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ESTIMATES ON THE FIRST TWO POLY-LAPLACIAN EIGENVALUES ON SPHERICAL DOMAINS*

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Abstract In this article, we study the first two eigenvalues of the higher order buckling problem on a domain in the unit sphere. We obtain an estimate on the second eigenvalue in terms of the first eigenvalue. In particular, the estimate on first two eigenvalues of the higher order buckling problem of Huang, Li and Qi [5] is included in our results.

Key words Eigenvalue; spherical domain; buckling problem

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1 Introduction

Let Ω be a connected bounded domain in an n-dimensional complete Riemannian manifold M. Assume that Λ_i is the ith eigenvalue of the Dirichlet poly-Laplacian with order p (≥ 2):

$$\begin{cases} (-\Delta)^p u = \Lambda(-\Delta)u & \text{in } \Omega, \\ u = \dfrac{\partial u}{\partial \nu} = \cdots = \dfrac{\partial^{p-1} u}{\partial \nu^{p-1}} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Δ is the Laplacian in M and ν denotes the outward unit normal vector field of $\partial\Omega$. Let $0 < \Lambda_1 \leq \Lambda_2 \leq \Lambda_3 \leq \cdots \to +\infty$ denote the successive eigenvalues for (1.1), where each eigenvalue is repeated according to its multiplicity. When $p = 2$, it is well known that the eigenvalue problem (1.1) is called a buckling problem. Payne-Polya-Weinberger first studied the buckling problem when $p = 2$ on Euclidean domain in their famous articles; Hile-Ye improved Payne-Polya-Weinberger's result; Ashbaugh obtained the bound in [1]:

$$\sum_{i=1}^{n} \Lambda_{i+1} \leq (n+4)\Lambda_1. \tag{1.2}$$

The above inequality was improved to the following form in [9]:

$$\sum_{i=1}^{n} \Lambda_{i+1} + \frac{4(\Lambda_2 - \Lambda_1)}{n+4} \leq (n+4)\Lambda_1. \tag{1.3}$$

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Answering a long standing question by Payne-Polya-Weinberger, Cheng and Yang obtained in [13] that, for $p=2$ and $M=\mathbb{R}^n$,

$$\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)^2\le \frac{4(n+2)}{n^2}\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)\Lambda_i. \tag{1.4}$$

The inequality (1.4) was generalized to any p in [10].

In 2007, Wang and Xia [14] considered this problem when $p=2$ and $M=\mathbb{S}^n(1)$. They proved that, for any $\delta>0$,

$$2\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)^2\le \sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)^2\Big(\delta\Lambda_i+\frac{\delta^2(\Lambda_i-(n-2))}{4(\delta\Lambda_i+n-2)}\Big) +\frac{1}{\delta}\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)\Big(\Lambda_i+\frac{(n-2)^2}{4}\Big). \tag{1.5}$$

In a recent article, by introducing a new parameter and using Cauchy inequality, Huang, Li and Cao [6] obtained the following inequality stronger than (1.5), which is independent of δ:

$$\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)^2\Big(2+\frac{n-2}{\Lambda_i-(n-2)}\Big) \le 2\Big\{\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)^2\Big(\Lambda_i-\frac{n-2}{\Lambda_i-(n-2)}\Big)\Big\}^{\frac12}\Big\{\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)\Big(\Lambda_i+\frac{(n-2)^2}{4}\Big)\Big\}^{\frac12}. \tag{1.6}$$

Obviously, inequality (1.6) is better than inequality (1.5). Inequality (1.6) was also obtained as the formula (1.23) of [10]. For the first two buckling eigenvalues in the unit sphere $\mathbb{S}^n(1)$, Huang, Li and Qi [5] proved

$$\Lambda_2\le\Lambda_1+\Big(\frac{n(n-\Lambda_1)}{\Lambda_1}+2(n+2)\Big)\frac{4\Lambda_1+(n-2)^2}{(n+2)^2}. \tag{1.7}$$

We remark that, for the first two eigenvalues of problem (1.1) in the unit sphere $\mathbb{S}^n(1)$, the inequality (1.7) is sharper than inequality (1.6). For the related research and important improvement in eigenvalue problems, we refer to [1–3, 7, 8, 11, 12, 14–16] and the references therein.

In this article, we consider the first two eigenvalues of problem (1.1) with any integer $p(\ge 2)$ when $M=\mathbb{S}^n(1)$. We will prove the following results:

Theorem 1.1 Let Ω be a connected domain in the n-dimensional unit sphere $\mathbb{S}^n(1)$. Assume that Λ_i is the i-th eigenvalue of problem (1.1). Then, we have

$$\Lambda_2\le\Lambda_1+\Big\{\sum_{k=1}^{p}\binom{p}{p-k}n^{p-k}\Lambda_1^{\frac{k-1}{p-1}}+4[2^p-(p+1)]\sum_{k=0}^{p-2}\binom{p-2}{p-k-2}n^{p-k-2}\Lambda_1^{\frac{k}{p-1}} -\Lambda_1+n(n^{p-1}-\Lambda_1)\Lambda_1^{\frac{-1}{p-1}}\Big\}\frac{4\Lambda_1+(n-2)^2}{(n+2)^2}. \tag{1.8}$$

Corollary 1.1 Under the same assumption as in Theorem 1.1, we have

$$\Lambda_2\le\Lambda_1+\Big\{\sum_{k=1}^{p}\binom{p}{p-k}n^{p-k}\Lambda_1^{\frac{k-1}{p-1}}+4[2^p-(p+1)]\sum_{k=0}^{p-2}\binom{p-2}{p-k-2}n^{p-k-2}\Lambda_1^{\frac{k}{p-1}} -\Lambda_1\Big\}\frac{4\Lambda_1+(n-2)^2}{(n+2)^2}. \tag{1.9}$$

We remark that when $p = 2$, inequality (1.8) becomes inequality (1.7). Hence, our Theorem extends the result of Huang, Li and Qi [5].

2 Proof of Theorem

In this section, we prove the following proposition 2.1, which plays a key role in the proof of our main theorem. Let $x_1, x_2, \cdots, x_{n+1}$ be the standard Euclidean coordinate functions of $\mathbb{R}^{n+1}$. Then, the unit sphere is defined by

$$\mathbb{S}^n(1) = \Big\{(x_1, x_2, \cdots, x_{n+1}) \in \mathbb{R}^{n+1}\ ;\ \sum_{\alpha=1}^{n+1} x_\alpha^2 = 1\Big\}.$$

Denote by u_1 the orthonormal eigenfunction corresponding to Λ_1 of the eigenvalue problem (1.1), then we have $\int_\Omega \langle \nabla u_1, \nabla u_1\rangle = 1$. First, we have the following lemma.

Lemma 2.1 (Lemma 2.1 in [4])

$$\begin{aligned}(-\Delta)^p(u_1 x_\alpha) &= (-\Delta)^{p-1}\langle a_1 x_\alpha\rangle + (-\Delta)^{p-1}\langle \nabla b_1, \nabla x_\alpha\rangle \\ &= (-\Delta)^{p-2}\langle a_2 x_\alpha\rangle + (-\Delta)^{p-2}\langle \nabla b_2, \nabla x_\alpha\rangle \\ &= \cdots = a_p x_\alpha + \langle \nabla b_p, \nabla x_\alpha\rangle,\end{aligned}$$

where a_k and b_k are functions of $u_1, (-\Delta)u_1, \cdots, (-\Delta)^k u_1$ and satisfy $a_0 = u_1, b_0 = 0$, and

$$a_{k+1} = \mu a_k - 2(-\Delta)b_k, \quad b_{k+1} = -2a_k + \eta b_k.$$

Here, μ and η are defined by $\mu = (-\Delta) + n$, $\eta = (-\Delta) - (n-2)$, respectively.

Making use of the above Lemma, we can prove the following key proposition.

Proposition 2.1

$$\begin{aligned}&\sum_{\alpha=1}^{n+1}\int_\Omega x_\alpha u_1\{(-\Delta)^p(x_\alpha u_1) - x_\alpha(-\Delta)^p u_1\} \\ \le &\sum_{k=1}^{p}\binom{p}{p-k} n^{p-k}\Lambda_1^{\frac{k-1}{p-1}} - \Lambda_1 \\ &+4[2^p-(p+1)]\sum_{k=0}^{p-2}\binom{p-2}{p-k-2} n^{p-k-2}\Lambda_1^{\frac{k}{p-1}} + n^p\int_\Omega u_1^2.\end{aligned} \tag{2.1}$$

Proof From Lemma 2.1, it is seen that

$$\begin{aligned}&\sum_{\alpha=1}^{n+1}\int_\Omega x_\alpha u_1\{(-\Delta)^p(x_\alpha u_1) - x_\alpha(-\Delta)^p u_1\} \\ =&\sum_{\alpha=1}^{n+1}\Big\{\int_\Omega x_\alpha u_1\{a_p x_\alpha + \langle \nabla b_p, \nabla x_\alpha\rangle\} - \int_\Omega x_\alpha^2 u_1(-\Delta)^p u_1\Big\} \\ =&\int_\Omega a_p u_1 - \Lambda_1.\end{aligned} \tag{2.2}$$

It was shown in [4] that

$$\begin{aligned}\int_\Omega a_p u_1 &\le \int_\Omega u_1\mu^p a_0 + 4[2^p-(p+1)]\int_\Omega u_1(-\Delta)\mu^{p-2}a_0\\ &= \int_\Omega u_1\mu^p u_1 + 4[2^p-(p+1)]\int_\Omega u_1(-\Delta)\mu^{p-2}u_1.\end{aligned}$$

In contrast, from (2.6) in [10], it is proved that

$$\int_\Omega u_1(-\Delta)^k u_1 \le \Lambda_1^{\frac{k-1}{p-1}} \qquad \text{for} \quad k=1,2,\cdots,p. \tag{2.3}$$

It follows that

$$\begin{aligned}\int_\Omega a_p u_1 &= \int_\Omega u_1\mu^p u_1 + 4[2^p-(p+1)]\int_\Omega u_1(-\Delta)\mu^{p-2}u_1\\ &= \sum_{k=0}^{p}\binom{p}{p-k} n^{p-k}\int_\Omega u_1(-\Delta)^k u_1\\ &\quad +4[2^p-(p+1)]\sum_{k=0}^{p-2}\binom{p-2}{p-k-2} n^{p-k-2}\int_\Omega u_1(-\Delta)^{k+1}u_1\\ &\le \sum_{k=1}^{p}\binom{p}{p-k} n^{p-k}\Lambda_1^{\frac{k-1}{p-1}}\\ &\quad +4[2^p-(p+1)]\sum_{k=0}^{p-2}\binom{p-2}{p-k-2} n^{p-k-2}\Lambda_1^{\frac{k}{p-1}} + n^p\int_\Omega u_1^2.\end{aligned} \tag{2.4}$$

Applying (2.4) into (2.2) concludes the proof of Proposition 2.1.

Proof of main theorem Let

$$\varphi_\alpha = x_\alpha u_1 - C_\alpha u_1, \tag{2.5}$$

where $C_\alpha = \int_\Omega x_\alpha u_1(-\Delta)u_1$. Then, one gets $\varphi_\alpha = \frac{\partial\varphi_\alpha}{\partial\nu} = \cdots = \frac{\partial^{p-1}\varphi_\alpha}{\partial\nu^{p-1}} = 0$ on $\partial\Omega$ and

$$\int_\Omega \langle\nabla\varphi_\alpha, \nabla u_1\rangle = 0. \tag{2.6}$$

It follows from the Rayleigh-Ritz inequality that

$$\Lambda_2 \le \frac{\int_\Omega \varphi_\alpha(-\Delta)^p\varphi_\alpha}{\int_\Omega |\nabla\varphi_\alpha|^2}. \tag{2.7}$$

By the definition of φ_α, one finds that

$$\begin{aligned}\int_\Omega |\nabla\varphi_\alpha|^2 &= \int_\Omega \varphi_\alpha(-\Delta)\varphi_\alpha = \int_\Omega \varphi_\alpha(-\Delta)(x_\alpha u_1)\\ &= \int_\Omega \varphi_\alpha x_\alpha(-\Delta)u_1 - \int_\Omega \varphi_\alpha[\Delta(x_\alpha u_1) + x_\alpha(-\Delta)u_1].\end{aligned} \tag{2.8}$$

Note that

$$\begin{aligned}\int_\Omega \varphi_\alpha(-\Delta)^p\varphi_\alpha &= \int_\Omega \varphi_\alpha(-\Delta)^p(x_\alpha u_1)\\ &= \Lambda_1\int_\Omega \varphi_\alpha x_\alpha(-\Delta)u_1 + \int_\Omega \varphi_\alpha[(-\Delta)^p(x_\alpha u_1) - x_\alpha(-\Delta)^p u_1].\end{aligned} \tag{2.9}$$

It follows from (2.7), (2.8), and (2.9) that

$$(\Lambda_2-\Lambda_1)\int_\Omega|\nabla\varphi_\alpha|^2 \le \Lambda_1\int_\Omega \varphi_\alpha[\Delta(x_\alpha u_1)+x_\alpha(-\Delta)u_1] + \int_\Omega \varphi_\alpha[(-\Delta)^p(x_\alpha u_1)-x_\alpha(-\Delta)^p u_1]. \tag{2.10}$$

Again, using integration by parts, one gets

$$\begin{aligned}\Lambda_1\sum_{\alpha=1}^{n+1}\int_\Omega \varphi_\alpha[\Delta(x_\alpha u_1)+x_\alpha(-\Delta)u_1] &= \Lambda_1\sum_{\alpha=1}^{n+1}\int_\Omega x_\alpha u_1[\Delta(x_\alpha u_1)+x_\alpha(-\Delta)u_1]\\ &= \Lambda_1\sum_{\alpha=1}^{n+1}\int_\Omega x_\alpha u_1[-nx_\alpha u_1+2\langle\nabla x_\alpha,\nabla u_1\rangle]\\ &= -n\Lambda_1\int_\Omega u_1^2. \end{aligned}\tag{2.11}$$

Applying (2.11) to (2.10) yields

$$\begin{aligned}&(\Lambda_2-\Lambda_1)\sum_{\alpha=1}^{n+1}\int_\Omega|\nabla\varphi_\alpha|^2\\ \le& -n\Lambda_1\int_\Omega u_1^2+\sum_{\alpha=1}^{n+1}\int_\Omega\varphi_\alpha[(-\Delta)^p(x_\alpha u_1)-x_\alpha(-\Delta)^p u_1]\\ =& -n\Lambda_1\int_\Omega u_1^2+\sum_{\alpha=1}^{n+1}\int_\Omega x_\alpha u_1[(-\Delta)^p(x_\alpha u_1)-x_\alpha(-\Delta)^p u_1]\\ \le& \sum_{k=1}^{p}\binom{p}{p-k}n^{p-k}\Lambda_1^{\frac{k-1}{p-1}}+4[2^p-(p+1)]\sum_{k=0}^{p-2}\binom{p-2}{p-k-2}n^{p-k-2}\Lambda_1^{\frac{k}{p-1}}\\ &-\Lambda_1+n(n^{p-1}-\Lambda_1)\int_\Omega u_1^2, \end{aligned}\tag{2.12}$$

where the last inequality follows by Proposition 2.1.

Let

$$D_\alpha=\int_\Omega\langle\nabla\varphi_\alpha,\nabla\langle\nabla x_\alpha,\nabla u_1\rangle-\frac{n-2}{2}x_\alpha\nabla u_1\rangle.$$

Then,

$$\begin{aligned}\sum_{\alpha=1}^{n+1}D_\alpha &= \sum_{\alpha=1}^{n+1}\int_\Omega\langle\nabla(x_\alpha u_1),\nabla\langle\nabla x_\alpha,\nabla u_1\rangle-\frac{n-2}{2}x_\alpha\nabla u_1\rangle\\ &\quad-\sum_{\alpha=1}^{n+1}C_\alpha\int_\Omega\langle\nabla u_1,\nabla\langle\nabla x_\alpha,\nabla u_1\rangle-\frac{n-2}{2}x_\alpha\nabla u_1\rangle\\ &=\sum_{\alpha=1}^{n+1}\int_\Omega\langle\nabla(x_\alpha u_1),\nabla\langle\nabla x_\alpha,\nabla u_1\rangle-\frac{n-2}{2}x_\alpha\nabla u_1\rangle\\ &=-\frac{n+2}{2}.\end{aligned}$$

As

$$\sum_{\alpha=1}^{n+1}\int_\Omega|\nabla\langle\nabla x_\alpha,\nabla u_1\rangle-\frac{n-2}{2}x_\alpha\nabla u_1|^2$$

$$= \sum_{\alpha=1}^{n+1}\Big(\int_\Omega |\nabla\langle\nabla x_\alpha,\nabla u_1\rangle|^2-(n-2)\int_\Omega\langle\nabla\langle\nabla x_\alpha\nabla u_1\rangle,x_\alpha\nabla u_1\rangle+\frac{(n-2)^2}{4}\int_\Omega|x_\alpha\nabla u_1|^2\Big)$$
$$=\Lambda_1+\frac{(n-2)^2}{4},$$

it follows that

$$\begin{aligned}\frac{(n+2)^2}{4}&=\Big(\sum_{\alpha=1}^{n+1}D_\alpha\Big)^2\\ &\le\Big(\sum_{\alpha=1}^{n+1}\int_\Omega|\nabla\varphi_\alpha|^2\Big)\Big(\sum_{\alpha=1}^{n+1}\int_\Omega|\nabla\langle\nabla x_\alpha,\nabla u_1\rangle-\frac{n-2}{2}x_\alpha\nabla u_1|^2\Big)\\ &=\Big(\Lambda_1+\frac{(n-2)^2}{4}\Big)\sum_{\alpha=1}^{n+1}\int_\Omega|\nabla\varphi_\alpha|^2.\end{aligned}\tag{2.13}$$

By (2.13), we obtain

$$\sum_{\alpha=1}^{n+1}\int_\Omega|\nabla\varphi_\alpha|^2\ge\frac{(n+2)^2}{4\Lambda_1+(n-2)^2}.\tag{2.14}$$

Consequently, we obtain from (2.14) and (2.12)

$$\begin{aligned}\Lambda_2\le\Lambda_1+\Big\{&\sum_{k=1}^{p}\binom{p}{p-k}n^{p-k}\Lambda_1^{\frac{k-1}{p-1}}+4[2^p-(p+1)]\sum_{k=0}^{p-2}\binom{p-2}{p-k-2}n^{p-k-2}\Lambda_1^{\frac{k}{p-1}}\\ &-\Lambda_1+n(n^{p-1}-\Lambda_1)\int_\Omega u_1^2\Big\}\frac{4\Lambda_1+(n-2)^2}{(n+2)^2}.\end{aligned}\tag{2.15}$$

It is clear that $\Lambda_1\ge\lambda_1^{p-1}$ by the variational principle, where λ_1 is the first eigenvalue of the buckling problem of order two. Hence,

$$\Lambda_1(\Omega_1)\ge n^{p-1}.\tag{2.16}$$

Using the Schwarz inequality and inequality (2.3), we have

$$1=\Big(\int_\Omega|\nabla u_1|^2\Big)^2=\Big(\int_\Omega u_1(-\Delta)u_1\Big)^2\le\int_\Omega u_1^2\int_\Omega u_1\Delta^2u_1\le\Lambda_1^{\frac{1}{p-1}}\int_\Omega u_1^2.$$

Hence,

$$\int_\Omega u_1^2\ge\Lambda_1^{\frac{-1}{p-1}}.\tag{2.17}$$

Applying (2.16) and (2.17) into (2.15) yields

$$\begin{aligned}\Lambda_2\le\Lambda_1+\Big\{&\sum_{k=1}^{p}\binom{p}{p-k}n^{p-k}\Lambda_1^{\frac{k-1}{p-1}}+4[2^p-(p+1)]\sum_{k=0}^{p-2}\binom{p-2}{p-k-2}n^{p-k-2}\Lambda_1^{\frac{k}{p-1}}\\ &-\Lambda_1+n(n^{p-1}-\Lambda_1)\Lambda_1^{\frac{-1}{p-1}}\Big\}\frac{4\Lambda_1+(n-2)^2}{(n+2)^2}.\end{aligned}$$

This completes the proof of Theorem.

Proof of Corollary From inequality (2.16), we have

$$(n^{p-1}-\Lambda_1)\Lambda_1^{\frac{-1}{p-1}}\le 0.$$

Then, the Corollary follows.

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