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Inequalities for a Polynomial and Its Derivative

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Let $p(z)=\sum_{v=0}^{n} a_v z^v$ be a polynomial of degree n having no zeros in $|z|<k$, $k\geqslant 1$. Then we prove that for $1\leqslant r\leqslant k$, $\max_{|z|=r}|p'(z)|\leqslant n(r+k)^{n-1}/(1+k)^n \max_{|z|=1}|p(z)|$ and for $0\leqslant r\leqslant \lambda\leqslant 1$, $\max_{|z|=r}|p(z)|\geqslant (k+r)^n/(k+\lambda)^n \max_{|z|=\lambda}|p(z)|$. Both results are best possible. Besides, it has been shown that if in addition $p'(0)=0$, then the bound in the latter inequality can be considerably improved.

1. INTRODUCTION AND STATEMENT OF RESULTS

If $p(z)$ is a polynomial of degree n, then

$$\max_{|z|=1}|p'(z)|\leqslant n\max_{|z|=1}|p(z)| \tag{1.1}$$

and

$$\max_{|z|=r}|p(z)|\geqslant r^n\max_{|z|=1}|p(z)|, \qquad \text{for} \quad r\leqslant 1. \tag{1.2}$$

Inequality (1.1) is an immediate consequence of S. Bernstein's theorem on the derivative of a trigonometric polynomial (for references see [7]) and inequality (1.2) is due to Zarantonello and Varga [8].

In both (1.1) and (1.2) equality holds only for $p(z)=e^{i\gamma}z^n$, i.e., when all the zeros of $p(z)$ lie at the origin. Erdős conjectured and later Lax [4] verified that if $p(z)\neq 0$ in $|z|<1$ then (1.1) can be replaced by

$$\max_{|z|=1}|p'(z)|\leqslant\frac{n}{2}\max_{|z|=1}|p(z)| \tag{1.3}$$

and in (1.3), equality holds if all the zeros of $p(z)$ lie on $|z|=1$. Rivlin [6] proved that if $p(z)\neq 0$ in $|z|<1$ then

$$\max_{|z|=r}|p(z)|\geqslant\left(\frac{1+r}{2}\right)^n\max_{|z|=1}|p(z)| \qquad \text{for} \quad r\leqslant 1 \tag{1.4}$$

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which is much better than (1.2). Besides, equality in (1.4) holds for $p(z) = ((\alpha + \beta z)/2)^n$ where $|\alpha| = |\beta|$.

For the class of polynomials having no zeros in $|z| < k$, $k \geqslant 1$, inequality (1.3) was generalized by Malik [5]. He proved that if $p(z) \neq 0$ in $|z| < k$, $k \geqslant 1$ then

$$\max_{|z|=1} |p'(z)| \leqslant \frac{n}{1+k} \max_{|z|=1} |p(z)|. \tag{1.5}$$

Inequality (1.5) is sharp and the equality holds for $p(z) = ((z+k)/(1+k))^n$.

In this paper, we generalize inequalities (1.4) and (1.5) for the class of polynomials having no zero in $|z| < k$, $k \geqslant 1$ by proving the following best possible results:

THEOREM 1. *If $p(z) = \sum_{v=0}^{n} a_v z^v$ is a polynomial of degree n, such that it has no zeros in $|z| < k$, $k \geqslant 1$, then for $0 \leqslant r \leqslant \lambda \leqslant 1$,*

$$\max_{|z|=r} |p(z)| \geqslant \left(\frac{k+r}{k+\lambda}\right)^n \max_{|z|=\lambda} |p(z)|. \tag{1.6}$$

The result is best possible and equality holds for the polynomial $p(z) = ((z+k)/(\lambda+k))^n$.

For $k = 1$, Theorem 1 reduces to a result due to Govil [3] and for $k = r = 1$ it reduces to inequality (1.4) due to Rivlin [6].

If in Theorem 1, we also assume $p'(0) = 0$, then the bound in (1.6) can be considerably improved. In fact we are able to prove the following

THEOREM 2. *If $p(z) = \sum_{v=0}^{n} a_v z^v$ is a polynomial of degree n such that $p'(0) = 0$ and $p(z) \neq 0$ for $|z| < k$, $k \geqslant 1$, then for $0 \leqslant r \leqslant \lambda \leqslant 1$,*

$$\max_{|z|=r} |p(z)| \geqslant \left(\frac{k+r}{k+\lambda}\right)^n \left\{1 - \frac{(k-\lambda)(\lambda-r)n}{4k^3} \left(\frac{k+r}{k+\lambda}\right)^{n-1}\right\}^{-1} \max_{|z|=\lambda} |p(z)|. \tag{1.7}$$

Theorem 1 is best possible, however, if $0 \leqslant r < \lambda < 1$, then for any polynomial $p(z)$ having no zeros in $|z| < k$, $k \geqslant 1$, and $p'(0) = 0$, the bound obtained by Theorem 2 is sharper than the one obtained by Theorem 1.

Lastly, as a generalization of (1.5) we prove the following

THEOREM 3. *If $p(z) = \sum_{v=0}^{n} a_v z^v$ is a polynomial of degree n having no zeros in $|z| < k$, $k \geqslant 1$, then*

$$\max_{|z|=r} |p'(z)| \leqslant \frac{n(r+k)^{n-1}}{(1+k)^n} \max_{|z|=1} |p(z)|, \tag{1.8}$$

for $1 \leqslant r \leqslant k$. *The result is best possible and the equality holds for* $p(z) = ((z+k)/(1+k))^n$.

2. Lemmas

Lemma 1. *Let* $p(z) = a_0 + \sum_{v=\mu}^{n} a_v z^v$ *be a polynomial of degree n such that* $p(z) \neq 0$ *in* $|z| < k$, $k \geqslant 1$, *then*

$$\max_{|z|=1} |p'(z)| \leqslant \frac{n}{1+k^\mu} \max_{|z|=1} |p(z)|. \tag{2.1}$$

The equality holds for $p(z) = (z^\mu + k^\mu)^{n/\mu}$ *where n is a multiple of* μ.

The above lemma is due to Chan and Malik [2].

Lemma 2. *If* $p(z) = \sum_{v=0}^{n} a_v z^v$ *is a polynomial of degree n, having no zeros in* $|z| < k$, $k \geqslant 1$, *then*

$$\max_{|z|=r \leqslant 1} |p(z)| \geqslant \left(\frac{r+k}{1+k}\right)^n \max_{|z|=1} |p(z)|. \tag{2.2}$$

The result is best possible with equality for the polynomial $p(z) = ((z+k)/(1+k))^n$.

We omit the proof of the above lemma as it follows easily on using arguments similar to those used in Rivlin [6].

Lemma 3. *If* $p(z)$ *is a polynomial of degree n having no zeros in* $|z| < k$, $k \geqslant 1$, *then for* $1 \leqslant R \leqslant k^2$,

$$\max_{|z|=R} |p(z)| \leqslant \left(\frac{R+k}{1+k}\right)^n \max_{|z|=1} |p(z)|. \tag{2.3}$$

The result is sharp and equality occurs for the polynomial $p(z) = ((z+k)/(1+k))^n$.

The above lemma is due to Aziz and Mohammed [1].

3. Proof of the Theorems

Proof of Theorem 1. Since $r \leqslant \lambda \leqslant 1$ and the polynomial $p(z)$ has no zeros in $|z| < k$, $k \geqslant 1$, therefore the polynomial $P(z) = p(\lambda z)$ has no zeros in $|z|<k/\lambda$, $k/\lambda \geqslant 1$. Further, since $r/\lambda \leqslant 1$, therefore on using Lemma 2, we get

$$\max_{|z|=r/\lambda \leqslant 1} |P(z)| \geqslant \left(\frac{r/\lambda + k/\lambda}{1+k/\lambda}\right)^n \max_{|z|=1} |P(z)|$$

which implies

$$\max_{|z|=r/\lambda \leqslant 1} |p(\lambda z)| \geqslant \left(\frac{r+k}{\lambda+k}\right)^n \max_{|z|=1} |p(\lambda z)|$$

which is equivalent to

$$\max_{|z|=r} |p(z)| \geqslant \left(\frac{r+k}{\lambda+k}\right)^n \max_{|z|=\lambda} |p(z)| \qquad \text{for} \quad r \leqslant \lambda \leqslant 1,$$

and the proof of Theorem 1 is complete.

Proof of Theorem 2. If $p(z)$ has no zeros in $|z|<k$, $k \geqslant 1$, and $0 \leqslant t \leqslant 1$ then $P(z)=p(tz)$ has no zeros in $|z|<k/t$, $k/t \geqslant 1$. Also $P'(0)=tp'(0)=0$, hence on using Lemma 1, with $\mu=2$, we get

$$\max_{|z|=1} |P'(z)| \leqslant \frac{n}{1+k^2/t^2} \max_{|z|=1} |P(z)|$$

which is equivalent to

$$\max_{|z|=t} |p'(z)| \leqslant \frac{nt}{t^2+k^2} \max_{|z|=t} |p(z)|. \tag{3.1}$$

For $0 \leqslant r \leqslant \lambda \leqslant 1$, we have

$$p(\lambda e^{i\theta}) - p(re^{i\theta}) = \int_r^\lambda e^{i\theta} p'(te^{i\theta})\, dt$$

which implies

$$|p(\lambda e^{i\theta}) - p(re^{i\theta})| \leqslant \int_r^\lambda |p'(te^{i\theta})|\, dt$$

which on using inequality (3.1) and Theorem 1, gives

$$\begin{aligned} |p(\lambda e^{i\theta}) - p(re^{i\theta})| &\leqslant \int_r^\lambda \frac{nt}{t^2+k^2} \left(\frac{k+t}{k+r}\right)^n \max_{|z|=r} |p(z)|\, dt \\ &= \frac{n}{(k+r)^n} \max_{|z|=r} |p(z)| \int_r^\lambda \frac{t(k+t)^n}{k^2+t^2}\, dt. \end{aligned} \tag{3.2}$$

Now for $0 \leqslant r \leqslant \lambda \leqslant 1$, inequality (3.2) is equivalent to

$$\begin{aligned}
\max_{|z|=\lambda} |p(z)| &\leqslant \left\{1 + \frac{n}{(k+r)^n} \int_r^\lambda \frac{t(k+t)^n}{k^2+t^2}\, dt\right\} \max_{|z|=r} |p(z)| \\
&\leqslant \left\{1 + \frac{n}{(k+r)^n} \frac{\lambda(k+\lambda)}{k^2+\lambda^2} \int_r^\lambda (k+t)^{n-1}\, dt\right\} \max_{|z|=r} |p(z)| \\
&= \left[\frac{k^2-\lambda k}{k^2+\lambda^2} + \frac{(\lambda k+\lambda^2)(k+\lambda)^n}{(k+r)^n\,(k^2+\lambda^2)}\right] \max_{|z|=r} |p(z)| \\
&= \left(\frac{k+\lambda}{k+r}\right)^n \left[1 - \frac{(k^2-\lambda k)(\lambda-r)}{(k^2+\lambda^2)(k+\lambda)(1-(k+r)/(k+\lambda))}\right. \\
&\quad \left. \times \left\{1 - \left(\frac{k+r}{k+\lambda}\right)^n\right\}\right] \max_{|z|=r} |p(z)|
\end{aligned}$$

which on simplification gives

$$\begin{aligned}
\max_{|z|=\lambda} |p(z)| &\leqslant \left(\frac{k+\lambda}{k+r}\right)^n \left[1 - \frac{(k^2-\lambda k)(\lambda-r)n}{(k^2+\lambda^2)(k+\lambda)} \left(\frac{k+r}{k+\lambda}\right)^{n-1}\right] \max_{|z|=r} |p(z)| \\
&\leqslant \left(\frac{k+\lambda}{k+r}\right)^n \left[1 - \frac{(k^2-\lambda k)(\lambda-r)n}{4k^4} \left(\frac{k+r}{k+\lambda}\right)^{n-1}\right] \max_{|z|=r} |p(z)|
\end{aligned}$$

which in turn gives

$$\max_{|z|=r} |p(z)| \geqslant \left(\frac{k+r}{k+\lambda}\right)^n \left\{1 - \frac{(k^2-\lambda k)(\lambda-r)n}{4k^4} \left(\frac{k+r}{k+\lambda}\right)^{n-1}\right\}^{-1} \max_{|z|=\lambda} |p(z)|$$

and the proof of Theorem 2 is complete.

Proof of Theorem 3. If $p(z)$ has no zeros in $|z| < k$, $k \geqslant 1$, and $1 \leqslant r \leqslant k$ then $P(z) = p(rz)$ has no zeros in $|z| < k/r$, $k/r \geqslant 1$. Thus applying inequality (1.5) to the polynomial $P(z)$, we get

$$\max_{|z|=1} |P'(z)| \leqslant \frac{n}{1+k/r} \max_{|z|=1} |P(z)|$$

which implies

$$\max_{|z|=1} |p'(rz)| \leqslant \frac{n}{r+k} \max_{|z|=1} |p(rz)|$$

which is equivalent to

$$\max_{|z|=r} |p'(z)| \leqslant \frac{n}{r+k} \max_{|z|=r} |p(z)|. \tag{3.3}$$

For $1 \leqslant r \leqslant k$, inequality (3.3) when combined with Lemma 3, gives

$$\max_{|z|=r} |p'(z)| \leqslant \frac{n}{r+k}\left(\frac{r+k}{1+k}\right)^n \max_{|z|=1} |p(z)|$$

$$= \frac{n(r+k)^{n-1}}{(1+k)^n} \max_{|z|=1} |p(z)|$$

which proves the theorem.

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