REACTOR CRITICALITY IN TRANSPORT THEORY

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Communicated by Eugene P. Wigner, February 19, 1959

Recent work has indicated increasingly that, in nuclear reactor theory, the mathematical concepts of critical neutron distribution, effective multiplication factor k, and importance function, depend essentially on the *positivity* of the multiplicative process relating successive events in random neutron histories. This has already been demonstrated in both the discrete[1] and continuous[2] multigroup approximation. The purpose of this note is to sketch a proof of the same principle, in the context of neutron transport theory.[3]

Accordingly, we consider the vector position x and velocity u of emission of a neutron, after fission, as a point $\omega = (x, u)$ in *phase-space* Ω. Such a neutron may undergo scattering, fission, or absorption in its next interaction with a nucleus; we let S, F, A denote the associated (linear) operators $\omega \rightarrow \omega'$ on Ω. These define a random process on Ω; in an infinite reactor, for which escape is impossible, $S + F + A$ defines a Markoff process (hence a stochastic process). Unfortunately, the operators S, F, A are highly singular.

To treat them rigorously, we let $B_\sigma(\Omega)$ denote the Boolean σ-algebra of all Borel subsets W of Ω, with respect to the natural topology of Ω. We let $\mathfrak{L}(\Omega) = L_\sigma(B_\sigma(\Omega))$ be the linear space of all σ-additive valuations on $B_\sigma(\Omega)$. It follows[4] that $\mathfrak{L}(\Omega)$ is an abstract (L)-space, and in particular a Banach lattice; since $B_\sigma(\Omega)$ contains a continuum of atoms (points), it is highly nonseparable. Each valuation $v(W)$ refers to the expected number of neutrons emitted from W; it has[4] a unique decomposition $v = v^+ + v^- = \alpha d - \beta d'$ into disjoint nonnegative resp. nonpositive components. Hence, letting $\alpha = v^+(\Omega)$ and $\beta = v^-(\Omega)$, each $v \in \mathfrak{L}(\Omega)$ has a unique representation as a linear combination of *distributions*,[4] or nonnegative σ-additive valuations with $d(\Omega) = d'(\Omega) = 1$.

Intuition suggests that S, F, A should carry positive valuations into positive valuations. Moreover, if ν is the neutron yield for fission (assumed constant below), and νP: $\omega \rightarrow \omega'$, is the operator connecting fission neutrons emitted in successive generations, then we should expect

$$P = (I + S + S^2 + \ldots)F = (I - S)^{-1}F. \qquad (1)$$

To establish these intuitions rigorously, from the usual[3] definitions of S, F, A as singular integral kernels, we proceed as follows; the case of S is typical.

For each $\omega \in \Omega$, scattering from ω is only possible to a subvariety $V(\omega)$; the kernels of transport theory[3] are usually Riemann integrable on these $V(\omega)$, with respect to Lebesgue measure in $V(\omega)$. Hence, for any Borel set $W_1 \leq \Omega$, the integral

$$S(\omega \rightarrow W_1) = \int S(\omega, \omega')dm(\omega') \text{ over } V(\omega) \cap W, \qquad (2)$$

is defined and nonnegative, with $S(\omega \rightarrow \Omega) \leq 1$. For fixed W_1, moreover, $S(\omega \rightarrow W_1)$ is a Borel function of the $\omega \in \Omega$. Therefore, the Stieltjes integral

$$\int S(\omega \rightarrow W_1)dv(\omega) = v_S(W_1) \qquad (3)$$

is defined for $v \in \mathcal{L}(\Omega)$, as a σ-additive valuation $v_S \in \mathcal{L}(\Omega)$. Moreover, if $v \geqq 0$, it is obvious that

$$0 \leqq v_S, v_F, \qquad \text{and} \qquad \|v_S\| + \|v_F\| \leqq \|v\|.$$

Hence, the operators S, F, A belong to the Banach algebra[5] of all bounded linear operators on the Banach lattice $\mathcal{L}(\Omega)$. It follows that the expression (1) is well defined, and that $\|P\| \leqq 1$.

We now consider the linear probabilistic operator P; it defines a (linear) *multiplicative processs.*[6] For each P, ν, we shall define the concepts of effective multiplication factor k, critical neutron distribution, and importance factor. The definition will be based on an extension[7] of the Theorem of Jentzsch, which was derived with the present application in mind.

In the extension, $C = \mathcal{L}^+(\Omega)$ denotes the "cone" of nonnegative elements of $\mathcal{L}(W)$—i.e., the set of $v^+(W)$. The linear operator T is called "uniformly positive" on $\mathcal{L}(W)$ if and only if, for some fixed $e \in C$ and finite positive number K,

$$\lambda e \leqq fT \leqq K\lambda e \text{ for any } f \in C \text{ and some } \lambda = \lambda(f) > 0. \tag{4}$$

If some power P^r of a nonnegative P is uniformly positive, then[7] the neutron *distribution* $fP^n/\|fP^n\|$ tend to a fixed limiting *critical distribution* d_c as $n \to \infty$. Moreover,[7] for some unique *critical fission yield* ν_c,

$$\nu_c^n fP^n = f(\nu_c P)^n \to d_c, \tag{5}$$

in the sense of stochastic convergence. The effective multiplication factor $k = \nu/\nu_c$, very simply. Finally, P admits a unique invariant subspace S: $\sigma[f] = 0$, on which the spectral radius of P is less than $\gamma = \nu_c^{-1}$. The bounded, nonnegative linear functional σ is the *importance* functional of the transport model in question.

To apply the preceding argument to any particular reactor model, it suffices to demonstrate the uniform positivity of some P^r. We first consider *bounded* reactors, in the often treated[3] model of monoenergetic neutrons, and isotropic scattering (and fission) in the laboratory frame. In this case, we can replace Ω by the reactor domain X, ignoring the dependence of scattering on u. The classic device of smoothing integral kernels under iteration shows that P^4 is uniformly positive with respect to the density $e(x)$ of fissionable material, provided all cross sections are bounded above (no "black" control rods).

If one assumes that all cross sections Σ_S, Σ_f, Σ_c are bounded above, together with the probability of scattering through an angle θ (even as $\theta \downarrow 0°$), then one can easily show that each $S^n(\omega \to W_1)$ is uniformly bounded above in comparison with $m(W_1)$, $n \geqq 4$. If one assumes a limiting behavior for scattering of, and fission by, "slow" neutrons (as $u \to O$), then the upper bound will also be uniform in $m \geqq 4$. Hence $P^n(\omega \to W_1)$ will be uniformly bounded above by a constant multiple of $e(X_1)$, the amount of fissionable material in X_1, for subsets $W_1 = X_1 \times U$ consisting of all (x, u) with $x \in X_1$. If Σ_S is bounded away from zero, together with the probability of scattering through θ (even as $\theta \to 180°$), then a similar lower bound exists. If the fission yield is isotropic, or even just bounded above and below in direction over the neutron velocity-range considered, the verification of (4) for P^4 becomes trivial.

The preceding results refer to stationary nuclei (the usual zero-temperature approximation). Detailed proofs, together with a discussion of the extension to media at positive temperature ("Doppler effect"), will be presented elsewhere.

[1] Birkhoff, G., and R. S. Varga, *J. Soc. Ind. Appl. Math.*, Dec., 1958 (also Report WAPD-166, July, 1957).

[2] Habetler, G. J., and M. A. Martino, Report KAPL-1886, July, 1958.

[3] Davison, B., *Neutron Transport Theory* (Oxford, 1957).

[4] Birkhoff, G., *Lattice theory*, rev. ed. (New York, 1948), esp. Ch. 15, §§13–14. For stochastic convergence, see the first ed., p. 135, top. See also Doob, J. L., *Stochastic Process*, ch. 5, §5.

[5] Loomis, L., *Abstract Harmonic Analysis* (New York, 1953), ch. 4.

[6] As defined by Everett, C. J., and S. Ulam, these PROCEEDINGS, **34**, 403–7 (1948).

[7] Birkhoff, G., *Trans. Am. Math. Soc.*, **85**, 219–27 (1957), esp. Thms. 1, 3, 4. The ν_c of (5) is the γ^{-1} of Thm. 3, *op. cit.* For a different extension, less obviously related to the present note, see Krein, M. G., and M. A. Rutman, *Uspekhi Mat. Nauk.*, **3**, 3–95 (1948).

A CLASS OF DIOPHANTINE EQUATIONS

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Communicated by Deane Montgomery, February 3, 1959

Consider the sequence $\{s_m\}$ defined by

$$s_{n+h} + a_1 s_{n+h-1} + \ldots + a_h s_n = 0$$

where h is fixed while the a's and $s_1, \ldots, s_h$ are given rational integers. It is, in general, an unsolved problem to find *effectively* (especially if we use the Thue-Siegel-Roth method) all the solutions (assuming their number to be finite) for a given c, of the equation

$$s_m = c \qquad (m = 1, 2, 3, \ldots) \tag{1}$$

We consider a rather special case of this general problem in that we wish to find a *determinable* constant g_0 such that

$$|\alpha^g + \beta^g| > c \qquad \text{for} \qquad g > g_0 = g_0(c) \tag{2}$$

where g is a positive integer and α, β are the roots of $\chi^2 - \chi + \lambda = 0$. Here λ is a positive integer >1. D. J. Lewis, M. Dunton, S. Chowla, and I have solved this problem for the special case $\lambda = 2$ and the solution rests essentially on my Theorem 1 below. (So far it has not been clear how one can extend Theorem 1 to the case $\lambda > 2$.) Write

$$s(g) = \alpha^g + \beta^g, \qquad \alpha, \beta = \frac{1 \pm \sqrt{-7}}{2} \tag{3}$$

Then we have

THEOREM 1. *For given $\theta > 2$, the equation (in g) $s(g) = s(2^\theta)$ has only the trivial solution $g = 2^\theta$.*

One also has the result (which will not be proved in this paper).

THEOREM 2. *For given c, the equation (in g) $s(g) = c$ has at most two solutions.*

We need the following Lemmas for the proof of Theorem 1.