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Zygmund Spaces, Inviscid Limit and Uniqueness of Euler Flows

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Abstract: The paper improves the classical uniqueness result for the incompressible Euler system in the n dimensional case assuming that $\nabla u^E \in L_1(0, T; BMO(\Omega))$, only. Moreover the rate of the convergence for the inviscid limit of solutions to the Navier-Stokes equations is obtained, under the same regularity of the limit Eulerian flow. A key element of the proof is a logarithmic inequality between the Hardy and L_1 spaces which is a consequence of the basic properties of the Zygmund space $\mathbf{L} \ln \mathbf{L}$.

1. Introduction

The analysis of the evolutionary Euler system modeling the motion of incompressible flows in n dimensional bounded domains is the subject of this paper. We want to study the issue of uniqueness and the problem of the inviscid limit for the Navier-Stokes equations treated as an approximation of the system of inviscid flows.

The classical results [6] and [16] require that solutions to the Euler system belong at least to the class of regularity which guarantees that the velocity is in the space $u^E \in L_1(0, T; W^1_\infty(\Omega))$. Due to that fact we obtain the following estimate:

$$\left| \int_0^T \int_\Omega v \cdot \nabla u^E v \, dxdt \right| \leq C \|\nabla u^E\|_{L_1(0,T;L_\infty(\Omega))} \|v\|^2_{L_\infty(0,T;L_2(\Omega))}, \tag{1.1}$$

which is the core of methods in [6,16]. Given inequality (1.1), uniqueness of solutions to the Euler system follows from elementary energy estimates.

The goal of our paper is to improve the classical methods to the Euler system replacing L_∞ by the BMO-space. Because of relatively low regularity in the studied problem we cannot apply the properties of the BMO-space directly. A key element of our technique will be an application of properties of Zygmund spaces $\mathbf{L} \ln \mathbf{L}$ (see [20]). This analysis allows us to prove the following bound:

$$\|w\|_{\mathcal{H}^1(\Omega)} \leq C \|w\|_{L_1(\Omega)} \left[|\ln \|w\|_{L_1(\Omega)}| + \ln(1 + \|w\|_{L_\infty(\Omega)}) \right], \tag{1.2}$$

which measures the difference between the L_1 and Hardy space $\mathcal{H}^1$.

We will study the uniqueness criteria which play an important role in analysis based on weak solutions, where – by definition – high regularity is not admitted. Weak solutions allow considerations of a larger class of external/initial data to obtain information almost the same as for smooth data. Thanks to (1.2) we will be able to prove that the criteria for the incompressible Euler system should guarantee that $\nabla u^E \in L_1(0,T; BMO(\Omega))$, only, replacing the stronger condition from (1.1). Moreover the analysis will enable to consider the approximation of solutions to the Euler system by solutions to the Navier-Stokes equation with a small viscosity coefficient. The BMO-space is specially distinguished in two dimensions, because it is the target space for the imbedding $H^1(\mathbb{R}^2) \subset BMO(\mathbb{R}^2)$ (we cannot obtain L_∞ here). This case is of special interest to us, since we are able to point out good examples for which the L_∞-regularity with respect to spatial coordinates is too strong.

Our first result, being the fundamental tool of analysis of the Euler system, is the following

Theorem 1.1. *Let $f \in BMO(\mathbb{R}^n)$, $supp f$ – compact in $\mathbb{R}^n$ and $g \in L_1(\mathbb{R}^n) \cap L_\infty(\mathbb{R}^n)$, then*

$$\left|\int_{\mathbb{R}^n} fg\, dx\right| \le C\|f\|_{BMO(\mathbb{R}^n)}\|g\|_{L_1(\mathbb{R}^n)}\left[|\ln\|g\|_{L_1(\mathbb{R}^n)}| + \ln(1+\|g\|_{L_\infty(\mathbb{R}^n)})\right]. \tag{1.3}$$

Theorem 1.1 is a version of the logarithmic Sobolev inequality for the Hardy and L_1-spaces. The structure of (1.3) and its proof is essentially based on the properties of the Zygmund spaces $\mathbf{L}\ln\mathbf{L}$ – see [15,20]. Inequality (1.2) (or (1.3)) can be compared with a similar estimate between the L_∞- and BMO-spaces from [7]. The authors have shown that

$$\|f\|_{L_\infty(\Omega)} \le C\left[1+\|f\|_{BMO(\Omega)}(1+\ln^+\|f\|_{W^s_p(\Omega)})\right] \quad \text{for } s > \frac{n}{p}. \tag{1.4}$$

Inequality (1.4) helped to improve the classical result for the Euler system for the blow-up criteria replacing L_∞ by the BMO-space. In our problems we cannot apply estimate (1.4), however it remains a motivation for Theorem 1.1.

We want to apply Theorem 1.1 to analyze the Euler system

$$\begin{array}{ll} u^E_t + u^E\cdot\nabla u^E + \nabla p^E = 0 & \text{in } \Omega\times(0,T),\\ \operatorname{div} u^E = 0 & \text{in } \Omega\times(0,T),\\ \vec{n}\cdot u^E = 0 & \text{on } \partial\Omega\times(0,T),\\ u^E|_{t=0} = u_0 & \text{in } \Omega, \end{array} \tag{1.5}$$

where $u^E : \Omega\times(0,T) \to \mathbb{R}^n$ is the velocity field, $p^E : \Omega\times(0,T)\to\mathbb{R}$ is the pressure, $\vec{n}$ - unit, outward normal to $\partial\Omega$, $u_0 : \Omega \to \mathbb{R}^n$ - a divergence-free initial velocity field. We exclude the external forces (the r.h.s. of $(1.5)_1$), since it would not provide any new analytical difficulties, but only a few technical elementary estimates.

We prove the following result concerning the issue of uniqueness of solutions to system (1.5)

Theorem 1.2. *Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary. Let u^E_1 and u^E_2 be two solutions to the Euler system (1.5) with initial data u_0 such that*

$$\begin{gathered} \nabla u^E_1, \nabla u^E_2 \in L_1(0,T; BMO(\Omega)) \quad \textit{and} \\ u^E_1, u^E_2 \in L_\infty(0,T; L_{2+\sigma}(\Omega)) \ \textit{for given } \sigma > 0, \end{gathered} \tag{1.6}$$

then $u^E_1 \equiv u^E_2$.

The above result generalizes the classical theory. The proof of Theorem 1.2 is essentially based on Theorem 1.1 and an application of the Osgood theorem (see [5]) giving uniqueness in the ODE theory. Additionally, the properties of the BMO-space allow to exchange the gradient ∇u by the vorticity $rot\ u$ in the condition (1.6) which may simplify the application of Theorem 1.1.

An improvement of the classical results as [6] and [16] can be found in [18] and [16]. However the relaxation of the L_∞-regularity in the space (in [18] we even have a bit weaker space than BMO) forces that the regularity with respect to time is required to belong to the L_∞-class. Hence the regularity with respect to time has to be much stronger than in (1.1). Our approach enables to keep the weak condition in the L_1-norm in $(0, T)$. Moreover thanks to Theorem 1.1 we omit numerous technical estimates which often appear in results of that type.

Our last result concerns the inviscid limit for solutions to the Navier-Stokes equations under the slip boundary conditions

$$\begin{array}{ll} u_t^\nu + u^\nu \cdot \nabla u^\nu - \operatorname{div} \mathbb{T}(u^\nu, p^\nu) = 0 & \text{in } \Omega \times (0, T), \\ \operatorname{div} u^\nu = 0 & \text{in } \Omega \times (0, T), \\ \vec{n} \cdot \mathbb{T}(u^\nu, p) \cdot \vec{\tau} + \alpha\, u^\nu \cdot \vec{\tau} = 0 & \text{on } \partial\Omega \times (0, T), \\ n \cdot u^\nu = 0 & \text{on } \partial\Omega \times (0, T), \\ u^\nu|_{t=0} = u_0 & \text{in } \Omega, \end{array} \tag{1.7}$$

where $u^\nu : \Omega \times (0, T) \to \mathbb{R}^n$ is the velocity field, $p^\nu : \Omega \times (0, T) \to \mathbb{R}$ is the pressure, $\vec{n}$ - unit, outward normal to $\partial\Omega$, $\vec{\tau}$ - unit, tangent to $\partial\Omega$, $\mathbb{T}(u^\nu, p^\nu) = \nu\mathbb{D}(u) - p^\nu \,\mathrm{Id}$ - stress tensor, α - describes the friction coefficient of the boundary, $u_0 : \Omega \to \mathbb{R}$ - a divergence-free initial velocity field. At this point we could consider boundary conditions different than $(1.7)_{3,4}$, however by the results from [1,8,9], the form of (1.7) seems to be the most suitable for the issue. A comprehensive description of the properties of this type of boundary relations can be found in [4,10,11,19].

We prove the following

Theorem 1.3. *Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary, let u^ν be a solution to the Navier-Stokes system (1.7) and u^E be the solution to the Euler system (1.5) both with initial divergence-free data u_0. Fix $T > 0$, $\sigma > 0$ and consider ν such that $0 < \nu \le 1$. Assume that*

$$\begin{gathered} \|\nabla u^E(\cdot, t)\|_{BMO(\Omega)} \le f_0(t) \quad \text{and} \quad f_0 \in L_1(0, T), \\ \|\nabla u^\nu(\cdot, t)\|_{L_2(\Omega)} + \|\nabla u^E(\cdot, t)\|_{L_2(\Omega)} \le g_0(t) \quad \text{and} \quad g_0 \in L_2(0, T), \\ \|u^\nu(\cdot, t)\|_{L_{2+\sigma}(\Omega)} + \|u^E(\cdot, t)\|_{L_{2+\sigma}(\Omega)} \le h_0(t) \quad \text{and} \quad h_0 \in L_\infty(0, T). \end{gathered} \tag{1.8}$$

Then considering the inviscid limit of solutions to (1.7) we obtain

$$\sup_{0 \le t \le T} \|(u^\nu - u^E)(\cdot, t)\|_{L_2(\Omega)} \to 0 \quad \text{as} \quad \nu \to 0^+, \tag{1.9}$$

where the precise rate can be expressed by the properties of functions f_0, g_0 and h_0.

If we assume extra that

$$\sup_{0 \le t \le T} |f_0(t) + g_0^2(t)| \le M, \tag{1.10}$$

then we obtain the following explicit rate of convergence

$$\sup_{0 \le t \le T} \|(u^\nu - u^E)(\cdot, t)\|_{L_2(\Omega)} \le C \nu^{e^{-2MT}}. \tag{1.11}$$

Theorem 1.3 gives general conditions for the inviscid limit to solutions of the Navier-Stokes equations, provided very low (lowest known) conditions on the regularity of solutions to (1.7) with respect to the viscosity coefficient ν. The main disadvantage is that in the general case we are not able to construct solutions fulfilling (1.8). However in a special case of two dimensions (see [12] and [9] for the case with homogeneous boundary data) we find a class of solutions to (1.7) which fulfill assumptions (1.10). Then by Theorem 1.3 we obtain an explicit rate of convergence to solution of the Euler system given by (1.11). A similar result has been known only for the two dimensional case [2] in whole space under the classical assumption $\nabla u^E \in L_1(0,T;L_\infty(\mathbb{R}^2))$. The rate of convergence of $\sup_{0\le t\le T}\|u^\nu - u^E\|_{L_2(\mathbb{R}^2)}$ is estimated by $\sim\sqrt{\nu T}$, however the initial data considered in [2] correspond to a vortex patch – vorticity is localized to a bounded domain with smooth boundary.

Maybe there is hope to find a realization of (1.8) by some class of solutions to the Navier-Stokes equations, however the problem seems to be challenging.

In the proceeding, Ω will be always understood as a bounded subset of $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. Spaces $(L_p(\Omega),\|\cdot\|_{L_p(\Omega)})$, $(L_p(\mathbb{R}^n),\|\cdot\|_{L_p(\mathbb{R}^n)})$ for $p\in[1,\infty]$ denote the common Lebesgue spaces. Spaces $BMO(\mathbb{R}^n)$ and $BMO(\Omega)$ are understood as space of locally integrable functions with corresponding semi-norms

$$||f||_{BMO(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n, r>0}\frac{1}{|B(x,r)|}\int_{B(x,r)}|f(s)-\{f\}_{B(x,r)}|\,ds$$

and

$$||f||_{BMO(\Omega)} = \sup_{x\in\mathbb{R}^n, r>0}\frac{1}{|B(x,r)\cap\Omega|}\int_{B(x,r)\cap\Omega}|f(s)-\{f\}_{B(x,r)\cap\Omega}|\,ds,$$

where $\{f\}_A = \frac{1}{|A|}\int_A f(s)\,ds$, are finite.

By C we denote a generic constant that is independent from ν.

2. Proofs of Theorems

We start with the proof of the estimate (1.3) which will be used throughout the proceeding

Proof of Theorem 1.1. Consider $g\in\mathcal{H}^1(\mathbb{R}^n)$. By characterization of $\mathcal{H}^1(\mathbb{R}^n)$ we have

$$\|g\|_{\mathcal{H}^1(\mathbb{R}^n)} = \|g\|_{L_1(\mathbb{R}^n)} + \sum_{k=1}^n \|R_k g\|_{L_1(\mathbb{R}^n)},$$

where the Riesz transform is given in the usual way as $\mathcal{F}[R_k f_k] = \frac{\xi_k}{|\xi|}\mathcal{F}[f_k]$. Using the fact that $BMO(\mathbb{R}^n) = (\mathcal{H}^1(\mathbb{R}^n))^*$ we get

$$\begin{aligned}\left|\int_{\mathbb{R}^n} fg\,dx\right| &\le \|f\|_{BMO(\mathbb{R}^n)}\|g\|_{\mathcal{H}^1(\mathbb{R}^n)}\\ &\le \|f\|_{BMO(\mathbb{R}^n)}\left(\|g\|_{L_1(\mathbb{R}^n)} + \sum_{k=1}^n\|R_k g\|_{L_1(\mathbb{R}^n)}\right).\end{aligned} \tag{2.1}$$

For the characterization of the Hardy space $\mathcal{H}^1(\mathbb{R}^n)$ the reader may refer to [3,13,14]. Hence it suffices to obtain an estimate on the L_1-norm of $R_k g$. We use the classical Zygmund result that can be found in [15].

Proposition 2.1. *Let h be a sufficiently smooth non-negative function with bounded support. Then*

$$\|R_k h\|_{L_1(\mathbb{R}^n)} \le C + C\int_{\mathbb{R}^n} h \ln^+ h\, dx, \tag{2.2}$$

where $\ln^+ a = \max\{\ln a, 0\}$ *and the constant C depends on the measure of support of h.*

To apply Proposition 2.1 we split g into its positive and negative parts and consider only the positive part which for simplicity is denoted by g. Estimates for the negativepart are analogous. By elementary scaling we change inequality (2.2) to get

$$\|R_k g\|_{L_1(\mathbb{R}^n)} \le \lambda + C\int_{\mathbb{R}^n} g \ln^+ g/\lambda\, dx$$

for any $\lambda \in \mathbb{R}_+$. Consider $\ln^+ g/\lambda = \ln g - \ln\lambda$ for $g \ge \lambda$, then

$$\left|\ln(g|_{\{g\ge\lambda\}})\right| \le \left|\ln(1+\|g\|_{L_\infty(\mathbb{R}^n)})\right| + \left|\ln \frac{g}{\|g\|_{L_\infty(\mathbb{R}^n)}+1}|_{\{g\ge\lambda\}}\right|.$$

Since $\frac{1}{1+\|g\|_{L_\infty(\mathbb{R}^n)}} \le 1$ by properties of the logarithm we obtain

$$\left|\ln(1+\|g\|_{L_\infty(\mathbb{R}^n)})\right| + \left|\ln \frac{g}{\|g\|_{L_\infty(\mathbb{R}^n)}+1}|_{\{g\ge\lambda\}}\right| \le \left|\ln(1+\|g\|_{L_\infty(\mathbb{R}^n)})\right| + \left|\ln \frac{\lambda}{\|g\|_{L_\infty(\mathbb{R}^n)}+1}\right|.$$

Since it suffices to consider $\lambda \le \|g\|_{L_\infty(\mathbb{R}^n)}$, we get

$$|\ln(g|_{\{g\ge\lambda\}})| \le 2\ln(\|g\|_{L_\infty(\mathbb{R}^n)}+1) + |\ln\lambda|.$$

Choose $\lambda = \|g\|_{L_1(\mathbb{R}^n)}$. We then have

$$\begin{aligned}\|R_k g\|_{L_1(\mathbb{R}^n)} &\le c\|g\|_{L_1(\mathbb{R}^n)} + 2\int_{\mathbb{R}^n} g\left(\ln(\|g\|_{L_\infty(\mathbb{R}^n)}+1) + |\ln\|g\|_{L_1(\mathbb{R}^n)}|\right) dx\\ &\le c\|g\|_{L_1(\mathbb{R}^n)}\left(1 + 2\ln(\|g\|_{L_\infty(\mathbb{R}^n)}+1) + |\ln\|g\|_{L_1(\mathbb{R}^n)}|\right).\end{aligned} \tag{2.3}$$

Inequality (1.3) follows from inequalities (2.1),(2.3). □

Remark 2.1. Let Ω be a bounded subset of $\mathbb{R}^n$, $f \in BMO(\Omega)$, $g \in L_1(\Omega) \cap L_\infty(\Omega)$. Then

$$\left|\int_\Omega fg\, dx\right| \le C\|f\|_{BMO(\Omega)}\|g\|_{L_1(\Omega)}\left[|\ln\|g\|_{L_1(\Omega)}| + \ln(1+\|g\|_{L_\infty(\Omega)})\right]. \tag{2.4}$$

Proof. Extending f, g by 0 outside Ω we can apply Theorem 1.1. Now notice that for such extension $\|g\|_{L_1(\mathbb{R}^n)} = \|g\|_{L_1(\Omega)}$, $\|g\|_{L_\infty(\mathbb{R}^n)} = \|g\|_{L_\infty(\Omega)}$ and $\|f\|_{BMO(\mathbb{R}^n)} \le C\|f\|_{BMO(\Omega)}$. Such an extension of f is possible due to its local integrability. □

Inequality (1.3) is a key estimate in the proof of Theorem 1.2.

Proof of Theorem 1.2. Let $\{u_1^E, p_1^E\}$ and $\{u_2^E, p_2^E\}$ be two different solutions of (1.5). Subtracting we get the momentum equation in the form

$$(u_1^E - u_2^E) + u_1^E \cdot \nabla(u_1^E - u_2^E) + (u_1^E - u_2^E)\nabla u_2^E = -\nabla(p_1^E - p_2^E). \tag{2.5}$$

Multiplying both sides by $(u_1^E - u_2^E)$ and integrating over Ω we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_\Omega (u_1^E - u_2^E)^2\, dx + \int_\Omega u_1^E \nabla(u_1^E - u_2^E)(u_1^E - u_2^E)\, dx \\ &\quad + \int_\Omega (u_1^E - u_2^E)\nabla u_2^E (u_1^E - u_2^E)\, dx = -\int_\Omega (u_1^E - u_2^E)\nabla(p_1^E - p_2^E)\, dx. \end{aligned} \tag{2.6}$$

Integrating by parts, using boundary conditions and incompressibility of flow we reduce (2.6) to

$$\frac{1}{2}\frac{d}{dt}\int_\Omega (u_1^E - u_2^E)^2\, dx + \int_\Omega (u_1^E - u_2^E)\nabla u_2^E (u_1^E - u_2^E)\, dx = 0. \tag{2.7}$$

Let $\alpha = (u_1^E - u_2^E)^2$ and $\beta = \nabla u_2^E$. Notice that from the assumptions on u_1^E, u_2^E and (2.7) it follows that $\|u_1^E - u_2^E\|_{L_2(\Omega)} \in C([0, T])$. We split $\alpha = \alpha_m + \alpha_r$, where $|\alpha_m| = \min(|\alpha|, m)$ for some $m > 1$. Upon Theorem 1.1 and Remark 2.1 we get

$$\left|\int_\Omega |\alpha\beta|\, dx\right| \le C\|\beta\|_{BMO(\Omega)}\|\alpha_m\|_{L_1(\Omega)}(1+|\ln\|\alpha_m\|_{L_1(\Omega)}|+\ln(1+m))+\int_\Omega |\alpha_r\beta|\, dx. \tag{2.8}$$

Denote

$$f(t) = 2C\|\beta(t)\|_{BMO},\;\; g(t) = \int_\Omega |\alpha_r\beta|\, dx,\;\; x(t) = \|u_1^E - u_2^E\|^2_{L_2(\Omega)}.$$

Consider $x(t)$ small enough so that the function $|x(t)\ln x(t)|$ is increasing, which by the continuity of $x(t)$ is equivalent to restricting our attention to sufficiently small T_0. Then (2.8) can be restated as follows:

$$\left|\int_\Omega |\alpha\beta|\, dx\right| \le C\|\beta\|_{BMO(\Omega)}\|\alpha\|_{L_1(\Omega)}(1+|\ln\|\alpha\|_{L_1(\Omega)}|+\ln(1+m))+\int_\Omega |\alpha_r\beta|\, dx. \tag{2.9}$$

Thus from (2.9) we obtain the following differential inequality:

$$\begin{aligned} &\dot{x} \le f(t)x(t)(|\ln x(t)| + 1 + \ln(1+m)) + g(t), \\ &x(0) = 0. \end{aligned} \tag{2.10}$$

To find a good estimate on $x(t)$ we introduce the following equation:

$$\begin{aligned} &\dot{y} = f(t)y(t)(|\ln y(t)| + 1 + \ln(1+m)) + g(t), \\ &y(0) = 1/m, \end{aligned} \tag{2.11}$$

for some m large enough. From the Osgood existence theorem we know that there exists a unique local solution to (2.11). Additionally the r.h.s. of $(2.11)_1$ guarantees that $y(\cdot)$ is increasing. This implies that the solution of (2.11) majorizes $x(t)$, i.e.:

$$0 \le x(t) \le y(t) \le 1 \quad \text{for} \quad t \in [0, T_0].$$

Hence we investigate the behavior of solutions to (2.11). By Gronwall's inequality we get

$$\begin{aligned} y(t) &\leq \frac{1}{m}\exp\left(\int_0^t f(s)\,(|\ln y(s)| + 1 + \ln(1+m))\;ds\right) \\ &\quad + \int_0^t g(s)\exp\left(\int_s^t f(\tau)\,(|\ln y(\tau)| + 1 + \ln(1+m))\;d\tau\right)\,ds \\ &\leq \frac{1}{m}\exp\left((1+\ln(1+m))\int_0^t f(s)\,ds\right)\exp\left(\int_0^t f(s)|\ln y(s)|\,ds\right) \\ &\quad + \exp\left(\int_0^t f(\tau)|\ln y(\tau)|\,d\tau\right)\exp\left((1+\ln(1+m))\int_0^t f(\tau)\,d\tau\right)\int_0^t g(s)\,ds. \end{aligned} \tag{2.12}$$

Since $1 \geq y(t) \geq 1/m$ implies $|\ln y(t)| \leq \ln m$ we can estimate the right-hand-side of (2.12) (up to multiplication by some constant) by

$$\begin{aligned} &\frac{1}{m}(1+m)^{\int_0^t f(s)\,ds} m^{\int_0^t f(s)\,ds} + m^{\int_0^t f(s)\,ds}(1+m)^{\int_0^t f(s)\,ds}\int_0^t g(s)\,ds \\ &= (m(1+m))^{\int_0^t f(s)\,ds}\left(\frac{1}{m} + \int_0^t g(s)\,ds\right) \leq (2m^2)^{\int_0^t f(s)\,ds}\left(\frac{1}{m} + \int_0^t\int_\Omega |\alpha_r\beta|\,dxds\right). \end{aligned} \tag{2.13}$$

This shows that it suffices to control the part α_r. In this case estimates on the measure of the support come handy. Since $\alpha \in L_\infty(0,T;L_{1+\sigma/2}(\Omega))$ and $\beta \in L_1(0,T;BMO(\Omega))$ we have $\beta \in L_1(0,T;L_p(\Omega))$ for any $p < \infty$, hence by elementary Hölder's inequality

$$\int_\Omega |\alpha_r\beta|\,dx \leq \|\alpha_r\|_{L_{1+\sigma/4}(\Omega)}\|\beta\|_{L_{(1+\sigma/4)'}(\Omega)},$$

thus we obtain a bound

$$\begin{aligned} y(t) &\leq (2m^2)^{\int_0^t f(s)\,ds}\left(\frac{1}{m} + \int_0^t\int_\Omega |\alpha_r\beta|\,dxds\right) \\ &\leq (2m^2)^{\int_0^t f(s)\,ds}\left(\frac{1}{m} + \|\alpha_r\|_{L_\infty(0,T;L_{1+\sigma/4}(\Omega))}\|\beta\|_{L_1(0,T;L_{(1+\sigma/4)'}(\Omega))}\right). \end{aligned} \tag{2.14}$$

From the Chebyschev inequality we have

$$|supp\ \alpha_r| \leq \left(\frac{\|\alpha\|_{L_\infty(0,T;L_{1+\sigma/2}(\Omega))}}{m}\right)^{1+\sigma/2} \tag{2.15}$$

uniformly in time. Notice that by Hölder's inequality

$$\|\alpha_r\|_{L_{1+\sigma/4}(\Omega)} = \left(\int_{supp\ \alpha_r} |\alpha_r|^{1+\sigma/4}\,dx\right)^{\frac{1}{1+\sigma/4}} \leq |supp\ \alpha_r|^{\frac{2\sigma}{(4+\sigma)(2+\sigma)}} \cdot \|\alpha_r\|_{L_{1+\sigma/2}(\Omega)}. \tag{2.16}$$

Inequalities (2.16) and (2.15) imply

$$\|\alpha_r\|_{L_{1+\sigma/4}(\Omega)} \leq m^{-\frac{\sigma}{4+\sigma}} \cdot \|\alpha\|_{L_{1+\sigma/2}(\Omega)}^{1+\frac{\sigma}{(4+\sigma)}}, \tag{2.17}$$

hence

$$\|\alpha_r\|_{L_\infty(0,T;L_{1+\sigma/4}(\Omega))} \leq m^{-\frac{\sigma}{4+\sigma}} \cdot \|\alpha\|^{1+\frac{\sigma}{(4+\sigma)}}_{L_\infty(0,T;L_{1+\sigma/2}(\Omega))}. \tag{2.18}$$

The above estimate leads to the following:

$$y(t) \leq C \Big(m^{2\int_0^t f(s)\,ds-1} + m^{-\frac{\sigma}{4+\sigma}} \times \left(\|u_1^E\|_{L_\infty(0,T;L_{2+\sigma}(\Omega))} + \|u_2^E\|_{L_\infty(0,T;L_{2+\sigma}(\Omega))} \right)^{1+\frac{\sigma}{4+\sigma}} \Big). \tag{2.19}$$

Choose $0 < t_1 \leq T_0$ small enough so that $2\int_0^{t_1} f(s)\,ds - 1 < -\frac{\sigma}{4+\sigma}$, then we have

$$y(t) \leq C(DATA) m^{-\frac{\sigma}{4+\sigma}} \quad \text{for } 0 \leq t \leq t_1. \tag{2.20}$$

Letting $m \to \infty$ we get $y(t) = 0$ for $0 \leq t \leq t_1$ which implies $x(t) = 0$ for $0 \leq t \leq t_1$ hence $u_1^E = u_2^E$ for $0 \leq t \leq t_1$. We can continue this procedure starting at $t = t_1$ and get uniqueness for all $t \in [0, T]$. □

Estimate (1.3) can also be used to give insight into the rate of convergence in the inviscid limit of the system (1.7).

Proof of Theorem 1.3. Let $\{u^\nu, p^\nu\}$ and $\{u^E, p^E\}$ be solutions to problems (1.7) and (1.5) respectively. Subtracting the momentum equations we get

$$(u^\nu - u^E)_t - \nu\Delta u^\nu + u^\nu \cdot \nabla(u^\nu - u^E) + (u^\nu - u^E)\nabla u^E = -\nabla(p^\nu - p^E). \tag{2.21}$$

Multiplying both sides by $(u^\nu - u^E)$ and integrating over Ω we obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega (u^\nu - u^E)^2\,dx - \nu\int_\Omega (u^\nu - u^E)\Delta u^\nu\,dx + \int_\Omega (u^\nu - u^E)\nabla(u^\nu - u^E)u^\nu\,dx + \int_\Omega (u^\nu - u^E)\nabla u^E(u^\nu - u^E)\,dx = -\int_\Omega (u^\nu - u^E)\nabla(p^\nu - p^E)\,dx. \tag{2.22}$$

Integrating by parts, using boundary conditions with zero friction coefficient and incompressibility of flow we reduce (2.22) to

$$\frac{1}{2}\frac{d}{dt}\int_\Omega (u^\nu - u^E)^2\,dx + \int_\Omega (u^\nu - u^E)\nabla u^E(u^\nu - u^E)\,dx + \nu\int_\Omega \mathbf{D}u^\nu\mathbf{D}(u^\nu - u^E)\,dx = 0. \tag{2.23}$$

Let $\alpha = (u^\nu - u^E)^2$, $\beta = \nabla u^E$. By Theorem 1.1 and Remark 2.1 we have

$$\left|\int_\Omega |\alpha\beta|\,dx\right| \leq C\|\beta\|_{BMO}\|\alpha\|_{L_1(\Omega)}(1+|\ln\|\alpha\|_{L_1(\Omega)}| + \ln(1+\|\alpha\|_{L_\infty}(\Omega))). \tag{2.24}$$

Let $\alpha = \alpha_\nu + \alpha_r$ so that $|\alpha_\nu| = \min(\frac{1}{\nu}, |\alpha|)$. Proceeding as in the proof of Theorem 1.2 we denote $f(t) = 2C\|\beta(t)\|_{BMO(\Omega)}$, $g(t) = \int_\Omega |\alpha_r\beta|\,dx$, $x(t) = \|u^\nu - u^E\|^2_{L_2(\Omega)}$. From (2.24) we get the following inequality:

$$\dot{x} \leq f(t)x(t)\left(|\ln x(t)| + 1 + \ln\left(1 + \frac{1}{\nu}\right)\right) + g(t) + \nu g_0(t)^2, \quad x(0) = 0. \tag{2.25}$$

To find a good estimate on $x(t)$ we introduce the following equation:

$$\begin{aligned}\dot{y} &= f(t)y(t)\left(|\ln y(t)| + 1 + \ln\left(1 + \frac{1}{\nu}\right)\right) + g(t) + \nu g_0(t)^2,\\ y(0) &= \nu\end{aligned} \tag{2.26}$$

for some ν sufficiently small. From the Osgood existence theorem we know that there exists a unique local solution to (2.26). The solution of (2.26) majorizes $x(t)$, i.e.:

$$0 \le x(t) \le y(t) \quad \text{for} \quad t \in [0, T_0],$$

where T_0 is chosen by similar rules as in the proof of Theorem 1.2. From (2.26) by Gronwall's inequality we have

$$\begin{aligned}y(t) &\le \nu \exp\left(\int_0^t f(s)\left[|\ln y(s)| + 1 + \ln\left(1 + \frac{1}{\nu}\right)\right] ds\right)\\ &\quad + \int_0^t \left(g(s) + g_0(s)^2\nu\right) \exp\left(\int_s^t f(\tau)\left[|\ln y(\tau)| + 1 + \ln\left(1 + \frac{1}{\nu}\right) d\tau\right]\right) ds\\ &\le \nu \exp\left(\left(1 + \ln\left(1 + \frac{1}{\nu}\right)\right)\int_0^t f(s)\, ds\right) \exp\left(\int_0^t f(s)|\ln y(s)|\, ds\right)\\ &\quad + \exp\left(\left(1 + \ln\left(1 + \frac{1}{\nu}\right)\right)\int_0^t f(\tau)\, d\tau\right) \exp\left(\int_0^t f(s)|\ln y(s)|\, ds\right)\\ &\quad \cdot \int_0^t (g(s) + g_0(s)^2\nu)\, ds.\end{aligned} \tag{2.27}$$

The condition $y(t) \ge \nu$ for sufficiently small ν gives $|\ln y(t)| \le -\ln\nu = \ln\frac{1}{\nu}$. Also let ν be small enough so that $\frac{1}{\nu}\left(1 + \frac{1}{\nu}\right) \le \frac{2}{\nu^2}$, thus we can estimate the right-hand-side of (2.27) (up to a constant factor) by

$$\begin{aligned}&\nu\left(\frac{2}{\nu^2}\right)^{\int_0^t f(s)\, ds} + \left(\frac{2}{\nu^2}\right)^{\int_0^t f(s)\, ds} \int_0^t (g(s) + \nu g_0(s)^2)\, ds\\ &= \left(\frac{2}{\nu^2}\right)^{\int_0^t f(s)\, ds} \left(\nu + \int_0^t \int_\Omega |\alpha_r \beta|\, dx ds + \nu \int_0^t g_0(s)^2\, ds\, dt\right).\end{aligned} \tag{2.28}$$

Since $\alpha \in L_\infty(0, T; L_{1+\sigma/2}(\Omega))$ and $\beta \in L_1(0, T; BMO(\Omega))$, hence $\beta \in L_1(0, T; L_p(\Omega))$ for any $p < \infty$, thus we have

$$\int_\Omega |\alpha_r \beta|\, dx \le \|\alpha_r\|_{L_{1+\sigma/4}(\Omega)} \|\beta\|_{L_{(1+\sigma/4)'}(\Omega)},$$

thus

$$\begin{aligned}&\left(\tfrac{2}{\nu^2}\right)^{\int_0^t f(s)\, ds} \left(\nu + \int_0^t \int_\Omega |\alpha_r \beta|\, dx ds + \nu \int_0^t |g_0(s)|^2\, ds\right)\\ &\le \left(\tfrac{2}{\nu^2}\right)^{\int_0^t f(s)\, ds} \left(\nu + \|\alpha_r\|_{L_\infty(0,T;L_{1+\sigma/4}(\Omega))} \|\beta\|_{L_1(0,T;L_{(1+\sigma/4)'}(\Omega))} + \nu\|g_0\|^2_{L_2(0,T)}\right).\end{aligned} \tag{2.29}$$

From the Chebyshev inequality we notice that

$$|supp\ \alpha_r| \leq \left(\nu \|\alpha\|_{L_\infty(0,T;L_{1+\sigma/2}(\Omega))}\right)^{1+\sigma/2}, \tag{2.30}$$

uniformly in time. By Hölder's inequality

$$\|\alpha_r\|_{L_{1+\sigma/4}(\Omega)} = \left(\int_{supp\ \alpha_r} |\alpha_r|^{1+\sigma/4}\, dx\right)^{\frac{1}{1+\sigma/4}} \leq |supp\ \alpha_r|^{\frac{2\sigma}{(4+\sigma)(2+\sigma)}} \cdot \|\alpha_r\|_{L_{1+\sigma/2}(\Omega)}. \tag{2.31}$$

Inequalities (2.31) and (2.30) imply

$$\|\alpha_r\|_{L_{1+\sigma/4}(\Omega)} \leq \nu^{\frac{\sigma}{4+\sigma}} \cdot \|\alpha\|^{1+\frac{\sigma}{4+\sigma}}_{L_{1+\sigma/2}(\Omega)}, \tag{2.32}$$

hence

$$\|\alpha_r\|_{L_\infty(0,T;L_{1+\sigma/4}(\Omega))} \leq \nu^{\frac{\sigma}{4+\sigma}} \cdot \|\alpha\|^{1+\frac{\sigma}{4+\sigma}}_{L_\infty(0,T;L_{1+\sigma/2}(\Omega))}. \tag{2.33}$$

As in the proof of Theorem 1.2 the above implies the following estimate

$$y(t) \leq C\left(\nu^{1-2\int_0^t f(s)\,ds} + \nu^{\frac{4}{4+\sigma}}\left(\|u^\nu\|_{L_\infty(0,T;L_{2+\sigma}(\Omega))} + \|u^E\|_{L_\infty(0,T;L_{2+\sigma}(\Omega))}\right)^{1+\frac{\sigma}{4+\sigma}}\right) + C\nu^{1-2\int_0^t f(s)\,ds}\left(\|\nabla u^\nu\|^2_{L_1(0,T;L_2(\Omega))} + \|\nabla u^\nu\|^2_{L_1(0,T;L_2(\Omega))}\right). \tag{2.34}$$

Choose $0 < t_1 \leq T_0$ small enough so that $1 - 2\int_0^{t_1} f(s)\,ds < \frac{\sigma}{4+\sigma}$, then for $0 \leq t \leq t_1$ there is

$$y(t) \leq C(DATA)\nu^{\frac{\sigma}{4+\sigma}}. \tag{2.35}$$

Consider now, $t_1 \leq t \leq T$ and a problem analogous to (2.26) but with the initial condition $y(t_1) = \nu^{\frac{\sigma}{4+\sigma}}$. Repeating all above estimates we pick $t_1 < t_2 \leq T$ such that $\sup_{t_1 \leq t \leq t_2} \|u^\nu - u^E\|_{L_2(\Omega)} \leq C\nu^{\frac{\sigma}{8+2\sigma}}$. Due to integrability of $f(t)$ iterating the procedure we eventually cover the whole interval $[0, T]$. This way we obtain the explicit rate of the convergence which depends mainly on the structure of integrability of function f_0. Thus we proved (1.9).

An additional condition (1.10) improves the result and gives an explicit uniform rate of convergence. The basic estimate presented above gives

$$x(t) \leq C\nu\left(\frac{2}{\nu^2}\right)^{Mt} \tag{2.36}$$

with M as in (1.10). Fix some $n \in \mathbb{N}$ and consider $0 \leq t \leq T/(2Mn)$. We have $\sup_{0\leq t\leq T/(2Mn)} \|u^\nu - u^E\|_{L_2(\Omega)} \leq C\nu^{1-T/n}$. The time interval has been divided into $2Mn$ parts and repeating the estimate we get

$$\sup_{0\leq t\leq T} \|u^\nu - u^E\|_{L_2(\Omega)} \leq C\nu^{(1-T/n)^{2Mn}}, \tag{2.37}$$

and taking limit $n \to \infty$ we obtain

$$\sup_{0\leq t\leq T} \|u^\nu - u^E\|_{L_2(\Omega)} \leq C\nu^{e^{-2MT}}. \tag{2.38}$$

Theorem 1.3 is proved. □

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