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Integro-Differential Delay Equations of Positive Type

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1. INTRODUCTORY REMARKS

In [*1*] the authors investigated the asymptotic behavior of solutions of the differential delay-equation

$$\frac{dy}{dx} = \phi(x)\, y(x-1). \tag{1}$$

In this paper, we shall generalize some of the results of [*1*] to the more general Stieltjes integro-differential equation of Myschkis [*2*]

$$\frac{dy}{dx} = \int_{s=0}^{\infty} y(x-s)\, dr(x, s). \tag{2}$$

The integroDE (2) will be said to be of *positive* type (called "unstable" by Myschkis), when

$$r(x, s) \text{ is a nondecreasing function of } s. \tag{3}$$

Note that if $r(x, s) = \phi(x)$ when $s > 1$, and 0 when $s \leqslant 1$, then Eq. (2) reduces to the differential equation (1).

In addition to the monotonicity of $r(x, s)$, we shall assume with Myschkis that there exist positive constants δ and Δ such that

$$\begin{aligned} r(x, s) &\equiv 0 && \text{when} \quad s \leqslant \delta \\ r(x, s) &\equiv R(x) > 0 && \text{when} \quad s > \Delta. \end{aligned} \tag{4}$$

By a change of scale, with no loss in generality we can assume $\delta = 1$, so that for $N = \Delta/\delta > 0$ (not necessarily an integer), we have

$$r(x, s) = 0 \qquad \text{when} \qquad s \leqslant 1.$$

$$r(x, s) = R(x) > 0 \qquad \text{when} \qquad s > N. \tag{5}$$

Finally, let $r(x, s)$ be bounded; thus we shall assume that there exists $R = \text{const.} < \infty$ such that

$$\int_{s=0}^{\infty} dr(x, s) = R(x) \leqslant R. \tag{6}$$

The preceding conditions are clearly fulfilled in the case treated in [*1*], if $\phi(x)$ is bounded, with $\Delta = 1$ and $R(x) = \phi(x)$. Under the more general conditions described above, it is proved that Eq. (2) has one and (up to a constant numerical factor $c > 0$) only one *positive* solution on $(-\infty, \infty)$. Various other related results are proved incidentally.

Because of the delay in the argument of the integrand in (2), to speak of a solution for $x > a$ requires us to impose "initial values" on the solution for $x - s < a$. However, only that part of the initial values defined over $[a - N, a]$ affects the solution beyond a. Thus a solution over $[a, \infty)$ is actually defined over $[a - N, \infty)$. Myschkis [*2*, p. 11] proves that for given continuous $y(x)$ on $x \leqslant a$, there is one and only one continuous function which satisfies (2) for all $x > a$ ("the" solution of the initial value problem).

In the remainder of this paper, we shall assume that $r(x, s)$ satisfies conditions (3)-(6).

2. Applications of the Theory of Positive Operators

Denote by P_a the set of non-negative continuous functions $y(x)$ defined on $[a - N, \infty)$, which satisfy (2) for all $x \geqslant a$, and by $P_a{}^{+}$ the set of all positive such functions. These sets will be nonempty, by Myschkis' existence theorem. Moreover P_a is a convex cone in the Archimedean directed vector space [*3*, p. 37] S_a of *all* continuous functions on $[a - N, \infty)$ which satisfy (2) on $[a, \infty)$. Now consider the linear transformation $L_a f \equiv \bar{f}$, with domain S_{a-N} and range in S_a, defined by

$$L_a f \equiv f(a) + \int_a^x dt \int_0^{\infty} f(t - s)\, dr(t, s) = \bar{f}(x). \tag{7}$$

Over the interval $[a, \infty)$

$$L_a f \equiv f(x) \tag{8}$$

whence $\bar{f} \in S_a$, i.e., $L_a S_{a-N} \subset S_a$.

Note that, because of the monotonicity on $[a-N, \infty)$ of any $f \in P_{a-N}$, if $f(x) \not\equiv 0$ then $f(x) > 0$ for any $x \geqslant a$. Thus we have

$$L_a[P_{a-N} - 0] \subset P_a{}^+. \tag{8'}$$

The positivity of L_a allows us to use Hilbert's projective quasi-metric [3, p. 45], defined for functions on any interval J by

$$\theta(f, g; J) = \ln \left\{ \frac{\sup_J [f(x)/g(x)]}{\inf_J [f(x)/g(x)]} \right\}$$

$$= \ln \left\{ \sup_{x,y \in J} \left[\frac{f(x)\, g(y)}{g(x) f(y)} \right] \right\} \geqslant 0. \tag{9}$$

We first apply this to the case $J = [a-N, \infty)$, which refers to the domain of L_a. We prove, for $f, g \in P_{a-n}^+$

LEMMA 1. $\theta(f, g; [a-N, \infty)) = \theta(f, g; [a-N, a])$.

Proof. Let $m_a = \min_{[a-N,a]} (f/g)$, $M_a = \max_{[a-N,a]} (f/g)$. On $[a-N, a]$, $m_a g(x) \leqslant f(x) \leqslant M_a g(x)$. Then for any $x \in [a, a+1]$,

$$m_a g(x) = m_a g(a) + m_a \int_a^x dt \int_1^\infty g(t-s)\, dr(t, s)$$

$$\leqslant f(a) + \int_a^x dt \int_1^\infty f(t-s)\, dr(t, s)$$

$$= f(x).$$

Applying this argument repeatedly, we find that $m_a g(x) \leqslant f(x)$ for any $x \in [a, \infty)$; i.e., $m_a \leqslant f(x)/g(x)$. Hence $m_a \leqslant \inf_{[a,\infty)}[f(x)/g(x)]$. Similarly $M_a \geqslant \sup_{[a,\infty)}[f(x)/g(x)]$, from which Lemma 1 follows. Note also that

$$m_a \leqslant m_{a+N} \leqslant M_{a+N} \leqslant M_a .$$

LEMMA 2. *The projective diameter D of the range of the nontrivial non-negative solutions of (2) in $x \geqslant a - N$ under L_a (i.e., $L_a[P_{a-N} - 0]$) satisfies*

$$D \equiv \sup\, [\theta(\bar{f}, \bar{g}; [a, \infty)) \mid f, g \in P_{a-N} - 0] \leqslant n \ln (1 + R) \tag{9'}$$

for any integer $n \geqslant N$.

Proof. For any $f, g \in P_{a-N} - 0$, from (8') we have $L_a f$, $L_a g \in P_a{}^+$. Applying (8) and Lemma 1 we obtain

$$D = \sup\, [\theta(f, g; [a, \infty)) \mid f, g \in P_a{}^+]$$

and

$$\theta(f, g; [a, \infty)) = \theta(f, g; [a, a + N]).$$

But

$$f(a) \leqslant f(x) \leqslant (1 + R) f(a) \qquad \text{on} \qquad [a, a + 1],$$

$$f(a) \leqslant f(x) \leqslant (1 + R)^n f(a) \qquad \text{on} \qquad [a, a + N] = I,$$

and similarly for $g(x)$. Clearly then,

$$\frac{f(x)\, g(a)}{g(x)\, f(a)} \leqslant (1 + R)^n$$

since $g(a) \leqslant g(x)$. The choice of a above is arbitrary. Hence the preceding inequality is valid with a replaced by any $y \in I$. Therefore

$$\theta\,(f, g; I) = \ln \left\{ \sup_{x,y \in I} \left(\frac{f(x)\, g(y)}{g(x)\, f(y)} \right) \right\} \leqslant n \ln (1 + R), \tag{10}$$

whence

$$D = \sup\,[\theta(f, g; I)] \leqslant n \ln (1 + R) \qquad \text{Q.E.D.}$$

We can now apply the contraction mapping principle of [3, Corollary, Theorem 3]. This states that if D is bounded, then L_a contracts all projective distances by a factor at most

$$\tanh\,(D/4) \leqslant \tanh\,[(n/4) \ln (1 + R)] \equiv \gamma.$$

With the help of (8), this gives us

LEMMA 3. *Let* $f, g \in P_{a-N}^{+}$. *Then*

$$\theta(f, g; [a, \infty)) \leqslant \gamma \theta(f, g; [a - N, \infty)). \tag{11}$$

By repeated applications of (11) and making use of (10) we obtain the following result.

LEMMA 4. *Let* $f, g \in P_a{}^{+}$. *Then for any positive integer* k, *and any integer* $n \geqslant N$,

$$\theta\,(f, g; [a + kN, \infty)) \leqslant \gamma^k n \ln (1 + R).$$

3. NON-OSCILLATORY SOLUTIONS

By a non-oscillatory solution $f(x)$ is meant one whose zeros are bounded above. Then either $\pm f(x) \in P_a{}^{+}$ for sufficiently large a, and we can now apply the results of Section 2.

THEOREM 1. *Let $f(x)$ and $g(x)$ be any two non-oscillatory solutions of* (2). *Then*

$$\lim_{x\to\infty}\left[\frac{f(x)}{g(x)}\right] = \text{const.} \neq 0.$$

Proof. Without loss in generality, we can assume that $f, g \in P_a^+$ for all sufficiently large a. Then from Lemma 4,

$$\lim_{k\to\infty} \theta(f, g; [a + kN, \infty)) = 0.$$

Invoking Lemma 1 and the theorem on nested intervals, we obtain the conclusion stated.

THEOREM 2. *If $p(x)$ is a non-oscillatory solution of* (2), *then a second solution $y(x)$ is oscillatory if and only if $y(x) = o(p(x))$ as $x \to \infty$.*

Proof. Let $y(x)$ be any oscillatory solution of (2); let $f^+(x)$ and $f^-(x)$ be the positive and negative components of the initial values of $y(x)$ on $[-N, 0]$; and let $z^+(x)$ and $z^-(x)$ be the solutions of (2) on $[0, \infty)$ having these initial values. Then by Theorem 1, $z^+(x) \sim Ap(x)$ and $z^-(x) \sim -Bp(x)$ for suitable positive constants A and B, $p(x)$ being any non-oscillatory (e.g., positive) solution of (2). If $A \neq B$, then $y(x) = z^+(x) + z^-(x) \sim (A - B)\,p(x)$ is non-oscillatory. Conversely, by Theorem 1, if $y(x)$ is non-oscillatory then $A \neq B$.

COROLLARY. *The sum of any two oscillatory solutions of* (2) *is itself oscillatory; and the sum of any oscillatory and any non-oscillatory solution is non-oscillatory.*

THEOREM 3. *Let $f(x)$ and $g(x)$ be any two positive solutions of* (2) *on* $(-\infty, \infty)$. *Then $g(x) = cf(x)$ for some positive constant c.*

Proof. Since $f, g \in P^+_{a-kN}$ for all integers k, we have from Lemma 4

$$\theta(f, g)\,|_{x\geqslant a} \leqslant \gamma^k n \ln(1 + R).$$

Letting $k \to +\infty$, we find that $\theta(f, g)\,|_{x\geqslant a} = 0$. From (9), the definition of θ, there exists a constant c such that $g(x) = cf(x)$ for all $x \geqslant a$. But since a is arbitrary, this means that $g(x) = cf(x)$ for all x.

This (essential) uniqueness theorem suggests an existence theorem. Such is indeed the case.

THEOREM 4. ***Equation* (2) *has a positive solution in* $(-\infty, +\infty)$.**

Proof. First we recall that if $f(x)$ satisfies (2), then $L_{-a}f = f(x)$ when $x \in [-a, b]$, *and conversely*, for any $a > 0$ and any $b > 0$. Now for any integer $k > 0$, consider that solution $f_k(x) \in S_{-k}$, defined initially as $f_k(x) \equiv c_k = \text{const.}$ for $x \leqslant -k$, with c_k so selected that $f_k(0) = 1$. Then on $[-k, b]$, $f_k(x) \leqslant e^{bR}$ for all k, since $f_k(x)$ is increasing and

$$\frac{df_k(x)}{dx} = \int_{s=0}^{\infty} f_k(x-s)\, dr(x, s) \leqslant f_k(x)\, R.$$

Therefore, since

$$f_k(x) = f_k(-k) + \int_{-k}^{x} dt \int_{0}^{\infty} f_k(t-s)\, dr(t, s),$$

for any $x_1, x_2 \in [-k, b]$ we have

$$| f_k(x_2) - f_k(x_1) | \leqslant e^{bR} R\, | x_2 - x_1 | .$$

Thus for $k \geqslant a$, the f_k 's are equicontinuous in $[-a, b]$ as well as uniformly bounded there. From Arzela's Theorem then, there is a limiting function $f_\infty(x)$ defined over $[-a, b]$ for each $a > 0$. This function is positive on $(-\infty, a]$ and satisfies (2). For, the linear operator L_{-a}, restricted to solutions over $[-a, b]$, is bounded with respect to the norm $\| f \| = \sup_{[-a,b]} | f(x) |$:

$$\| L_{-a} \| = \sup_{\|f\| \leqslant 1} \left\| f(-a) + \int_{-a}^{x} dt \int_{0}^{\infty} f(t-s)\, dr(t, s) \right\|$$
$$\leqslant 1 + (b + a)\, R,$$

and hence continuous. Thus

$$f_\infty(x) = \lim_{k\to\infty} f_k(x) = \lim_{k\to\infty} L_{-a} f_k(x) = L_{-a} f_\infty(x),$$

whence $f_\infty(x)$ satisfies (2) in $[-a, b]$ for each $a > 0$, $b > 0$.

The significance of this theorem may be gauged by the fact that, except in this (unstable) case, there appear to be no general theorems asserting the existence of nontrivial solutions over the entire x-axis.

4. Further Results

In this section we apply more elementary techniques to obtain a comparison theorem and two on order of growth.

THEOREM 5. *Let $y_1(x)$ and $y_2(x)$ be solutions of*

$$y_i'(x) = \int_{s=0}^{\infty} y_i(x - s)\, dr_i(x, s),$$

where the $r_i(x)$ satisfy (4)-(6) *and* $|dr_1| \leqslant dr_2$. *If* $|y_1(x)| \leqslant y_2(x)$ *on* $[-N, 0]$, *then* $|y_1(x)| \leqslant y_2(x)$ *for all* $x > 0$.

Proof. For any $x \in [0, N]$, we have

$$\begin{aligned} |y_1(x)| &= \left| y_1(0) + \int_0^x dt \int_0^\infty y_1(t - s)\, dr_1(t, s) \right| \\ &\leqslant y_2(0) + \int_0^x dt \int_0^\infty y_2(t - s)\, dr_2(t, s) \\ &= y_2(x). \end{aligned}$$

The extension to all intervals $[kN, (k + 1) N]$ follows similarly. Note that $r_1(x, s)$ need not be monotonic.

In the following theorems, recall Eq. (6):

$$R(x) = \int_0^\infty dr(x, s).$$

THEOREM 6. *If* $\int_0^\infty R(x)\, dx = \infty$, *then every positive solution* $y(x)$ *of* (2) *tends to* ∞ *as* $x \to \infty$.

Proof. For all $x \geqslant 0$,

$$\begin{aligned} y(x) &= y(0) + \int_0^x dt \int_0^\infty y(t - s)\, dr(t, s) \\ &\geqslant y(0) + y(-N) \int_0^x R(t)\, dt. \end{aligned}$$

THEOREM 7. *If* $\int_0^\infty R(t)\, dt < \infty$, *then every solution of* (2) *approaches a finite limit as* $x \to \infty$.

Proof. Let Y_n be the maximum absolute value of $y(x)$ on $[(n - 1) N, nN]$. Then since

$$y(x) = y(nN) + \int_{nN}^{x} \int_{s=0}^{\infty} y(t - s)\, dr(t, s)\, dt$$

when $nN \leqslant x \leqslant (n + 1) N$, we have

$$\begin{aligned} Y_{n+1} &\leqslant Y_n \left(1 + \int_{nN}^{(n+1)N} \int_{s=0}^{\infty} dr(t, s)\, dt\right) \\ &= Y_n \left(1 + \int_{nN}^{(n+1)N} R(t)\, dt\right). \end{aligned}$$

Hence, by induction on n, since $1 + r \leqslant e^r$ $(r > 0)$,

$$Y_{n+1} \leqslant Y_0 \exp \left[\int_0^{nN+N} R(t)\, dt \right];$$

and so every solution is bounded by some constant M. It follows from (2) that if $x < x_1$, then

$$| y(x) - y(x_1) | \leqslant M \int_x^{x_1} R(t)\, dt.$$

But from the hypothesis, this tends to zero as $x \to \infty$.

We note that the same proof is valid under the weaker hypothesis

$$\int_0^\infty V(r(x, s))\, dx < \infty,$$

where the integrand is the total variation of (the not necessarily monotonic) $r(x, s)$.

References

1. Birkhoff, G., and Kotin, L. Asymptotic behavior of solutions of first order linear differential-delay equations. *J. Math. Anal. Appl.*, to appear.
2. Myschkis, A. D. "Lineare Differentialgleichungen mit nacheilendem Argument." Deutscher Verlag Wiss., 1955.
3. Birkhoff, G. Uniformly semi-primitive multiplicative processes. *Trans. Amer. Math. Soc.* **104** (1962), 37-51.