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Results in Mathematics

Inequalities for the Coefficients of Meromorphic Starlike Functions with Nonzero Pole

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Abstract. In this article we consider functions f that are meromorphic and univalent in the unit disc $\mathbb{D}$ with pole at the point $z = p \in (0, 1)$ and having a Taylor expansion at the origin of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n(f) z^n, \quad |z| < p.$$

The class of functions that satisfy the above conditions and map the unit disc such that $\overline{\mathbb{C}} \setminus f(\mathbb{D})$ is starlike with respect to a point w_0 $(\neq 0, \infty)$ will be denoted by $\Sigma^*(p, w_0)$. We generalize and sharpen an inequality for $a_2(f)$, $f \in \Sigma^*(p, w_0)$, proved by Miller (Proc Am Math Soc 80:607–613, 1980) by use of the coefficients $a_n(f)$, $n \geq 3$.

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1. Introduction

Let $S(p)$ be the class of univalent meromorphic mappings $f : \mathbb{D} \to \overline{\mathbb{C}}$ with a simple pole at $z = p$, $p \in (0, 1)$, and with the normalization

$$f(z) = z + \sum_{n=2}^{\infty} a_n(f) z^n, \quad |z| < p. \tag{1.1}$$

In this article we are concerned about functions in $S(p)$ such that $\overline{\mathbb{C}} \setminus f(\mathbb{D})$ is a starlike set with respect to a point $w_0 \neq 0, \infty$. We denote the class of such

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functions by $\Sigma^*(p,w_0)$. The class $\Sigma^*(p,w_0)$ has been introduced first by Miller in [7]. We now recall the following analytic characterization for functions in $\Sigma^*(p,w_0)$.

Theorem A. *$f\in\Sigma^*(p,w_0)$ if and only if there is a probability measure $\mu(\zeta)$ on $\partial\mathbb{D}=\{\zeta:|\zeta|=1\}$ so that*

$$f(z)=w_0+\frac{pw_0}{(z-p)(1-zp)}\exp\left(\int\limits_{\partial\mathbb{D}}2\log(1-\zeta z)\,d\mu(\zeta)\right) \tag{1.2}$$

where

$$w_0=-\frac{1}{p+1/p-2\int_{|\zeta|=1}\zeta d\mu(\zeta)}. \tag{1.3}$$

The necessary part of this characterization was established by Miller in [8] and the sufficiency part by Yuh Lin in [12, Theorem 1]. In addition, Yuh Lin proved that the closed disc Ω_p where

$$\Omega_p=\left\{w:\left|w+\frac{p(1+p^2)}{(1-p^2)^2}\right|\le\frac{2p^2}{(1-p^2)^2}\right\}, \tag{1.4}$$

describes the *region of all starlike centers* for all functions in $\Sigma^*(p,w_0)$ (compare [12, Corollary, p. 518]). See also [3, Corollary 3.1] for an alternative proof of this fact and some more interesting results for functions in $\Sigma^*(p,w_0)$. We state here that the following analytic characterization for functions in $\Sigma^*(p,w_0)$, proved by Yuh Lin and Owa in [11], is an equivalent form of the analytic characterization which is presented in the Theorem A.

Theorem B. *The function f belongs to the class $\Sigma^*(p,w_0)$ if and only if there exists a function $\omega:\mathbb{D}\to\overline{\mathbb{D}}$ holomorphic in $\mathbb{D}$ such that $f(0)=f'(0)-1=0$ and*

$$\frac{1+z\omega(z)}{1-z\omega(z)}=\frac{-zf'(z)}{f(z)-w_0}-\frac{p}{z-p}+\frac{pz}{1-pz},\quad z\in\mathbb{D}. \tag{1.5}$$

We now recall the following result of J. Miller concerning inequalities for the Taylor coefficients of functions in $\Sigma^*(p,w_0)$ (see f.i. [8, Theorem 8] and also compare [6, p. 257, Theorem 54]).

Theorem C. *Let $f\in\Sigma^*(p,w_0)$ and have the expansion at the origin of the form (1.1). Then the following inequalities*

$$\left|a_2(f)-\frac{1+p^2}{p}-w_0\right|\le|w_0| \tag{1.6}$$

and for $n\ge 2$

$$\left|a_{n+1}(f)-\left(\frac{1+p^2}{p}\right)a_n(f)+a_{n-1}(f)\right|\le\frac{2|w_0|}{n+1} \tag{1.7}$$

are true. The inequality (1.6) *is sharp for the function*

$$\begin{aligned} f = f_2(z) &= \frac{z}{(1-z/p)(1-zp)} \\ &= w_0\left(1 - \frac{1+z^2}{(1-z/p)(1-zp)}\right); \quad w_0 = \frac{-p}{1+p^2}. \end{aligned} \tag{1.8}$$

and equality occurs in (1.7) *for the functions*

$$f = f_{n+1}(z) = w_0\left(1 - \frac{(1+z^{n+1})^{2/(n+1)}}{(1-z/p)(1-zp)}\right), \quad w_0 = -p/(1+p^2). \tag{1.9}$$

In [4, Theorem 2.1], Ponnusamy and the authors of the present article determined the exact region of variability of $a_2(f)$, $f \in \Sigma^*(p, w_0)$. Now an use of this result will yield that (1.6) describes the exact region of variability for $a_2(f)$, $f \in \Sigma^*(p, w_0)$ if and only if $w_0 = -p/(1+p^2)$.

In this paper, we prove generalizations of the inequality (1.6) involving the coefficients $a_n(f)$, $n \geq 3$.

2. Results

We need little more preparation to state our first result. We start with the following Lemma due to Clunie (see [5] and compare [9, Theorem 2.2]) which he proved for the case of holomorphic functions in $\mathbb{D}$. This result was later generalized to the case of meromorphic functions in [2, Lemma 2.1]. We will use it in the following form for our purpose.

Lemma 1. *Let the functions F and G be meromorphic in $\mathbb{D}$ and holomorphic in a neighborhood of the origin, where they have expansions*

$$F(z) = \sum_{n=0}^{\infty} A_n z^n \quad \text{and} \quad G(z) = \sum_{n=0}^{\infty} B_n z^n. \tag{2.1}$$

If there exists a function $\omega : \mathbb{D} \to \overline{\mathbb{D}}$ holomorphic in $\mathbb{D}$ such that $F(z) = \omega(z)G(z), z \in \mathbb{D}$, then for all $n \in \mathbb{N} \cup \{0\}$ the inequalities

$$\sum_{k=0}^{n} |A_k|^2 \leq \sum_{k=0}^{n} |B_k|^2 \tag{2.2}$$

are valid.

We are now in a position to state the first result.

Theorem 1. *Let $f \in \Sigma^*(p, w_0)$ and have the expansion* (1.1) *at the origin. Then for any $n \in \mathbb{N} \setminus \{1\}$, the following inequality is valid:*

$$\left|a_2(f) - \frac{1+p^2}{p} - w_0\right|^2 + \sum_{k=2}^{n-1} k \left|a_{k+1}(f) - \left(\frac{1+p^2}{p}\right) a_k(f) + a_{k-1}(f)\right|^2$$
$$+ \frac{(n+1)^2}{4} \left|a_{n+1}(f) - \left(\frac{1+p^2}{p}\right) a_n(f) + a_{n-1}(f)\right|^2 \le |w_0|^2. \tag{2.3}$$

In (2.3) equality is attained if

$$f = \begin{cases} f_2 & \text{for any} \quad n \ge 2, \\ f_{n+1} & \text{for fixed} \quad n \ge 2; \end{cases}$$

where f_2 and f_{n+1} are given in (1.8) and (1.9) respectively.

Proof. We solve the Eq. (1.5) in Theorem B in the form

$$F(z) = \omega(z)\, G(z), \quad z \in \mathbb{D};$$

where

$$F(z) = f'(z)\left(\frac{p(1+z^2)}{1+p^2} - z\right) + (f(z) - w_0)\left(\frac{2pz}{1+p^2} - 1\right) \quad \text{and}$$
$$G(z) = zf'(z)\left(\frac{p(1+z^2)}{1+p^2} - z\right) - (f(z) - w_0)\left(\frac{2p}{1+p^2} - z\right).$$

As the above functions F and G are meromorphic in $\mathbb{D}$ and holomorphic in a neighborhood of the origin, we have the Taylor expansions of F and G of the form (2.1), where the Taylor coefficients A_k and B_k's are being computed as follows:

$$A_0 = \frac{p}{1+p^2} + w_0, \quad A_1 = \left(\frac{2p}{1+p^2}\right) a_2(f) - 2 - \left(\frac{2p}{1+p^2}\right) w_0,$$
$$A_k = (k+1)\left(\frac{p}{1+p^2}\right)\left(a_{k+1}(f) - \left(\frac{1+p^2}{p}\right) a_k(f) + a_{k-1}(f)\right), \quad k \ge 2; \text{ and}$$
$$B_0 = \left(\frac{2p}{1+p^2}\right) w_0, \quad B_1 = -A_0, \quad B_k = \left(\frac{k-2}{k}\right) A_{k-1}, \quad k \ge 2.$$

We note that for $k \ge 2$, we have

$$|A_k|^2 - |B_{k+1}|^2 = k\left(\frac{2p}{1+p^2}\right)^2 \left|a_{k+1}(f) - \left(\frac{1+p^2}{p}\right) a_k(f) + a_{k-1}(f)\right|^2.$$

Now, an application of the inequality (2.2) using the above computation yields the inequality (2.3). The proof of the theorem will be complete if we can prove our assertion on sharpness of this inequality. To this end, we first consider the coefficients

$$a_n(f_2) = \frac{1}{p^{n-1}} \sum_{k=0}^{n-1} p^{2k}, \quad n \ge 2.$$

Now from a little computation using these coefficients, we see that

$$a_2(f_2) = \frac{1+p^2}{p} \quad \text{and} \quad a_{k+1}(f_2) - \left(\frac{1+p^2}{p}\right) a_k(f_2) + a_{k-1}(f_2) = 0 \quad \text{for } k \ge 2.$$

This proves the first assertion on equality in the inequality (2.3). We now turn on to the remaining cases of equality in the inequality (2.3). To fulfill our aim, we consider the coefficients $a_k(f_m)$, $(m \geq 3,\ k \geq 1)$ of

$$f_m(z) = -\frac{p}{1+p^2}\left(1 - \left[\sum_{j=0}^{\infty}\binom{2/m}{j} z^{jm}\right]\left[\sum_{j=0}^{\infty} a_{j+1}(f_2) z^j\right]\right), \tag{2.4}$$

and we discuss the following cases.

Case (i) $[k=2,\ m=n+1 \geq 3]$: For these values of k and n, we see that

$$a_2(f_{n+1}) - \frac{1+p^2}{p} - w_0 = \left(\frac{p}{1+p^2}\right) a_3(f_2) - \frac{1+p^2+p^4}{p(1+p^2)} = 0.$$

Case (ii) $[3 \leq k \leq n,\ m = n+1 \geq 3]$: In this case an use of the coefficients of f_m from (2.4) will result

$$\begin{aligned} & a_k(f_{n+1}) - \left(\frac{1+p^2}{p}\right) a_{k-1}(f_{n+1}) + a_{k-2}(f_{n+1}) \\ & = \left(\frac{p}{1+p^2}\right)\left[a_{k+1}(f_2) - \left(\frac{1+p^2}{p}\right) a_k(f_2) + a_{k-1}(f_2)\right] = 0. \end{aligned}$$

Case (iii) $[k = n+1,\ m = n+1 \geq 3]$: In this case we get

$$\begin{aligned} & a_{n+1}(f_{n+1}) - \left(\frac{1+p^2}{p}\right) a_n(f_{n+1}) + a_{n-1}(f_{n+1}) \\ & = \frac{2p}{(1+p^2)(n+1)} + \left(\frac{p}{1+p^2}\right)\left[a_{k+2}(f_2) - \left(\frac{1+p^2}{p}\right) a_{k+1}(f_2) + a_k(f_2)\right] \\ & = \frac{2p}{(1+p^2)(n+1)} = \frac{2|w_0|}{n+1}. \end{aligned}$$

Hence, we see that all terms but the last one on the left hand side of (2.3) vanish for $f = f_{n+1}$ and that the last one equals $|w_0|^2$, which proves the sharpness of the inequality (2.3). It remains to prove that $f_n \in \Sigma^*(p, -p/(1+p^2))$, $n \geq 2$. To show this, we compute

$$\frac{-zf_n'(z)}{f_n(z) - w_0} - \frac{p}{z-p} + \frac{pz}{1-pz} = \frac{1+z^n}{1-z^n}.$$

Now, the proof is complete from the characterization formula (1.5) by letting $\omega(z) = z^{n-1}$, $z \in \mathbb{D}$. □

An application of the above theorem will directly yield the inequalities (1.6) and (1.7) of Theorem C, proved by J. Miller:

Corollary 1. *Let $f \in \Sigma^*(p, w_0)$ have the expansion (1.1) at the origin. Then the inequalities (1.6) and (1.7) are valid and they are sharp.*

Remark. We mention here that the inequality (2.2) is equivalent to (1.4) for the case $n = 0$, which is the set of all possible starlike centers.

If we let $n \to \infty$ in (2.3), we get the following generalization of Theorem 1.

Theorem 2. *Let $f \in \Sigma^*(p, w_0)$ have the expansion (1.1) at the origin. Then the following inequality holds*

$$\left|a_2(f) - \frac{1+p^2}{p} - w_0\right|^2 + \sum_{k=2}^{\infty} k \left|a_{k+1}(f) - \left(\frac{1+p^2}{p}\right) a_k(f) + a_{k-1}(f)\right|^2 \leq |w_0|^2. \tag{2.5}$$

Equality is attained in the above inequality for any $f = f_m := f_{n+1}$, $n+1 = m \geq 2$, where f_{n+1} is defined in (1.9).

Proof. We need only to prove the assertion on equality in (2.5). The proof of the Theorem 1 makes clear that equality in (2.5) occurs for $f = f_2$. Hence we have to consider the Taylor coefficients of f_m, $m \geq 3$. Again the proof of Theorem 1 shows that the first term of the left hand side of (2.5) vanishes always. In the following computations we use the representation (2.4) of f_m and the fact that the sum

$$S_j = \sum_{i=0}^{j-1} \binom{2/m}{i} \left(a_{k+2-im}(f_2) - \left(\frac{1+p^2}{p}\right) a_{k+1-im}(f_2) + a_{k-im}(f_2) \right)$$

vanishes for $k > (j-1)m$, $j \in \mathbb{N}$. We now consider the following three cases in order to fulfill our aim:

Case (i) $[(j-1)m < k < k+1 < jm]$: In this case, we have

$$a_{k+1}(f_m) - \left(\frac{1+p^2}{p}\right) a_k(f_m) + a_{k-1}(f_m) = \left(\frac{p}{1+p^2}\right) S_j = 0.$$

Case (ii) $[k+1 = jm]$: Here we have

$$a_{k+1}(f_m) - \left(\frac{1+p^2}{p}\right) a_k(f_m) + a_{k-1}(f_m) = \left(\frac{p}{1+p^2}\right) \left[\binom{2/m}{j} + S_j\right] = \left(\frac{p}{1+p^2}\right) \binom{2/m}{j}.$$

Case (iii) $[k = jm]$: A computation in this case reveals that

$$\begin{aligned} &a_{k+1}(f_m) - \left(\frac{1+p^2}{p}\right) a_k(f_m) + a_{k-1}(f_m) \\ &= \left(\frac{p}{1+p^2}\right) \left[\binom{2/m}{j} a_2(f_2) - \left(\frac{1+p^2}{p}\right) \binom{2/m}{j} + S_j\right] \\ &= 0. \end{aligned}$$

Since $w_0 = -p/(1+p^2)$, the sharpness of the inequality (2.5) will be established by the validity of the following identity

$$\sum_{j=1}^{\infty}(jm-1)\binom{2/m}{j}^2 = 1.$$

Equivalently, the above identity may be expressed as

$$\sum_{j=1}^{\infty}(jm-1)\frac{[(-2/m)_j]^2}{(1)_j\, j!} = 1, \tag{2.6}$$

where $(a)_j := a(a+1)\cdots(a+j-1)$ denotes the Pochhammer symbol. To prove this identity we introduce the following hypergeometric function

$$F\left(-2/m,-2/m,1;z^m\right) = \sum_{j=0}^{\infty}\frac{[(-2/m)_j]^2}{(1)_j\; j!}\; z^{jm},$$

and we see that (2.6) is the same as

$$\lim_{z\to 1}\ \frac{d}{dz}\ \left[\left(\frac{1}{z}\right)F\left(-2/m,-2/m,1;\ z^m\right)-\left(\frac{1}{z}\right)\right] = 1,$$

i.e.

$$\lim_{z\to 1}\ \frac{d}{dz}\ \left[\left(\frac{1}{z}\right)F\left(-2/m,-2/m,1;\ z^m\right)\right] = 0. \tag{2.7}$$

We know that (compare [10, Theorem 18, p. 49]) if $\mathrm{Re}\,(c-a-b)>0$ and c is neither zero nor a negative integer, then

$$\lim_{z\to 1}\ F(a,b,c;z) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}.$$

Using the above formula we compute

$$\begin{aligned}
&\lim_{z\to 1}\ \frac{d}{dz}\ \left[\left(\frac{1}{z}\right)F\left(-2/m,-2/m,1;\ z^m\right)\right]\\
&\quad= \lim_{z\to 1}\ \left(\frac{-1}{z^2}\right)F\left(-2/m,-2/m,1;\ z^m\right)) +\\
&\lim_{z\to 1}\ \left(\frac{m}{z}\right)\left(\frac{4}{m^2}\right)F\left(1-2/m,1-2/m,2;\ z^m\right)\\
&\quad= -\frac{\Gamma(1)\Gamma\left(1+4/m\right)}{[\Gamma(1+2/m)]^2}+\left(\frac{4}{m}\right)\frac{\Gamma(2)\Gamma(4/m)}{[\Gamma(1+2/m)]^2} = 0.
\end{aligned}$$

This completes the proof of the Theorem 2. □

Let us consider the class of concave univalent functions denoted by $Co(p)$. These functions belong to the class $S(p)$ and additionally they map the unit disc $\mathbb{D}$ onto the complement of a compact convex set in the complex plane. For more details about the class of such functions one can see the articles [1–3]. For these functions, we now readily have the following:

Corollary 2. *Let $f \in Co(p)$ have the expansion (1.1). Then the inequalities (2.3) and (2.5) follow with $w_0 = -p/(1+p^2)$. Both the inequalities are sharp for the function $f = f_2$.*

Proof. In [8, theorem 6], Miller proved that $Co(p) \subsetneq \Sigma^*(p, -p/(1+p^2))$. Now the Corollary is a direct consequence of this result. □

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