\Sigma$  only (for all values of the parameter  $p \geq 1$ ) if the gradient  $|\nabla f|$  in the right-hand side of the Poincaré inequalities in  $W^{1,p}(\mathbb{R}^n)$  is replaced by the more refined concept of the Hardy–Littlewood maximal function of the gradient  $|\nabla f|$ , as will be explained later.

Actually Sobolev was averaging the gradients, not the difference quotients, but his fundamental papers [75], [76], [77] led to the discovery of concepts, ideas and new mathematical facts of unprecedented importance for higher analysis, differential and partial differential equations and applications.

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<sup>5)</sup>functions in  $W^{1,p}$  are discontinuous and have values a.e. only

In  $\mathbb{R}^n$  the relation of the class  $MQ^{1,1}(\mathbb{R}^n)$  with the class of smooth functions and classical Sobolev spaces  $W^{1,1}(\mathbb{R}^n)$  is described by the elementary (well known)

**Lemma 1.1.** *For  $f \in C^1(\mathbb{R}^n)$  and any ball  $B(x, r)$  the inequality holds*

$$\begin{aligned} \int_{B(x,r)} \frac{|f(z) - f(x)|}{|z - x|} dz &= \int_0^1 \int_{B(x,r)} |\nabla f(x + (z - x)t)| dz dt \\ &\leq \int_0^1 \int_{B(x,tr)} |\nabla f(z)| dz dt. \end{aligned} \quad (1.22)$$

*Proof.* Use (1.17) and the change of variable:  $\zeta = x + (z - x)t$ ,  $d\zeta = t^n dz$ , transforming the ball  $B(x, r)$  into  $B(x, tr)$ .  $\square$

For functions in the classical Sobolev space  $W_{\text{loc}}^{1,p}(\mathbb{R}^n)$ ,  $1 \leq p \leq \infty$ , invoking the definition of the Hardy–Littlewood maximal gradient  $\mathcal{M}(|\nabla f|)(x)$  (recalled below) we get the following corollary.

**Corollary 1.3.** *For all  $x \in \mathbb{R}^n$  the inequality*

$$MQf(x) \leq \mathcal{M}(|\nabla f|)(x). \quad (1.23)$$

*holds; in particular,  $W^{1,p}(\mathbb{R}^n) \subset MQ^{1,p}(\mathbb{R}^n)$  for  $1 < p \leq \infty$ .*

The inverse inequality and inverse inclusion, though essentially true, are somewhat more delicate [19], [22], [26], [29], [25], [42].

Since the concept of the Hardy–Littlewood maximal function [49] is very helpful to describe important consequences of (1.22) [41], [46], we recall it briefly.

The centered Hardy–Littlewood maximal function of  $f \in L_{\text{loc}}^1(\mathbb{R}^n)$  is defined

as the supremum of the averages

$$\mathcal{M}(f)(x) = \sup_{r>0} \int_{B(x,r)} |f| dy.$$

The uncentered Hardy–Littlewood maximal function is defined as

$$M(f)(x) = \sup_r \int_{x \in B(y,r)} |f| dz,$$

i.e. when the averaging is performed over all open balls that contain the point  $x$ .

Obviously  $\mathcal{M}f(x) \leq Mf(x) \leq 2^n \mathcal{M}f(x)$  and both  $\mathcal{M}f$  and  $Mf$  are lower semicontinuous, though, roughly speaking,  $Mf$  is more regular (“smoother”) than  $\mathcal{M}f$ , e.g.  $Mf$  is even continuous.

For functions on the real line, both right  $M_R f$  and left  $M_L f$  are useful maximal functions and play an important role in many subtle questions of classical analysis on the real line  $\mathbb{R}$ :

$$M_R f(x) = \sup_{r>0} \int_x^{x+r} |f(y)| dy, \quad M_L f(x) = \sup_{r>0} \int_{x-r}^x |f(y)| dy.$$

For functions on  $\mathbb{R}^n$  considered as the Cartesian products  $\mathbb{R}^n = \mathbb{R} \times \dots \times \mathbb{R}$ ,  $n$  times, also iterated maximal functions [55], [41] and Hardy–Littlewood maximal functions with respect to cubes, rectangular maximal functions, dyadic maximal functions  $M_d f(x)$  are very useful in a variety of contexts.

**Proposition 1.3.** *For  $1 < p \leq \infty$  the pointwise inequality (1.9) characterises the Sobolev space  $W^{1,p}(\mathbb{R}^n)$  and  $W_{loc}^{1,p}(\mathbb{R}^n)$  (and  $MQ^{1,p}(\mathbb{R}^n)$  as well).*

*Proof.* Proposition 1.3 and its various proofs were well understood (and published) at the time when [18] and [42] first appeared. However, the proof of the inequality (1.9) for the class  $MQ^{1,p}(\mathbb{R}^n)$  (Proposition 1.1 above) is, apparently,

published here first<sup>6</sup>). We skip here the details of the well known proof that the inequality (1.9) for a function  $f \in L^p_{\text{loc}}(\mathbb{R}^n)$  implies that  $f \in W^{1,p}_{\text{loc}}(\mathbb{R}^n)$ .  $\square$

Before recalling the rather not trivial implications of the pointwise Sobolev inequality (1.9), the local Poincaré type pointwise inequalities (1.13), (1.14), (1.15) and the Poincaré integral inequalities (1.20), (1.21) we show that the concept of the mean-quotient spaces  $MQ^{1,p}_{\text{loc}}(\mathbb{R}^n)$  is meaningful in the general metric measure space context and, via Proposition 1.1 above, under some mild geometric conditions (doubling measure and overlapping condition (1.8), (1.12)) it leads directly to the metric Sobolev–Poincaré type vector spaces.

Thus, given a triple  $(X, d, \mu)$ : a complete metric measure space  $X$  with distance  $d = d(x, y)$ ,  $x, y \in X$  and a Borel regular measure  $\mu$ , exactly as in (1.1) and (1.2) and the Definition 1.1 above, we define the (restricted) maximal mean quotient ( $d(z, x) = |z - x|$  in (1.2))

$$MQ_R f(x) = \sup_{r < R} \int_{B(x,r)} \frac{|f(z) - f(x)|}{d(z,x)} d\mu \quad (1.24)$$

and the maximal mean difference quotient at  $x \in X$

$$MQf(x) = \sup_{R > 0} MQ_R f(x). \quad (1.25)$$

The requirement  $MQf \in L^p(X, \mu)$ ,  $p \geq 1$ , where  $L^p(X, \mu)$  denotes the Lebesgue space of the measure space  $(X, \mu)$ , defines then the vector space  $MQ^{1,p}(X, d, \mu)$ . If the geometric measure conditions, analogous to (1.8) or (1.12) hold, the proof above of the Proposition 1.1 with appropriate changes is valid. We obtain

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<sup>6</sup>The present proof of Proposition 1.1, together with Lemma 1.1 above, also explicitly shows that the complete proof of Lemma 1 in [16] does not need any references to “known facts of harmonic analysis” [79], [80]. It is a direct and elementary consequence of the accepted definitions of the spaces  $MQ^{1,p}$  and the Sobolev space  $W^{1,p}(\mathbb{R}^n)$ .

**Corollary 1.4.** *The linear vector space  $MQ^{1,p}(X, d, \mu)$  has the structure of a metric Sobolev space with intrinsic metric gradient  $MQf(x) \in L^p(X, \mu)$ .*

If the triple  $(X, d, \mu)$  satisfies some kind of doubling condition or Ahlfors  $Q$ -regular measure condition, the theory of Hajlasz–Koskela and their followers applies [42], [43], [44], [45], [49], [51], [52], [60].

In particular, the analogues of our Propositions 1.1, 1.2, 1.3 above and the local Poincaré-type inequalities described above in Corollaries 1.1, 1.2, 1.3, 1.4 hold. These imply far reaching, mostly not trivial, consequences with proofs which can be traced back to many authors.

We give below a short list of some of the results that can be deduced.

1. Refinements of the Lebesgue differentiation theorems for  $L^1(X, \mu)$  functions for Sobolev  $W^{1,p}(X, d, \mu)$  and  $MQ^{1,p}(X, d, \mu)$  spaces. Approximate differentiability in the Euclidean case.
2. Hölder continuity of functions in  $MQ^{1,p}(X, d, \mu)$  spaces for  $p > s$ , Sobolev imbedding  $MQ^{1,p}(X, d, \mu) \hookrightarrow C_{\text{loc}}^\alpha$ ,  $p > s$ ,  $\alpha = 1 - \frac{s}{p}$ ,  $\alpha > 0$  ( $s$ —Hausdorff dimension of  $(X, d, \mu)$ ; in the Euclidean case  $X = \mathbb{R}^n$ ,  $s = n$ ).
3. Differentiability in the Euclidean case for  $p > n$ , A. P. Calderon theorem [27], J. Cheeger type approach in measure metric case for the triple  $(X, d, \mu)$  [2], [6], [21], [28], [49], [57], [79].
4. Luzin type approximation by Lipschitz or  $C^1$  functions [48].
5. The Hajlasz–Koskela Theorem [45] on the general local Sobolev imbedding theorem of  $MQ^{1,p}(X, d, \mu)$  spaces for  $p < s$  into  $L^{p^*}$  spaces,  $p^* = \frac{ps}{s-p}$ .

## 2 Higher order Sobolev spaces

For Sobolev space  $W^{m,p}(\mathbb{R}^n)$ ,  $m \geq 2$ ,  $1 \leq p \leq \infty$  higher degree Taylor polynomials  $T^k f(y; x)$ ,  $0 \leq k \leq m - 1$ , centered at  $x$  and the Taylor–Whitney remainders  $R^k f(y; x)$  come into play.

Thus, by definition, for  $f \in W^{m,p}(\mathbb{R}^n)$  we have

$$T^k f(y, x) = f_0(x) + f_1(x)(y - x) + \dots + f_k(x) \frac{(y - x)^k}{k!} \quad (2.1)$$

and

$$R^k f(y, x) = f(y) - T^k f(y, x) \quad (2.2)$$

with

$$f_0(x) \equiv f(x)$$

and

$$f_k(x) = D_x^k f(x), \quad k = 1, \dots, m - 1 \quad (2.3)$$

understood as the generalized Sobolev, weak or distributional, derivatives of the function  $f$ .

In the formulas (2.1)–(2.3) we use standard polylinear algebra and multidimensional analysis notations for the  $n$ -dimensional vector space.

It is convenient to call the variables  $x$  in the formulas (2.1)–(2.3) as *field variables* and the variables  $y$  as *space variables*:  $y \in \mathbb{R}^n$ . Although in our basic model case of Sobolev spaces:  $W^{m,p}(\mathbb{R}^n)$  both  $x$  and  $y$  vary in the whole space  $\mathbb{R}^n$ ,  $(x, y) \in \mathbb{R}_x^n \times \mathbb{R}_y^n$ , in most interesting cases, considered in the general theory and in applications, the field variables are restricted to a subset  $\Sigma \subset \mathbb{R}^n$ , a subdomain of  $\mathbb{R}^n$ , open or closed or just on arbitrary closed subset of  $\mathbb{R}_x^n$ . Both variables  $x$  and  $y$  in the Taylor–Whitney remainders  $R^k f(y, x)$ , and  $R_i^k f(y, x)$  introduced later, are considered to be field variables and take their values in the

subset  $\Sigma$ . For general Sobolev spaces the linear operator coefficients  $f_0, \dots, f_k$ ,  $k \leq m - 1$  have their values in the Lebesgue spaces  $L^p(\mathbb{R}^n)$ ,  $(L^p(\Sigma))$ .

The operator valued coefficients  $f_k$  in (2.1) and (2.2) need not satisfy the restricting conditions (2.3). In this general case we call the collection  $F = \{f_0(x), \dots, f_k(x)\}$ ,  $x \in \Sigma$  a *Whitney  $k$ -jet  $F$* , defined on  $\Sigma$  and the expressions (2.1) and (2.2) are termed the *Taylor–Whitney fields* and *Taylor–Whitney remainders of the  $k$ -jet  $F$* . For convenience the set of  $k$ -jets on  $\Sigma$  is denoted as:  $J^k(\Sigma)$ .

Taylor–Whitney fields  $T^k F(y, x)$  can be differentiated in  $y$ , without any restrictions and can be nicely described in terms of the formal jet derivatives  $D_l : J^k \rightarrow J^{k-|l|}$ ,  $D_l F = \{f_l, f_{1+l}, \dots, f_k\}$

$$D_y^l T^k F(y, x) = T^{k-|l|}(D_l F)(y, x) \quad (2.4)$$

and their Taylor–Whitney remainders

$$\begin{aligned} R^{k-|l|}(D_l F)(y, x) &= R_l^k F(y, x), \quad l = 0, 1, \dots, k - 1, \\ R^k F(y, x) &= R_0^k F(y, x) \end{aligned} \quad (2.5)$$

connected by the formulas (2.2) for the  $l$ -th component of the  $k$ -jet  $F$  or the first (i.e. zero) component of  $D_l F$

$$R^{k-|l|}(D_l F)(y, x) + T^{k-|l|}(D_l F)(y, x) \equiv f_l(y). \quad (2.6)$$

The recalled formulas (2.1)–(2.6) and their admissible derivatives with respect to the space variables, generate what has been called the Taylor-algebra:  $\mathfrak{T}_m(\mathbb{R}^n)$ . It has been introduced mainly by G. Glaeser [40] and B. Malgrange [63] as an algebraic tool to describe the H. Whitney [83]–[86] theory of continuously

differentiable functions  $C^k(\Sigma)$ , on arbitrary closed subsets of  $\mathbb{R}^n$ .

As shown in a series of papers, starting [18], [8], [12] etc., up to the present one, these concepts, properly adjusted, are very useful in the geometric understanding of the theory of Sobolev spaces as well.

Given a function  $f \in W^{m,p}(\mathbb{R}^n)$  represented by its  $(m-1)$  jet

$$F = \{f(x), \dots, f^{(i)}(x), \dots, f^{(m-1)}(x)\}, \quad f^{(i)} = D^i f$$

we define the  $m$ -th order restricted mean quotient

$$M_R Q^m F(x) = \sup_{r \leq R} \int_{B(x,r)} \frac{|R^{m-1} F(z,x)|}{|z-x|^m} dz. \quad (2.7)$$

We also define  $m$ -th maximal mean quotient

$$MQ^m f(x) = \sup_{R>0} M_R Q^m f(x). \quad (2.8)$$

In general the following fact holds:

**Proposition 2.1.** *For a function  $f \in W^{m-1,p}(\mathbb{R}^n)$ ,  $1 < p \leq \infty$ , the quantities  $MQ_R^m f(x)$  completely control the Sobolev smoothness of the function  $f$  in  $W^{m,p}(\mathbb{R}^n)$ .*

As it stands for  $m \geq 2$  Proposition 2.1 is rather vague and needs some comments: The case  $m = 1$  is completely clarified by our Proposition 1.1 in Section 1: we start with a function  $f \in L^p(\mathbb{R}^n)$  and get simple sufficient condition (obviously also necessary) for  $f$  to be in the Sobolev space  $W^{1,p}(\mathbb{R}^n)$  with the full control of the norm<sup>7)</sup>  $\|f\|_{W^{1,p}(\mathbb{R}^n)}$ :  $MQ^1(f) \in L^p(\mathbb{R}^n)$ ,  $\|f\|_{L^{1,p}(\mathbb{R}^n)} \leq C \|MQ^1 f\|_{L^p}$  with an absolute constant  $C$ , depending on  $n$  only.

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<sup>7)</sup> Here  $L^{1,p}(\mathbb{R}^n)$  stands for the homogeneous space  $W^{1,p}(\mathbb{R}^n)$  with the semi-norm  $\|f\|_{L^{1,p}} = \|\nabla f\|_{L^p}$ .

For  $m \geq 2$  the  $m$ -jet  $F$  representing  $f \in W^{m-1,p}(\mathbb{R}^n)$  allows us to define the quantity  $M_R Q^m f(x)$ .

To extend Proposition 2.1 to arbitrary functions in the Lebesgue space  $L^p(\mathbb{R}^n)$  directly, some other process e.g. Newton–Lagrange interpolation, described in [16], see also [82], is needed. We shall come back to this topic in more detail in [17].

Now we briefly show how to handle the case  $m = 2$  with the elementary geometric tools used for the case  $m = 1$  in Section 1 and some simple considerations in the Taylor algebra  $\mathfrak{T}_2(\mathbb{R}^n)$ , see [14].

**Lemma 2.1.** *For arbitrary  $f \in W^{2,1}(\mathbb{R}^n)$ , almost all  $x, y \in \mathbb{R}^n$  and  $z \in \Sigma_r(x, y)$ ,  $r = |x - y|$  the pointwise inequality holds*

$$\frac{|\mathbb{R}^1 f(y, x)|}{|y - x|^2} \leq \frac{|\mathbb{R}^1 f(z, y)|}{|z - y|^2} + \frac{|\mathbb{R}^1 f(z, x)|}{|z - x|^2} + \frac{|f'(z) - f'(x)|}{|z - x|}. \quad (2.9)$$

This is an analogue of the estimate (1.7) in Section 1.

*Proof.* To avoid the troublesome discussion of the set  $N_f$  of points in  $\mathbb{R}^n$ , excluded by the words “almost all” above, we prove here the lemma for  $f \in C^1(\mathbb{R}^n)$ . We have,  $|N_f| = 0$ : the Lebesgue  $n$ -measure of  $N_f$  is zero. For  $f \in C^1(\mathbb{R}^n)$ , the proof reduces to simple algebraic calculations in the Taylor algebra  $\mathfrak{T}_1(f)$  of order 1 and elementary geometry of the sphere in  $\mathbb{R}^n$  already exploited the discussion of Sobolev spaces  $W^{1,p}(\mathbb{R}^n)$ .

For the triple of points  $(x, y, z) \in \Sigma \times \Sigma \times \Sigma$  we introduce the expressions

$$\begin{aligned} Rf(y, x) &= f(y) - f(x) \\ R^1 f(y, x) &= f(y) - f(x) - f'(x)(y - x) \end{aligned} \quad (2.10)$$

and analogous ones for the pairs  $(x, z)$  and  $(y, z)$ .

In the Taylor algebra  $\mathfrak{T}_2(f)$  we have ( $R \equiv R^0$ )

$$R(y, x) + R(x, y) = 0 \quad (2.11_1)$$

$$R(y, x) - R(y, z) = -R(x, z) \quad (2.11_2)$$

and in the Taylor algebra  $\mathfrak{T}_2(f)$  we have ( $R \equiv R^0$ )

$$R^1 f(y, x) + R^1 f(x, y) = \langle f'(y) - f'(x), y - x \rangle \quad (2.12_1)$$

$$\begin{aligned} P^1(x, y, z) &\equiv R^1 f(y, x) - R^1 f(y, z) \\ &= -R^1 f(z, x) + \langle f'(z) - f'(x), y - z \rangle \end{aligned} \quad (2.12_2)$$

Now (2.12<sub>2</sub>) for  $z \in \Sigma_r(x, y)$ ,  $r = |x - y|$  gives

$$\frac{|\mathbb{R}^1 f(y, x)|}{|y - x|^2} \leq \frac{|\mathbb{R}^1 f(z, y)|}{|z - y|^2} + \frac{|\mathbb{R}^1 f(z, x)|}{|z - x|^2} + \frac{|f'(z) - f'(x)|}{|z - x|}. \quad (2.13)$$

which completes the proof.  $\square$

It is also important to note the following consequence of (2.12<sub>2</sub>):

$$D_y P^1(x, y, z) \equiv f'(z) - f'(x). \quad (2.14)$$

Notice now that the integral means

$$\int_B \frac{|P^1(x, y, z)|}{|z - y|^2} dz \quad \text{and} \quad \int_B \frac{|P^1(x, y, z)|}{|x - z|^2} dy$$

over balls  $B$  in the space  $\mathbb{R}_y^n$  are controlled by the values of  $MQ^2 f(y)$  and  $MQ^2 f(x)$ . In consequence, by the Markov inequalities, see [12] and [14], the values of the quotients  $\frac{|f'(z) - f'(x)|}{|z - x|}$  are estimated from above by the maximal functions of the gradient  $\nabla^2 f$  pointwise. Thus we see also that in this somehow more

subtle approach to pointwise inequalities for Sobolev functions  $f \in W^{m,p}(\mathbb{R}^n)$ ,  $p > 1$ ,  $m \geq 2$ , the iterated maximal functions of the gradients appear naturally. The fundamental fact is that they all are, in the sense of  $L^p$  norms for  $p > 1$ , estimated from above by  $L^p$  norms of the Hardy–Littlewood maximal functions  $M(|\nabla^m f|)(x)$  or their iterates. In the existing literature this seems not to have been exposed clearly enough.

Now, by averaging of (2.13) over  $\Sigma_r$  we obtain the pointwise control of the left-hand side of (2.13) by the mean maximal quotients  $MQ^2 f$  and their Hardy–Littlewood maximal functions  $M(MQ^2 f)$  evaluated at the points  $x$  and  $y$ , as required in Proposition 2.1.

We remark also that the averages

$$\int_B \frac{|f'(z) - f'(x)|}{|z - x|} dz$$

in (2.9) can be also handled for functions  $f$  in  $W^{2,1}(\mathbb{R}^n)$ , or  $f' \in W^{1,1}(\mathbb{R}^n)$  by an inductive procedure with respect to  $m$ :  $f \in W^{2,p}(\mathbb{R}^n) \rightarrow f' \in W^{1,p}(\mathbb{R}^n)$  and  $f' \in MQ^{1,p}(\mathbb{R}^n)$ ,  $p \geq 1$ , that is the resulting inclusion

$$\int_B \frac{|f'(z) - f'(x)|}{|z - x|} dz \in L^p(\mathbb{R}^n)$$

holds.

In this way we get yet another approach to the proof of the Proposition 2.1.

### 3 Final comments

Paragraph 2 above is the first introductory step in a novel approach to the Sobolev pointwise inequalities characterizing the spaces  $W^{m,p}(\mathbb{R}^n)$  for  $m \geq 2$ . It will be continued in [17], along the road sketched in the introduction. In

particular we hope to analyze in [17] the simplest, most natural and transparent proofs of the inequality (0.26) expressed in terms of the divided differences and finite differences, see [15], [16], [80].

Some of the results and ideas developed in [20], [22] will be included in [17] as well.

We hope also to extend our approach to fractional Sobolev spaces, Besov type spaces and related trace and extension problems.

Stabilization to a polynomial of the functions in the homogeneous Sobolev spaces  $\mathring{W}^{m,p}(\mathbb{R}^n)$  will be also considered. See [1], [5], [88].

Clearly, the remarks at the end of paragraph 1 and the list of problems quoted there, should be considered as a forecast for a more detailed and thorough discussion in [17] based on the results, methods and proofs scattered in some of the references in our list at the end of this paper.

The proofs mentioned require a rather subtle refinement of the concept of Hardy–Littlewood maximal function [21], [41], [45], [49], [51], [54], [37], [53], [60], [62], upper gradient, higher order upper gradients, weak maximal functions, weak and strong inverse Hölder inequalities, etc. Here we refer to [21] where some of these refinements were elaborated mainly in connection with the requirements of the analytical problems of quasiconformal mappings theory, though some of them may be traced back even to some earlier works.

In the early eighties of the last century the paper [21] was rather widely studied in Finland and, later, quite many of the concepts and methods used in [21], were somehow “pushed out” to the “folklore”!

They occupy a prominent role, though, in many papers of the Finnish School from the period 1990–2005 quoted in our reference list. As an example see the proof of A. Calderón theorem [27] on total differentiability of Sobolev functions in  $W^{1,p}(\mathbb{R}^n)$  for  $p > n$ , given in [49], originally in [21], see also [6], [38], [67], [79].

Sobolev space theory discussed along the novel approach propagated, among others, in this paper and in [22], suggests, and stimulates, asking many new, apparently interesting and non-trivial questions. Here is an example: in an “admissible” family  $F$  of interpolation schemes  $Z$ , invariant under affine transformations of the Euclidean space  $\mathbb{R}^n$ , for a function  $f$  in the Sobolev space  $W^{m,p}(\mathbb{R}^n)$ , describe an “optimal” (?) one, “approximately optimal” (?!), for its “numerical efficiency” (?) or “cost of its computational implementation” (?)<sup>8)</sup>

As a working conjecture for this problem we state the following inequality for the Sobolev smoothness density  $g_{f,Z}(x)$

$$g_{f,Z}(x) \leq C_{n,m} g_{f,TW}(x), \quad (3.1)$$

where  $g_{f,TW}(x)$  is the notation of the Sobolev smoothness of the function  $f$  for the Taylor–Whitney interpolation, as in (2.2) or in [18], and  $g_{f,Z}$  is calculated in analogy with (0.3), see also [22];  $C_{n,m}$  is an absolute constant to be found.

Even for the case of Sobolev functions on the real line the inequalities of type (3.1) and the constant in (3.1) are not known [29], [31], [22].

In the “enormous” literature on Sobolev spaces and their applications there can hardly be found any references concerning this and related questions — even on a segment of the real line! — [4], [21], [26], [70].

Referring again to the “novelty” of the approach to the general theory of Sobolev spaces, emerging from this paper and others recent related, let me express the hope that this will be a useful contribution in the direction of producing a concise, readable exposition of the Sobolev theory, with minimal necessary technicalities<sup>9)</sup> and not missing any typical properties and concepts currently

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<sup>8)</sup>question marks mean that the suggested concepts *are to be defined as an essential part of the research process*, requiring maybe a rather deep, new insight.

<sup>9)</sup>Man soll die Dinge so einfach machen wie möglich — aber nicht einfacher (A. Einstein); translation for sciences (H. Triebel): One should present assertions as simple as possible — but not simpler.

used in geometrical, analytical and interdisciplinary applications. Any critical remarks, suggestions, corrections and new references will be very welcome.

Despite the rapidly growing list of recent monographic publications<sup>10)</sup> on topics of Sobolev spaces theory some farther effort seems to be worthwhile.

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<sup>10)</sup>A. and Yu. Brudnyi's I, II, 2011, V. G. Mazya 2011, W. Yuan, W. Sickel, D. Yang 2010, S. Kislyakov, N. Kruglyak 2013, L. Pick, A. Kufner, O. John, S. Fucik I, II 2013.

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