usually quite homogeneous in constitution, show little tendency for subdivision into local populations and give rise to few inter-specific hybrids. They indicate that mutation and the early isolation of the new types are important means of increase in the number of species. In any event, insular species are simpler to interpret, and an understanding of such forms may be basic to an understanding of continental species.

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ON THE INTEGRATION OF OPERATORS

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Suppose one is given a system of ordinary differential equations having the special form

$$dx_i/dt = \sum_{h=1}^{r}\rho_h(t)\cdot X_i^h(x_1, \cdots, x_n) \quad [i = 1, \cdots, n]. \tag{1}$$

We propose the question: how can one find a vector-field Z such that if $x(t)$ is any solution of (1), if $y(t)$ is any solution of

$$dy_i/dt = Z_i(y_1, \cdots, y_n) \tag{2}$$

and if $x_i(0) = y_i(0)$ for $i = 1, \ldots, n$, then $x_i(1) = y_i(1)$ for $i = 1, \ldots, n$?

If one regards the vector-fields $X^1, \ldots, X^r$ as infinitesimal transformations in the sense of Lie, one sees that this question is a special instance of the following more general problem.

Let $X(t) = \sum_{h=1}^{r} \rho_h(t) \cdot X^h$ be any variable linear combination of several fixed (not necessarily linear or commutative) operators X^h operating over a specified period of time—say the interval $[0, 1]$. What single fixed operator Z, operating constantly over the same period, will produce the same net effect?

Under many conditions which will be stated below, Z is the sum of a formal series

$$Z = \gamma_1 Z + \gamma_2 Z_2 + \gamma_3 Z_3 + \ldots \qquad (3)$$

whose terms are products $\gamma_i Z_i$ of the Poisson brackets Z_i in the X^h of different lengths* $w(Z_i)$, by scalars γ_i. Moreover each γ_i can be computed from the $\rho_h(t)$, by first finding their first derivatives $\rho_h'(t)$, and then performing at most $w(Z_i)$ multiplications and quadratures.

It follows that if the $\rho_h(t)$ are polynomials in t and $e^{\lambda_\kappa t}$—such polynomials can always be written as sums of monomials $c_j x^{n_j} e^{\lambda_j t}$—then the γ_i can be computed by rational operations on the numbers c_j, n_j, λ_j followed by reading off from a table of exponentials. For instance, this can be done in the especially interesting case that the $\rho_h(t)$ are simply periodic functions (the λ_j are imaginary).

In the commutative case, ignoring terms whose coefficients γ_i vanish, (3) assumes the trivial form

$$Z = \rho_1(1) \cdot X^1 + \ldots + \rho_r(1) \cdot X^r \qquad (4)$$

and Z is simply the (linear) *average* of the $X(t)$.

If $r = 2$, then the coefficient of $Z_3 = [X^1, X^2]$ is essentially the area integral $\int (\rho_1 \rho_2' - \rho_2 \rho_1') dt$. The coefficients of the terms of higher order give one a wholly new series of interesting affine covariant path-integrals, of similar polynomials in the ρ_h and ρ_h'.

If $r = 2$, setting $\rho_1(t) = 1$ and $\rho_2(t) = 0$ on $[0,1]$, and setting $\rho_1(t) = 0$ and $\rho_2(t) = 1$ on $[1,2]$, the series (3) gives (locally) the function of composition of any Lie group under canonical parameters.

The series (3) gives a Z with the required properties in three important cases.

The first is the case that the X^h (together with their commutators) generate a finite continuous group—as is always the case if they are linear. In this case (3) yields a solution provided the X^h are near enough the null operator.

The second is the case that the X^h are analytical operators expressible in the form

$$X_i^h(x_1, \ldots, x_n) = \sum_j c_{ij}^h x_j + \sum_{j i \kappa} c_{ij\kappa}^h x_j x_\kappa + \ldots \qquad (5)$$

that is, leaving the origin fixed. In this case a similar restriction applies.

The third is the case that the X^h are linear operators on any Banach space, and have a small enough "modulus" (in the sense of Banach).

In all three cases, the restrictions on the X^h depend on the $\rho_h(t)$, and can be removed entirely provided the X^h generate a group G, the wth term of whose "lower central series"† tends to zero as $w \longrightarrow \infty$.

The proofs of the above statements will be published elsewhere.

* The "length" of a Poisson bracket is simply the number of (equal or distinct) letters in it. Thus $[[X, Y], X]$ is of "length" three; X is of "length" one.

† By the wth term of the lower central series, is meant the invariant subgroup generated by brackets of lengths $\geqq w$.

GROUPS WHICH CONTAIN AN ABELIAN SUBGROUP OF PRIME INDEX

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When a group G contains a subgroup of index 2 this subgroup is invariant under G but when it contains a subgroup of index p, p being an odd prime number, this subgroup H is not necessarily invariant under G. We shall first prove that when H is abelian and non-invariant under G then G contains an invariant subgroup of order p and also an invariant subgroup whose order is divisible by p and involves no other prime factor except factors of $p - 1$. To prove the former of these facts it should be noted that the p conjugates of H under G are transformed by G either according to the metocyclic group of order $p(p - 1)$ or according to an invariant subgroup thereof for the following reasons: If two such conjugates have a common operator this appears in the central of G since it is transformed into itself by the operators of each of these two conjugate subgroups and therefore also by an operator of order p. An operator of G can therefore not transform one of these p conjugates into itself unless it is contained therein.

As these p conjugates are transformed under G according to a transitive group of degree p and of class $p - 1$ they must be transformed thereunder either according to the metacyclic group of order $p(p - 1)$ or according to a transition subgroup thereof. Hence G is isomorphic with such a group with respect to a subgroup contained in its central. Let s be an operator of lowest order among those whose orders are powers of p and which correspond to an operator of order p in the given quotient group. The number of the operators whose order is the same as that of s and which