General properties of harmonic mappings

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Heinz's Lemma

Let f maps the unit disk $\mathbb{D}$ harmonically onto itself, with $f(0) = 0$. Then

$$|f_z(0)|^2 + |f_{\bar{z}}(0)|^2 \geq c$$

for some absolute constant $c > 0$.

Proof

The following approximation argument shows it is enough to assume that f extends to a homeomorphism of $\mathbb{D}$ onto itself. For $0 < r < 1$, let $D_r \subset \mathbb{D}$ be the preimage under f of the disk $|w| < r$, and let φ be a conformal mapping of $\mathbb{D}$ onto D_r with $\varphi(0) = 0$.

Then $g = \frac{1}{r} f \circ \varphi$ maps $\mathbb{D}$ harmonically onto itself with $g(0) = 0$ (since $f(0) = 0$), and g extends homeomorphically on the closure of $\mathbb{D}$. But

$$g_z(0) = \frac{1}{r} f_z(0)\varphi^{'}(0) \quad \text{and} \quad g_{\bar{z}}(0) = \frac{1}{r} f_{\bar{z}}(0)\overline{\varphi^{'}(0)},$$

so if it can be shown that $|g_z(0)|^2 + |g_{\bar{z}}(0)|^2 \geq 2$, because the Schwarz lemma gives $|\varphi^{'}(0)| \leq 1$. In view of the symmetry of the expression to be estimated, it may also be assumed without loss of generality that f is sense-preserving.

Under these assumptions it is clear that f has a Poisson representation:

$$f(z) = \frac{1}{r}\int_0^{2\pi} \frac{1-|z|^2}{|e^{it}-z|^2} f(e^{it})dt\,,$$

where $f(e^{it}) = e^{i\theta(t)}$ and $\theta(t)$ is continuous and strictly increasing on the interval $[0, 2\pi]$, with $\theta(2\pi) - \theta(0) = 2\pi$. In standard notation, f has the espansion

$$f(z) = \sum_{n=1}^{\infty}(a_n z^n + \overline{b}_n \overline{z}^n), \quad a_1 = f_z(0), \quad b_1 = f_{\overline{z}}(0).$$

The Poisson kernel is

$$\frac{1-|z|^2}{|e^{it}-z|^2} = \mathrm{Re}\left\{\frac{e^{it}+z}{e^{it}-z}\right\} = 1+\sum_{n=1}^{\infty}(e^{-int}z^n+e_n^{int}\bar{z}^n).$$

Thus, the Poisson representation provides the formulas

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} e^{-int}e^{i\theta(t)}dt$$

$$b_n = \frac{1}{2\pi}\int_0^{2\pi} e^{-int}e^{-i\theta(t)}dt\,.$$

Integrating by parts now gives

$$2\pi n a_n = \int_0^{2\pi} e^{-int} e^{i\theta(t)} dt$$

$$2\pi n a_n = -\int_0^{2\pi} e^{-int} e^{-i\theta(t)} dt\,.$$

On the other hand, it follows from Parseval's relation that

$$\sum_{n=1}^{\infty} (|a_n|^2 + |b_n|^2) = \frac{1}{2\pi} \int_0^{2\pi} |f(e^{it})|^2 dt = 1\,.$$

Thus, it must be shown that

$$\sum_{n=2}^{\infty}(|a_n|^2 + |b_n|^2) \leq 1 - c\,, \quad c > 0\,.$$

The trivial estimates $|a_n| \leq \frac{1}{n}$ and $|b_n| \leq \frac{1}{n}$ are not good enough for this purpose. Instead, the formula for a_n and b_n will be combined to obtain an improved estimate for $|a_n|^2 + |b_n|^2$. A short calculation leads to the expression

$$2\pi^2 n^2(|a_n|^2 + |b_n|^2) =$$

$$\begin{aligned}
&= \int_0^{2\pi}\int_0^{2\pi} \cos n(s-t)\cos(\theta(s)-\theta(t))d\theta(s)d\theta(t) \\
&= \int_0^{2\pi} d\theta(t)\int_0^{2\pi} |d_s \sin(\theta(s)-\theta(t))| \\
&= \int_0^{2\pi} d\theta(t)\int_0^{2\pi} |d\sin\,\theta| = 8\pi\,.
\end{aligned}$$

Applying this estimate, one finds

$$\sum_{n=1}^{\infty}(|a_n|^2 + |b_n|^2) \le \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{2\pi}{3} - \frac{4}{\pi}\,,$$

The desired inequality with $c = 1 - \frac{2\pi}{3} + \frac{4}{\pi} = 0.1778\ldots$. Heinz's inequality now follows from Parseval's relation. ■

Theorem

The only harmonic mappings of $\mathbb{C}$ onto $\mathbb{C}$ are the affine mappings $f(z) = \alpha z + \gamma + \beta \bar{z}$, where α, β and γ are the complex constants and $|\alpha| \neq |\beta|$.

Proof

Let $f = h + \bar{g}$ map $\mathbb{C}$ harmonically onto $\mathbb{C}$, and without loss of generality that f is sense-preserving. This means that $|g^{'}(z)| < |h^{'}(z)|$, $|g^{'}(z)|/|h^{'}(z)| < 1$ for all $z \in \mathbb{C}$.

Hence, by Liouville's theorem $|g^{'}(z)|/|h^{'}(z)| \equiv b$ for some complex constant b with $|b| < 1$. Integration gives

$$g(z) = b\,h(z) + c\,,$$

where c is a constant. Thus, f has the form

$$f = h + \bar{c} + \bar{b}\bar{h} = F \circ h\,,$$

where F is (invertible) affine mapping. It follows that $h = F^{-1} \circ f$ maps $\mathbb{C}$ univalently onto $\mathbb{C}$. But h is analytic, so it must have the form $h(z) = \alpha z + \beta$ for some complex constants α and β. This shows that f is affine mapping ■

Rado's Theorem

There is no harmonic mapping of unit disk $\mathbb{D}$ onto $\mathbb{C}$.

Proof

The following argument actually gives a stronger quantitative form of Rado's theorem. Suppose that f maps $\mathbb{D}$ harmonically onto a domain $\Omega \subset \mathbb{C}$ which contains a disk Δ_R of radius R. Assume without loss of generality that $f(0) = 0$ and $\Delta_R = \{w \in \mathbb{C} : |w| < R\}$. Denote by D_R the subdomain of $\mathbb{D}$ for which $f(D_R) = \Delta_R$.

Let φ be a conformal mapping of $\mathbb{D}$ onto D_R with $\varphi(0) = 0$. Then $F = \frac{1}{R} f \circ \varphi$ maps $\mathbb{D}$ harmonically onto $\mathbb{D}$, with $F(0) = 0$, so the Heinz lemma says that

$$|F_\xi(0)|^2 + |F_{\bar{\xi}}(0)|^2 \geq c > 0\,,$$

where c is an absolute constant. But a calculation gives

$$F_\xi(0) = \frac{1}{R} f_z(0)\varphi^{'}(0), \quad F_{\bar{\xi}}(0) = \frac{1}{R} f_{\bar{z}}(0)\overline{\varphi'(0)}\,.$$

Since $|\varphi'(0)| \leq 1$ by the Schwarz lemma, it follows that

$$c\,R^2 \leq |f_z(0)|^2 + |f_{\bar{z}}(0)|^2\,.$$

In particular, the range of f can not contain disks of arbitrarily large radius centered at the origin. ■

Corollary

No proper subdomain of the plane can be mapped harmonically onto the whole plane.

Proof

Suppose there exists a harmonic mapping f of some simply connected domain $\Omega \neq \mathbb{C}$ onto $\mathbb{C}$. By the Riemann mapping theorem, there is a conformal mapping φ of $\mathbb{D}$ onto Ω. Thus, the composition $f \circ \varphi$ maps $\mathbb{D}$ harmonically onto $\mathbb{C}$, in violation of Rado's theorem. ■

Lewy's Theorem

If f is a complex-valued harmonic function that is locally univalent in a domain $D \subset \mathbb{C}$, then its Jacobian $J_f(z)$ is different from 0 for all $z \in D$.

Remark on Lewy's Theorem

In fact, Lewy's theorem is false in dimensions higher than two. The following counterexample is due to J. C. Wood. Consider the plynomial map $f : \mathbb{R}^3 \to \mathbb{R}^3$ **defined by** $f(x, y, z) = (u, v, w)$**, where**

$$u = x^3 - 3xz^2 + yz, \quad v = y - 3xz, \quad w = z\,.$$

It is immediately verified that each component of f **is harmonic function in** $\mathbb{R}^3$**. To see that** f **is univalent, suppose that**

$$f(x_1, y_1, z_1) = f(x_2, y_2, z_2) = (u, v, w)$$

for some pair of points (x_1, y_1, z_1) and (x_2, y_2, z_2) in $\mathbb{R}^3$. Then obviously $w = z_1 = z_2$ and

$$v = y_1 - 3x_1 w = y_2 - 3x_2 w\,,$$

which implies

$$u = x_1^3 + w(y_1 - 3x_1 w) = x_2^3 + w(y_2 - 3x_2 w)\,.$$

It follows that

$$x_1^3 + vw = x_2^3 + vw\,,$$

so that $x_1 = x_2$ and $y_1 = y_2$. This proves the univalence of f.

The calculations actually show that the mapping $(u, v, w) \longmapsto (x, y, z)$ defined by

$$x = (u - vw)^{1/3}, \quad y = v + 3w(u - vw)^{1/3}, \quad z = w\,,$$

is an inverse for f. Thus, f is a (univalent)harmonic mapping of $\mathbb{R}^3$ onto $\mathbb{R}^3$. On the other hand, a straighforward calculation reveals that f has the Jacobian

$$J_f(x, y, z) = 3x^2\,,$$

which vanish on the plane $x = 0$. Hence, Lewy's theorem is false in $\mathbb{R}^3$ and, therefore, in $\mathbb{R}^n$ for all $n \geq 3$.

Definition

A function f_0 harmonic in the $\mathbb{D}$ is said to be *subordinate* to a harmonic function f if it has the form $f_0(z) = f(\omega(z))$ for some function ω analytic and univalent in $\mathbb{D}$ with the properties $|\omega(z)| < 1$ and $\omega(0) = 0$. If f is a harmonic mapping of $\mathbb{D}$ (a *univalent* harmonic function), and if Ω is a simply connected domain such that $f(0) \in \Omega \subset f(\mathbb{D})$, then there is a harmonic mapping f_0 of $\mathbb{D}$ onto Ω, subordinate to f. Furthermore, this mapping f_0 is unique up to rotation of the disk.

To see this, we appeal to the Riemann mapping theorem and choose ω to be the conformal mapping of $\mathbb{D}$ onto $f^{-1}(\Omega)$ with $\Omega(0) = 0$ and $\Omega^{'}(0) > 0$. The subordinate function $f_0 = f \circ \omega$ is then a harmonic mapping of $\mathbb{D}$ onto Ω, and it is uniquely determined by the normalization $\omega^{'}(0) > 0$. Under these circumstances we will call f_0 the harmonic mapping subordinate to f, corresponding to the domain $\Omega \subset f(\mathbb{D})$.

Approximation Theorem

Let f be a univalent harmonic function in $\mathbb{D}$, and let $\{\Omega_n\}$ be a sequence of simply connected domains with the property $f(0) \in \Omega_n \subset f(\mathbb{D})$ that converges to $f(\mathbb{D})$. Then the corresponding sequence of subordinate functions f_n converges to f locally uniformly in $\mathbb{D}$.

Proof

The subordinate functions have the form $f_n(z) = f(\omega_n(z))$, where ω_n are the analytic univalent functions defined earlier. The hypothesis that $\Omega_n \to f(\mathbb{D})$ is easily seen to imply that $\omega_n(\mathbb{D}) \to \mathbb{D}$. Thus, it follows from the Carathéodeory convergence theorem that $\omega_n(z) \to z$ locally uniformly in $\mathbb{D}$. Thus, $f_n(z) \to f(z)$ locally uniformly. ■

References

[1] **Peter Duren:** *Harmonic mappings in the plane* .

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