In concluding I wish to express my thanks to Dr. A. N. Milgram and Dr. A. Wald for several suggestions simplifying the formulation and the proofs of the theorems presented in this paper.

[1] *Proc. Nat. Acad. Sci.*, **23**, 246–248 (1937) and **25**, 474–478 (1939), and Ergebnisse eines mathematischen Kolloquiums, **8**, Vienna (1937).

AN ERGODIC THEOREM FOR GENERAL SEMI-GROUPS

BY GARRETT BIRKHOFF

DEPARTMENT OF MATHEMATICS, HARVARD UNIVERSITY

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1. *A New Definition of Limit.*—By an "ergodic theorem," one means a theorem which asserts that the *averages* of the transforms of an element converge to a limit.

As has been pointed out to the author by L. Alaoglu, the formulation of an ergodic theorem for general groups or semi-groups requires a general notion of limit.[1] We shall first give a new definition of limit, which seems to be appropriate for formulating ergodic theorems.

Let $\{x_\alpha\}$ be any set of elements of any topological space, and suppose a transitive relation $x_\alpha < x_\beta$ (read, x_β is a successor of x_α) is defined on this set. We shall say that $\{x_\alpha\}$ "converges" to a limit a (in symbols, $x_\alpha \rightarrow a$) if and only if the following condition holds. Given any neighborhood $U(a)$ of a, every x_α must have a successor x_β, all of whose successors lie in $U(a)$.

It is easy to prove that such a generalized "directed set" converges to at most one limit. Also, the above definition specializes to Moore-Smith convergence in the usual sense if the set has the property of Moore-Smith.[2]

2. *An Example.*—The suitability of the above definition of convergence for general ergodic theorems follows from a very simple construction.

Let G be any semi-group of linear operators on topological linear space $\mathfrak{X}$, and let ξ be any element of $\mathfrak{X}$. We can order the means of the transforms ξT_a of ζ as follows. We shall say that one mean is a "successor" of another mean, if and only if it is a mean of its transforms.

Stated in another way, denote by $\Sigma c_a \xi T_a$ any mean of transforms of ξ under transformations T_a of G. By the "successors" of this mean, we intend the means

$$\sum_b c_b' (\sum_a c_a \xi T_a) T_b = \sum_{a,b} c_a' c_b (\xi T_a T_b).$$

Analogously, one can call the successors of a distribution function over any semi-group, its convolutions with other distribution functions; but the above example is typical.

3. *Special Case.*—We shall deal below with the case that G is a semi-group of linear isometries or contractions, and $\mathfrak{X}$ is a so-called Euclidean space of finite or infinite dimensions, such as Hilbert space.[3]

Absolutely no restriction will be placed on G otherwise, and the argument will be elementary throughout; we only use the fact that the transformations of G do not increase norm, and are closed under transformation-multiplication.

Let $\mathfrak{F}$ denote the set of *fix-points* of $\mathfrak{X}$—i.e., of elements ξ satisfying $\xi T_a = \xi$ for every T_a of G.

LEMMA 1: $\mathfrak{F}$ is a closed subspace.

The proof is trivial; $\mathfrak{F}$ is a subspace since the T_a are linear operators, and closed since they are continuous.

LEMMA 2: The orthogonal complement $\mathfrak{F}'$ of $\mathfrak{F}$ is transformed into itself by every element of G.

This curious fact can be proved as follows. Let $\xi \epsilon \mathfrak{F}'$ be given; we can assume $|\xi| = 1$ without losing generality. Write $\xi T_a = \eta + \zeta$, where $\eta \epsilon \mathfrak{F}'$ and $\zeta \epsilon \mathfrak{F}$. This is possible since $\mathfrak{F}$ and $\mathfrak{F}'$ are complementary subspaces. Now form $\zeta + c\xi$, where c is a variable scalar. Expanding, we will have for all c,

$$(\zeta + c\xi)T_a = \zeta T_a + c\xi T_a = \zeta + c\eta + c\zeta$$

whence

$$|(\zeta + c\xi)T_a|^2 \geqq (1 + c)^2 |\zeta|^2.$$

But since T_a is an isometry or contraction, we have

$$|(\zeta + c\xi)T_a|^2 \leqq |\zeta + c\xi|^2 = |\zeta|^2 + c^2|\xi|^2 = |\zeta|^2 + c^2.$$

Combining, we get the identity in c,

$$(1 + 2c + c^2)|\zeta|^2 \leqq |\zeta|^2 + c^2$$

and so $|\zeta|^2 \leqq c^2/(2c + c^2)$ for all c. From this, letting c tend to zero, we conclude $|\zeta|^2 = 0$, $\zeta = 0$ and ξT_a is an element η of $\mathfrak{F}'$, q. e. d.

LEMMA 3: Any closed convex set $\mathfrak{S}$ of $\mathfrak{X}$ has a unique point nearest 0.

PROOF: This depends on uniform convexity. In fact, let M be the infimum of the norms of the elements of $\mathfrak{S}$, and let η and η' be two elements of $\mathfrak{S}$ whose norms are at most $M + \epsilon$; without loss of generality, we can assume $|\eta'| \leqq |\eta| \leqq 1$. Then the norm of $\frac{1}{2}(\eta + \eta')$ is at most $|\eta| - \frac{1}{4}|\eta - \eta'|^2 = M + \epsilon - \frac{1}{4}|\eta - \eta'|^2$, and at least M since $\mathfrak{S}$ is convex. Therefore $|\eta - \eta'|^2 \leqq 4\epsilon$, and so $|\eta - \eta'|$ tends to 0 uniformly with ϵ. It follows that any sequence of elements of $\mathfrak{S}$ with norms bounded by $M + 1/n$, converges metrically to an element of $\mathfrak{S}$ of norm M, and that this is unique.

LEMMA 4: Let ξ be any element of $\mathfrak{F}'$. Then the closure $\mathfrak{S}$ of the convex hull of the transforms of ξ contains 0.

PROOF: By Lemma 2, $\mathfrak{S}$ is a subset of $\mathfrak{F}'$. Hence it is sufficient to show that $\mathfrak{S}$ contains a fix-point, for such an element will be in both $\mathfrak{F}$ and $\mathfrak{F}'$, and so 0.

But since $(\sum_a c_a\xi T_a)T_b = \sum_a c_a\xi(T_aT_b)$, clearly the convex hull is carried into itself by every $T_b \epsilon G$; and since every such T_b is continuous, so is its closure $\mathfrak{S}$. But since every T_b is an isometry or contraction, it carries the (unique) nearest point of $\mathfrak{S}$ into a point of $\mathfrak{S}$ equally near 0—hence into itself.

4. MAIN THEOREM. With the aid of the lemmas proved above, it is easy to prove the following

ERGODIC THEOREM: Let G be any semi-group of linear operators on a finite or infinite-dimensional Euclidean space, which do not increase distance. Then the *means* of the transforms of any element of the space *converge* in the sense of §§1–2 to a *fix-point.*

PROOF: In the notation of §3, any element ξ can be written $\eta + \zeta$ ($\eta\epsilon\mathfrak{F}$, $\zeta\epsilon\mathfrak{F}'$). Since we are dealing with linear operators, any transform ξT_a of ξ can be written $\eta + \zeta'$, where by Lemma 2 $\zeta'\epsilon\mathfrak{F}'$. Since $\mathfrak{F}'$ is convex, any mean of transforms of ξ can be written in the same form. Hence by Lemma 4, it has a "successor" in the sense of §§1–2 (i.e., a mean of transforms) of the form $\eta + \zeta''$, where $|\zeta''|$ is arbitrarily small. Finally, since η is a fix-point and we are dealing with isometries and contractions, all "successors" of this mean will be equally near η.

In summary, the means of the transforms of ξ "converge" to its orthogonal projection on $\mathfrak{F}$!

5. EXTENSION OF MAIN THEOREM. In a sequel, the above theorem will be extended to a class of transformations of the space (L) and its finite-dimensional analogues; this will apply to the theory of dependent probabilities.

A detailed study of the general case, of the definition of convergence given above and of its relation to the theory of almost periodic functions over general groups, will be made in a joint paper by L. Alaoglu and the author. In this paper an example will be given showing that the theorem above no longer holds either for isometries and contractions of general uniformly convex spaces, or for semi-groups of linear operators of bounded modulus on Euclidean spaces.

[1] This is not the case for the cyclic groups and semi-groups usually considered, or even for the case of n-parameter groups treated by N. Dunford, "An Ergodic Theorem for n-Parameter Groups," *Proc. Nat. Acad. Sci.*, **25**, 195–196 (1939), and Wiener.

[2] Cf., "Moore-Smith Convergence in General Topology," *Ann. Math.*, **38**, 39–56 (1937).

[3] In the sense of M. H. Stone, *Linear Transformations in Hilbert Space*, New York, 1932. Our restriction on G is simply that $|\xi T| \leqq |\xi|$ for all $T \epsilon G$ and $\xi \epsilon \mathfrak{X}$.