permutation of this transitive group. All of these permutations besides the identity are of degree n and the given cross-cut has an order which is divisible by α since this cross-cut is an invariant subgroup of the given transitive permutation group. A necessary and sufficient condition that the subgroup composed of all the permutations of this group which omit a given letter is maximal is that it cannot be extended by a permutation of the group so that the extended group is intransitive.

[1] Miller, Blichfeldt, Dickon, *Finite Groups*, John Wiley & Sons, p. 71 (1916).

METRIC FOUNDATIONS OF GEOMETRY

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1. Introduction.—In the following, a set of metric postulates will be developed for n-dimensional Euclidean, hyperbolic and spherical geometry in the large. It is well known that any such space S satisfies the following postulates:

I. S is a metric space.

II. Any two points of S can be joined by a straight segment; and any two sufficiently short straight segments issuing from the same point a of the same length, have either no other points or all other points in common.

III. Any isometry between portions of S can be extended to a self-isometry of all of S.

What is not realized is that, conversely, any space satisfying these postulates is isometric with one of the models described above. A similar result has been proved by Busemann,[1] but under additional hypotheses including finite compactness which excluded the spherical case.

In the author's opinion, however, the interest of the results below is not so much in the actual result mentioned as in the elementary character of the proofs of the theorems stated. In particular, the arguments in the bounded case (n-dimensional spherical geometry) are of a new type.

2. Preliminary Definitions and Results.—It is well known that, in any metric space, one can define such things as spheres, straight segments, geodesics (curves locally straight), bounded sets, diameters, cluster points, homeomorphisms, isometries and so on. The definitions of these terms will not be repeated here. We also note that Postulates I–II hold in any

Riemannian geometry; thus it follows that it is Postulate III of Pasch which distinguishes Euclidean, hyperbolic and spherical geometry from other Riemannian geometries.

THEOREM 1. *Through any two points passes an infinite geodesic. This is a straight line if space has infinite diameter; if space has finite diameter d, its segments of length d are straight. Any geodesic is homogeneous, and any two geodesics are isometric.*

DEFINITION. If the distance between two points is the diameter of space, they will be called antipodal.

THEOREM 2. *If two points can be joined by more than one straight segment, they are antipodal, and space is bounded. Conversely, if space is bounded, then every geodesic is periodic, of period twice the diameter of space.*

THEOREM 3 (DIGRESSION). *If the second part of Postulate II is replaced by the hypothesis that there exist two points joined by only one straight segment, then either Postulate II holds or space is topologically one-dimensional and not locally compact.*

3. Angles and Perpendicularity.—By an "angle" is meant the configuration formed by a point and two half-lines issuing from that point; straight angles and equal (isometric) angles are also easy to define.

THEOREM 4. *If two straight lines intersect, the opposite angles are equal.*

THEOREM 5. *The correspondence which reflects each geodesic through a point a in a is an isometry.*

It is called reflection in the point a.

DEFINITION. An angle is a "right angle" if the cross formed by extending the half-lines beyond the vertex admits eight isometries.

THEOREM 6. *The segment from a point p to the midpoint of a segment of a line L makes a right angle with L if and only if p is equidistant from the end-points of the segment.*

THEOREM 7. *In the unbounded case, one and only one perpendicular can be dropped from a point p to a geodesic L. In the bounded case, either every line through p is perpendicular to L, or the perpendiculars are equally spaced.*

THEOREM 8. *Two triangles are isometric if their sides are equal in pairs (unless one side is of length d), or if two sides and the included angle are equal.*

THEOREM 9. *All right angles are equal.*

4. Flats and Subspaces.—A "subspace" of S is defined as a subset which also satisfies Postulates I–III relative to the metric in S; a "flat," as a subset which contains, with any two non-antipodal points, the unique geodesic through them.

Clearly any geodesic is a subspace, any subspace is a flat and any intersection of flats is a flat. It is also almost trivial that the set of all points left invariant by any isometry is a topologically closed flat.

DEFINITION. A "hyperplane" is the set of all points equidistant from two suitable fixed points.

THEOREM 10. *Any hyperplane divides space in two, and is a maximal flat. Any two hyperplanes are isometric.*

THEOREM 11. *There is just one isometry besides the identity which leaves a given hyperplane pointwise invariant.*

This isometry is called *reflection* in the hyperplane. In somewhat the same vein, one can prove that the group of isometries leaving a given hyperplane setwise invariant is transitive on the points of the hyperplane, and also on those at any given positive distance from the hyperplane.

THEOREM 12. *In the case S has finite diameter d, there are just two points whose distance from a given hyperplane is $^1/_2d$; these are antipodal.*

THEOREM 13. *In the bounded case each point admits one and only one antipodal point.*

This basic fact yields many corollaries.

THEOREM 14. *Any hyperplane is a subspace, or consists of two antipodal points.*

The facts just proved suggest modifying our definitions of subspace and flat to simplify the statement of theorems, as follows. In the bounded case, the set consisting of two antipodal points will be admitted as a degenerate subspace, and a flat will be required to contain with each point, its antipode (as it does unless it consists of just one point).

5. *Orthocrosses and Dimension.*—By an "orthocross" we mean a set of mutually perpendicular lines (called "axes") through a common vertex. A maximal orthocross will be called a "rectangular basis."

By Theorem 9, any one-one correspondence between the axes of two orthocrosses can be induced by an isometry; also, any space has a rectangular basis. The following is less trivial.

THEOREM 15. *Any two rectangular bases of S contain the same finite number of axes.*

This integer is called the *dimension* of S. It is not hard to prove that the dimension of any hyperplane of S is one less than the dimension of S; this fact enables one to make arguments by induction on dimension.

DEFINITION. By $p \perp q$ from o, we mean that reflection in o carries p into a point at the same distance from q.

With one exception, this means that the angle made by the lines $\overline{op}$ and $\overline{oq}$ is a right angle.

Now let o be fixed; for any set R, we let R^* denote the set of all p such that $p \perp r$ from o for all $r \in R$. General arguments show[2] that the correspondence $R \rightarrow (R^*)^*$ is a closure operation, while if R is "orthoclosed" in the sense that $R = (R^*)^*$, then R and its "polar" R^* are complements in the lattice of all orthoclosed sets.

DEFINITION. Two orthocrosses are "complementary" if they have the same vertex and their sum is a rectangular basis.

LEMMA. If R and R^* are polar orthoclosed sets, then R and R^* contain complementary orthocrosses with vertex o.

THEOREM 16. *Let θ and θ' be complementary orthocrosses. Then the intersection of all flats containing θ is the polar $(\theta')^*$ of θ', the orthoclosure $(\theta^*)^*$ of θ and a subspace.*

THEOREM 17. *Let R and R^* be polar from o. There exists one and only one isometry ϕ which leaves R pointwise invariant and reflects R^* in o. It is involutory.*

The isometry ϕ is called *reflection in* R.

THEOREM 18. *The only isometries of period two are reflections in subspaces and the "antipodal" involution, which carries each point into its antipode.*

THEOREM 19. *Apart from trivial exceptions, the following conditions on a set T are equivalent: T is a flat, T is a subspace, T is orthoclosed, T consists of the fixpoints of a suitable isometry.*

6. *Lattice-Theoretic Structure.*—We now derive the known[3] lattice-theoretic structure of a space from our postulates, without reference to coordinate representation.

THEOREM 20. *The lattice of subspaces of any unbounded n-dimensional space is a matroid lattice of dimension $n + 1$.*

Using the polarity operation of complementation, and the known fact that any self-dual matroid lattice is modular, we get the following sharper result.

THEOREM 21. *The sublattice of those subspaces which contain any fixed point o is an abstract projective geometry.*

In the bounded case, one can sharpen the results by an ingenious special definition. Define $p \rho q$ to mean that the distance from p to q is half the diameter of space. Then construct polars T^* with respect to *this* relation. It is easy to show that for any set T, T^* is an intersection of hyperplanes, and hence a subspace—and, conversely, that if T is a subspace, then $T = (T^*)^*$. The complementation operation is no longer restricted to subspaces through o.

THEOREM 22. *If space is bounded, the subspaces form an abstract projective geometry.*

7. *Further Results.*—The author plans to develop the methods used in proving the above theorems far more extensively, and to publish detailed proofs of all results elsewhere. But some further remarks should be added to the present note.

In the first place, elementary inductive arguments based on the preceding results prove that in the unbounded case, S is homeomorphic with Euclidean n-space, and in the bounded case, with the n-sphere—whence in any case, any closed bounded portion of S is compact. Moreover the appropriate properties of parallelism follow even more immediately.

What is more important, definitions of the sum and of the relative magnitude of two angles can be set up, by restricting one's attention to two-dimensional subspaces. One can then prove that angles are *quantities*, which can be measured by real numbers modulo 360. Following this, one can follow standard methods in separating the three cases of Euclidean, hyperbolic and spherical geometry according to the sum of the three angles of a triangle. By standard methods,[4] one can then set up the trigonometric functions and establish the isometry in the large with the standard geometries.

[1] Busemann, H., "On Leibniz's Definition of Planes," *Am. Jour.*, **63**, 101–111 (1941).

[2] Cf. the author's *Lattice Theory*, New York, 1940, §32.

[3] Menger, K., "New Foundations of Projective and Affine Geometry," *Annals of Math.*, **37**, 456–482 (1936).

[4] For the standard methods referred to, cf. Coolidge, J. L., *The Elements of Non-Euclidean Geometry*, Oxford, 1909, Chaps. III–IV.

LINEAL ELEMENT TRANSFORMATIONS WHICH PRESERVE THE ISOTHERMAL CHARACTER

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1. *Introduction.*—It is a famous theorem in the theory of functions of a complex variable that the conformal transformations are the only *point* correspondences of the plane which carry all isothermal families into isothermal families of curves. Although this result is well known, it appears that there are no known generalizations in the plane. In the present paper, it is our intent to generalize this result to *lineal element* transformations. We also present a discussion of a new system of curves associated with any arbitrary field: the additive-multiplicative trajectories. The isogonal trajectories lead to the very interesting Cesaro-Scheffer theorem, whereas these new trajectories possess the Property I which first presented itself in my study of dynamical trajectories. See our two italicized theorems below.

2. *Isothermal Families into Isothermal Families.*—A lineal element consists of a point and a direction through that point. Any such element may be most conveniently denoted by the number triplet (x, y, θ), where (x, y) are the cartesian coördinates of the point and θ is the inclination of the direction. We shall call the totality of ∞^3 lineal elements of the plane the *opulence*.