A NEW THEORY OF VORTEX STREETS

BY GARRETT BIRKHOFF

DEPARTMENT OF MATHEMATICS, HARVARD UNIVERSITY

Communicated by J. H. Van Vleck, March 20, 1952

It is well known empirically that, at intermediate Reynolds numbers R, the wake behind an obstacle moving through a stream is most easily described as consisting of two rows of vortices, symmetrically staggered, with longitudinal spacing a and transverse spacing h.

If the vorticity of each vortex is κ, then von Kármán[1] has shown that such an array leads to the complex potential

$$W = i\kappa \operatorname{logsin} \frac{\pi}{h}\left(z - \frac{ih}{2}\right) - i\kappa \operatorname{logsin} \frac{\pi}{a}\left(z - \frac{a}{2} + \frac{ih}{2}\right), \qquad (1)$$

and has computed the drag coefficient of such an array.

Evidently, the array (1) involves three arbitrary parameters: a, h and κ. Von Kármán showed that, in a certain sense, arrays with $h/a = 0.281$ were the least unstable, and has observed that such a spacing ratio is typical of what is observed.

A second parameter can be estimated, following Heisenberg and Prandtl,[2] by considering the rate K at which vorticity is discharged from each side of the boundary layer. This estimate is much less sharp, however, as it involves an empirical factor β, which expresses the proportion of the vorticity discharged by the boundary layer, which is absorbed into the wake.

It is the purpose of this note to outline a new theory of vortex streets, permitting *a priori* theoretical estimates for all three parameters.

A central rôle is played in this theory by the following result, valid for any plane flow satisfying the Navier-Stokes equations: the *mean transverse moment of the vorticity per unit length,*

$$M = \lim_{L \to \infty} \frac{1}{2L} \int_{-L}^{L} dx \int_{-\infty}^{\infty} y\zeta \, dy = \lim_{L \to \infty} \frac{1}{2L} \int_{-L}^{L} \int_{-\infty}^{\infty} y \, d\kappa, \qquad (2)$$

is a constant in time.

If viscosity is neglected (as in the von Kármán theory), then since the mean absolute vorticity per unit length

$$2K^* = \lim_{L \to \infty} \frac{1}{2L} \int_{-L}^{L} \int_{-\infty}^{\infty} |d\kappa| \qquad (3)$$

is also constant in time, this implies that the *mean transverse spacing* $h^* = M/K^*$ *is constant in time.* Thus, although periodicity may be unstable, the mean transverse spacing of vortices is highly stable. Since the mean longitudinal spacing (if it exists) is trivially invariant, we come out with the conclusion that *the ratio of mean spacings is constant in time, in a non-viscous fluid.* Thus it implies that the spacing-ratio $h/a = 0.281$ cannot be regarded as a stable (or nearly stable) end-product of other initial spacing ratios, as originally suggested by von Kármán[1] (pp. 54–55).

Since the effect of viscosity is to decrease K^*, the observed tendency of h/a to increase downstream is also explained.

Further, the constancy of M gives a basis for predicting roughly the (mean) transverse spacing h. The boundary layer discharges at an initial rate[2] $K_1 = {}^1/_2(1 + C_W)v^2$, where v is the flow speed and C_W is the wake underpressure coefficient; moreover the initial transverse spacing d is about equal to the diameter of the obstacle. Hence we may expect h to differ not greatly from d/β (or, from d^*/β, where $d^* > d$ is the wake breadth a little behind the obstacle).

Instead of accepting $h/a = 0.281$ as implied by stability considerations, I propose a direct (very rough) prediction of the Strouhal number $S = Nd/v$ on a fresh basis. It is a matter of observation that the periodicity of vortex-formation is associated with a periodic (period $\tau = 1/N$) transverse oscillation of the wake. Assuming the inertia of the wake as due to its mass, and the cross-force to be given by the steady-state cross-force of about $\pi\rho v^2\alpha$ per unit length, one gets the observed S if one takes the wake length as about $1.5d$.

Although the preceding discussion of S is very rough, I believe it contains a useful new idea, and might profitably be elaborated and refined. In particular, it agrees with the observed proportionality $S \alpha 1/b \sin \theta$, for a flat plate of breadth b, inclined at an angle θ with the stream.

Finally, it should be remarked that the tendency for vortex sheets to "roll up" and become concentrated is easily exaggerated. Thus energy arguments, closely associated with the mathematical theory of the logarithmic potential, show that (even in a non-viscous fluid) a line segment of constant vorticity cannot roll up into a circle whose diameter is less than about 45% of the original length of the segment. This limitation has not apparently been observed hitherto, though Hooker has discussed the diffusion of vorticity by viscosity.

The preceding results and ideas, and their relation to experimental evidence, will be discussed in greater detail elsewhere.

[1] von Kármán, Th., and Rubach, H., *Phys. Zeit.*, **13**, 49–59 (1912).

[2] Heisenberg, W., *Ibid.*, **23**, 363–366 (1922), and comments thereto on p. 366, by L. Prandtl.