COMMUNICATIONS IN ALGEBRA, 30(5), 2085–2115 (2002)

COMPLETELY REAL HULLS OF PARTIALLY ORDERED RINGS

Ralph Berr

Fachbereich Mathematik, Universität, Dortmund,
44221 Dortmund, FRG

INTRODUCTION

In this paper we investigate certain extensions of partially ordered rings. This purely algebraic approach is inspired by the search for rings of "rational functions" on semialgebraic sets which should play the same role as the ring of regular functions in the framework of algebraic geometry. Therefore, in order to motivate the consideration of the algebraic constructions below we first briefly describe the geometric background.

Let R be a real closed field and let $S \subset \mathbb{A}^n(R)$ be a closed semialgebraic set. We let $R[S]$ denote the ring of R-valued polynomial functions on S and $P[S] \subset R[S]$ the partial order of nonnegative functions on S. Then it is known that the irreducible algebraic subsets of S correspond bijectively to the $P[S]$-convex prime ideals of $R[S]$. Consequently, in order to study the geometry of algebraic subsets of S it might seem natural to investigate the locally ringed space $(X_S, \mathcal{O}_S)$, where $X_S \subset \operatorname{Spec} R[S]$ denotes the proconstructible subspace of the $P[S]$-convex prime ideals and $\mathcal{O}_S$ the restriction of the structure sheaf of the affine scheme $(\operatorname{Spec} R[S], \mathcal{O})$ to X_S. From the geometric point of view the space $(X_S, \mathcal{O}_S)$ has some convenient properties. For example, the points of S are in bijective correspondence with the maximal ideals of the ring $\mathcal{O}_S(X_S) \cong R[S]_{1+P[S]}$ of global sections on X_S. But there is also the serious problem that the ideal-theoretic structure of the ring $\mathcal{O}_S(X_S)$ is only loosely related with the underlying space X_S. To be precise,

Copyright © 2002 by Marcel Dekker, Inc. www.dekker.com

the space $\operatorname{Spec} \mathcal{O}_S(X_S)$ is isomorphic to the subspace $\operatorname{Gen}(X_S) \subset \operatorname{Spec} R[S]$ of the generalizations of X_S in $\operatorname{Spec} R[S]$.

In the next step one therefore has to ask whether the ring $R[S]$ admits an extension $\varphi : R[S] \longrightarrow A$ such that the induced map $\operatorname{Spec}(\varphi)$ between the Zariski-spectra gives us an isomorphism

$$\operatorname{Spec}(\varphi) : \operatorname{Spec} A \longrightarrow X_S$$

of spectral spaces. But in Sec. 2 we will show that in general such an extension does not exist. So we are left with the question whether there is at least an extension $\varphi : R[S] \longrightarrow A$ such that the induced map $\operatorname{Spec}(\varphi)$ is a bijection between $\operatorname{Spec} A$ and X_S. If such an extension exists, one has further to ask whether the affine Scheme $(\operatorname{Spec} A, \mathcal{O})$ is convenient from the semi-algebraic point of view.

At this point we meet the subject of this paper. A partially ordered ring $(\tilde{A}, \tilde{P})$ is called completely real if all prime ideals of $\tilde{A}$ are $\tilde{P}$-convex. In this paper we study completely real extensions

$$\varphi : (A, P) \longrightarrow (\tilde{A}, \tilde{P})$$

of a given partially ordered ring (A, P). A first relationship with the problem discussed above is given by the simple fact that for any completely real extension $\varphi : (R[S], P[S]) \longrightarrow (A, P)$ we get a morphism

$$\operatorname{Spec}(\varphi) : \operatorname{Spec} A \longrightarrow X_S.$$

One of the main results of this paper states that there are certain ‘‘minimal’’ completely real extensions of $(R[S], P[S])$ that have the following much stronger properties:

(1) The induced map $\operatorname{Spec}(\varphi) : \operatorname{Spec} A \longrightarrow X_S$ is bijective.
(2) The induced map $\operatorname{Sper}(\varphi)$ yields a homeomorphism between

$$\{\alpha \in \operatorname{Sper} A \mid P \subset \alpha\} \text{ and } \{\alpha \in \operatorname{Sper} R[S] \mid P[S] \subset \alpha\}.$$

(3) For every prime ideal $\wp \in \operatorname{Spec} A$ the homomorphism $k(\varphi^{-1}(\wp)) \longrightarrow k(\wp)$ of residue fields is an isomorphism.

This characterization indicates that completely real extensions might be of interest for semialgebraic geometry. On the other hand we have to mention that there does not exist—up to isomorphism—a unique ‘‘minimal’’ completely real extension of $(R[S], P[S])$. To be precise, there exists for every $n \in \mathbb{N}$ a "distinguished" completely real extension

$$\varphi_n : (R[S], P[S]) \longrightarrow (A_n, P_n)$$

and for $n \geq 2$ these extensions satisfy (1)–(3). If $\dim(S) \geq 2$ the rings (A_n, P_n) form a strictly decreasing sequence of completely real rings whose intersection is not completely real.

The results of Sec. 3 show that this fact is due to basic properties of certain monoreflective subcategories of the category of partially ordered rings. In the first two sections preliminary results on completely real rings and extensions are stated. In Sec. 4 and 5 the completely real extensions mentioned above are constructed and their properties are investigated. In the final section we return to the geometric situation. We show that the rings we have obtained so far are in fact of interest for semialgebraic geometry. A more detailed study of these rings in the framework of semialgebraic geometry will be done in a forthcoming paper.

1. COMPLETELY REAL RINGS

Throughout this paper let A be a reduced commmutative ring with 1 and $P \subset A$ a partial order, i.e.,

$$P + P \subset P,\ P \cdot P \subset P,\ A^2 \subset P \text{ and } P \cap -P = (0).$$

Recall that a prime ideal $\wp \subset A$ is P-convex if for all $p, q \in P$ we have: $p + q \in \wp \Rightarrow p, q \in \wp$. Given a partial order $P \subset A$ we let

$$\operatorname{Spec}(A, P) \subset \operatorname{Spec} A$$

denote the subspace of the P-convex prime ideals of A.

Definition 1.1. *Let $P \subset A$ be a partial order. Then (A, P) is called completely real if* $\operatorname{Spec} A = \operatorname{Spec}(A, P)$.

For convenience we first state without proof some simple characterizations of completely real rings that will be used throughout this paper. Given a partial order $P \subset A$ and $\wp \in \operatorname{Spec}(A, P)$ we let $P/\wp \subset A/\wp$ denote the image of P with respect to the canonical epimorphism $A \longrightarrow A/\wp$. Since $\wp$ is P-convex, $P/\wp$ is a partial order of $A/\wp$. Next let $S \subset A$ be a multiplicative subset. As usual we set

$$P_S = \left\{ \frac{p}{s^2} \middle| p \in P,\ s \in S \right\} \subset A_S.$$

Note that P_S is a partial order of A_S (see [4, Proposition 3.1.1]). From basic properties of convex ideals and [4, Proposition 3.4.1] one infers

Lemma 1.2. *Let $P \subset A$ be a partial order. Then the following statements are equivalent:*

(1) *(A, P) is completely real.*
(2) *$(A/\wp, P/\wp)$ is completely real for all $\wp \in$ Spec A.*
(3) *(A_S, P_S) is completely real for all multiplicative subsets $S \subset A$.*

Given a partial order $P \subset A$ and $a, b \in A$ we set

$$a \leq_P b :\Leftrightarrow a - b \in P.$$

If no confusion can arise we will often just write $\leq$.

Lemma 1.3. *Let $P \subset A$ be a partial order. Then the following statements are equivalent:*

(1) *(A, P) is completely real.*
(2) *Given $a, b \in A$ with $0 \leq a \leq b$, there exists $k \in \mathbb{N}$ with $a^k \in (b)$.*
(3) *Given $a, b \in A$ with $0 \leq a \leq b$, there exists $k \in \mathbb{N}$ with $a^k \in b \cdot \sqrt{(b)}$.*

Let (A, P) be completely real, $0 \leq a \leq b \in A$ and $c \in A$ with $a^k = cb$. Then in general c is not uniquely determined. Therefore the next fact shows that in order to construct a completely real hull of a partially orderd ring it is more convenient to work with condition (3).

Lemma 1.4. *Let (A, P) be completely real. Given $0 \leq a \leq b \in A$ and $k \in \mathbb{N}$ with $a^k \in b \cdot \sqrt{(b)}$, there exists a unique $c \in \sqrt{(b)}$ with $a^k = c \cdot b$.*

Proof. Let $c_1, c_2 \in \sqrt{(b)}$ with $a^k = c_1 b = c_2 b$ and let $\wp \in$ Spec A. If $b \in \wp$, then $c_1 - c_2 \in \wp$, since $c_1, c_2 \in \sqrt{(b)}$. If $b \notin \wp$, then $c_1 - c_2 \in \wp$, as $b(c_1 - c_2) = 0$. By assumption A is reduced. Hence $c_1 = c_2$. □

Next we give some examples of completely real rings. Most of them will be used later on.

Examples 1.5. For simplicity we make the following convention. Given a real closed field R, a set M and any ring $A \subset \mathrm{Abb}(M, R)$, we say that A is completely real if (A, A^+) is completely real where

$A^+ = \{f \in A \mid \forall x \in M : f(x) \geq 0\}$.

(1) Every real valuation ring V is completely real with respect to $\sum V^2$.
(2) Let K be a (formally) real field and let $H(K) \subset K$ be its real holomorphy ring. Then any subring $H \subset K$ which contains $H(K)$ is completely real with respect to $\sum H^2$. For details concerning the real holomorphy ring of a field see e.g., [1] or [10].
(3) Let X be a topological space. We denote by $\mathcal{C}_{\mathbb{R}}(X)$ the ring of $\mathbb{R}$-valued continuous functions on X. Then $\mathcal{C}_{\mathbb{R}}(X)$ is completely real (cf. [5,5.5]).
(4) Let $S \subset R^n$ be a semialgebraic subset. We let $\mathcal{C}(S)$ denote the ring of continuous semialgebraic functions on S. It is well-known that $\mathcal{C}(S)$ is completely real.
(5) Let $U \subset R^n$ be an open semialgebraic set. Let $\mathcal{C}^k(U) \subset \mathcal{C}(U)$ denote the ring of $\mathcal{C}^k$-semialgebraic functions on U. Given $f, g \in \mathcal{C}^k(U)$ with $0 \leq f \leq g$ one readily verifies that the function φ defined by

$$\varphi(u) := \begin{cases} \frac{f^{2k+1}(u)}{g(u)} & u \in U,\, g(u) \neq 0 \\ 0 & u \in U,\, g(u) = 0 \end{cases}$$

is in $\mathcal{C}^k(U)$. Hence $\mathcal{C}^k(U)$ is completely real.
(6) Let $X \subset \operatorname{Sper} A$ be proconstructible. As usual we denote by $\mathcal{C}(X)$ the ring of continuous semialgebraic functions on X and by $P(X) \subset \mathcal{C}(X)$ the partial order of nonnegative functions (see [12]). Then it is well-known that $(\mathcal{C}(X), P(X))$ is completely real.
(7) Let $P \subset A$ be a partial order and let

$$X = \{\alpha \in \operatorname{Sper} A \mid P \subset \alpha\}.$$

Since A is reduced the canonical homomorphism $\varphi : A \longrightarrow \mathcal{C}(X)$ is injective with $\varphi(P) \subset P(X)$. In particular, $(\mathcal{C}(X), P(X))$ is a completely real extension of (A, P). □

Let $\mathcal{L}_{\leq}$ be the extension of the language of rings by a binary relation "$\leq$". Then we may regard partially ordered rings in a canonical way as $\mathcal{L}_{\leq}$-structures. In this context it is natural to ask whether the class of completely real rings is an elementary class of $\mathcal{L}_{\leq}$-structures. In the next step we will show that this fails.

Proposition 1.6. *The class of completely real rings is not elementary.*

Proof. It is sufficient to show that the class of completely real rings is not closed under ultraproducts. Let R be a real closed field and let $k \in \mathbb{N}$. We consider

$$R_k = \mathcal{C}^k(R^2) \text{ and } R_k^+ = \{f \in R_k \mid \forall x \in R^2 : f(x) \geq 0\}.$$

Now let $X, Y \in R_k$ be the coordinate functions on $\mathbb{A}^2$. Then $0 \leq X^2 \leq X^2 + Y^2 \in R_k$, but

$$(*) \quad X^{2k} \notin (X^2 + Y^2) \cdot R_k.$$

Let U be any ultrafilter on $\mathbb{N}$ which contains the cofinite subsets of $\mathbb{N}$. In view of (1.5) (5) it is sufficient to show that the ring

$$(A, P) := \left(\prod_{k \in \mathbb{N}} (R_k, R_k^+)\right) / U$$

is not completely real. Let x, y be the images of the tuples $(X), (Y) \in \prod R_k$ with respect to the canonical projection

$$\pi : \prod_{k \in \mathbb{N}} (R_k, R_k^+) \longrightarrow (A, P).$$

Then $0 \leq_P x^2 \leq_P x^2 + y^2$. Let $n \in \mathbb{N}$. From $(*)$ we infer

$$\{k \in \mathbb{N} \mid k \geq n\} \subset \{k \in \mathbb{N} \mid X^{2n} \notin (X^2 + Y^2) \cdot R_k\} \in U.$$

Hence

$$x^{2n} \notin (x^2 + y^2) \cdot A$$

for all $n \in \mathbb{N}$. Therefore (A, P) is not completely real. $\square$

The proof of the last result shows that in order to get an elementary class $\mathcal{K}$ of completely real rings one has to fix some $n \in \mathbb{N}$ such that for all $(A, P) \in \mathcal{K}$ and all $a, b \in A$ one has

$$0 \leq a \leq b \in A \Rightarrow a^n \in b\sqrt{(b)}.$$

At the beginning of the next section we will see that exactly the same problem occurs when constructing completely real extensions of partially ordered rings. Moreover, it will turn out that indeed one has to deal with the completely real rings just mentioned.

2. COMPLETELY REAL EXTENSIONS

In this short section we state some basic properties of completely real extensions and introduce the notions which will be needed later on. Moreover, so far we have considered arbitrary partial orders of reduced commutative rings. For the sake of simplicity we now confine ourselves to a certain subclass which is important for applications to semialgebraic geometry.

Let (A, P) be partially ordered. Then

$$\mathrm{Sper}\,(A, P) := \{\alpha \in \mathrm{Sper}\, A \mid P \subset \alpha\}$$

is called the real spectrum of (A, P). It is a spectral space as it is closed in $\mathrm{Sper}\, A$. Given an order-preserving homomorphism $\varphi : (A, P) \longrightarrow (\tilde{A}, \tilde{P})$, then φ induces a morphism

$$\mathrm{Sper}\,(\varphi) : \mathrm{Sper}\,(\tilde{A}, \tilde{P}) \longrightarrow \mathrm{Sper}\,(A, P)$$

of spectral spaces. One readily verifies that the real spectrum is a functor from the category of partially ordered rings into the category of spectral spaces. Given a partial order $P \subset A$ we call

$$\mathrm{Sat}(P) = \bigcap_{\alpha \in \mathrm{Sper}\,(A,P)} \alpha$$

the saturated hull of P and we say that P is saturated if $P = \mathrm{Sat}(P)$. The objects of the category $\mathrm{PO/N}$ are the pairs (A, P) with A a reduced commutative ring with 1 and $P \subset A$ a saturated partial order. The morphisms in $\mathrm{PO/N}$ are the order-preserving ring homomorphisms. Finally let $\mathrm{CRR} \subset \mathrm{PO/N}$ denote the full subcategory of the completely real rings in $\mathrm{PO/N}$.

Let $(A, P) \in \mathrm{PO/N}$ and let $\varphi : (A, P) \longrightarrow (\tilde{A}, \tilde{P}) \in \mathrm{CRR}$ be a completely real extension, i.e., φ is a monomorphism. By (1.5)(7) we know that such an extension always exists. Now one might wonder whether $(\tilde{A}, \tilde{P})$ contains a minimal completely real overring of (A, P). But in general this fails.

Example 2.1. Let $\mathcal{C}(\mathbb{R}^2)$ be the ring of continuous semialgebraic functions on $\mathbb{R}^2$ and $A = \mathbb{R}[X, Y] \subset \mathcal{C}(\mathbb{R}^2)$. Then $(\mathcal{C}(\mathbb{R}^2), \mathcal{C}(\mathbb{R}^2)^+)$ is a completely real extension of (A, A^+). For $k \in \mathbb{N}$ let $\mathcal{C}^k(\mathbb{R}^2)$ denote the ring of $\mathcal{C}^k$-semialgebraic functions. By (1.5)(5) the rings $\mathcal{C}^k(\mathbb{R}^2)$ are completely real. But

$$\bigcap_{k\in\mathbb{N}} \mathcal{C}^k(\mathbb{R}^2) = \mathcal{N}(\mathbb{R}^2),$$

where $\mathcal{N}(\mathbb{R}^2)$ denotes the ring of Nash functions on $\mathbb{R}^2$ which is not completely real. □

The last example as well as the proof of (1.6) suggest to introduce the following notion.

Definition 2.2. *Let $n \in \mathbb{N}$. A partially ordered ring (A, P) is called completely real of exponent n if for all $0 \le a \le b$ there is $c \in \sqrt{(b)}$ with $a^n = c \cdot b$.*

Remark 2.3. For a similar condition in the context of *f*-rings see.[7] □

Given $n \in \mathbb{N}$ we let $\mathrm{CRR}_n \subset \mathrm{CRR}$ denote the full subcategory of the completely real rings of exponent n in CRR. We will need the fact that the partially ordered rings $(A, P) \in \mathrm{CRR}_n$ satisfy the following sharpening of (1.4) which becomes false if the partial order P is not saturated.

Lemma 2.4. *Let $(A, P) \in \mathrm{CRR}_n$ and $0 \le a \le b$. Then the unique element $c \in \sqrt{(b)}$ with $a^n = c \cdot b$ satisfies $0 \le c \le b^{n-1}$ and $c^n \in (b^{n-1})$.*

Proof. We first show $b^{n-1} - c \in P$. From $b - a \in P$ we deduce

$$b^n - a^n = (b - a)(b^{n-1} + b^{n-1}a + \cdots + a^{n-1}) \in P.$$

Let $\alpha \in \mathrm{Sper}\,(A, P)$. If $b \notin \mathrm{supp}(\alpha)$, then obviously $b^{n-1} - c \in \alpha$. If $b \in \mathrm{supp}(\alpha)$, then $c \in \sqrt{(b)}$ implies again $b^{n-1} - c \in \alpha$. Hence $b^{n-1} - c \in P$, as P is saturated. Similarily one deduces from $a^n = cb \in P$ and $c \in \sqrt{(b)}$ that $c \in P$. Thus $0 \le c \le b^{n-1}$. Since $(A, P) \in \mathrm{CRR}_n$ we finally get $c^n \in (b^{n-1})$. □

It is now easy to show:

Corollary 2.5. *The class of completely real rings of exponent n is elementary.*

Proof. It is well-known that the class Ob(PO/N) of the objects of the category PO/N is elementary. Let Σ be any set of $\mathcal{L}_{\ge}$-sentences with $\mathrm{Ob(PO/N)} = \mathrm{Mod}(\Sigma)$. Let

$$\Phi_n := \forall a \forall b \exists c \exists d [0 \le a \le b \Rightarrow (a^n = cb \wedge c^n = db^{n-1})]$$

and $\Sigma_n := \Sigma \cup \{\Phi_n\}$. Then (2.4) shows that $\mathrm{Mod}(\Sigma_n)$ is the class of the completely real rings of exponent n. □

As a second application of (2.4) we show that in contrast to (2.1) completely real extensions of exponent n have the following property:

Lemma 2.6. *Let $(A, P) \in \mathrm{PO/N}$ and let $\varphi : (A, P) \longrightarrow (\tilde{A}, \tilde{P}) \in \mathrm{CRR}_n$ be a monomorphism. Then $(\tilde{A}, \tilde{P})$ contains a minimal completely real extension of exponent n of (A, P).*

Proof. Given any subring $B \subset \tilde{A}$ we set $B^+ = B \cap \tilde{P}$ and we let $\mathcal{M}$ denote the set of pairs (B, B^+) such that B is an intermediate ring $\varphi(A) \subset B \subset \tilde{A}$ with $(B, B^+) \in \mathrm{CRR}_n$. Finally let

$$(A_n, A_n^+) := \bigcap_{(B,B^+)\in\mathcal{M}} (B, B^+).$$

Let $a, b \in A_n$ with $0 \le a \le b$. Given any $(B, B^+) \in \mathcal{M}$ there exists by (1.4) a unique $c_B \in \sqrt{b \cdot B}$ with $a^n = c_B \cdot b$. Moreover, the uniqueness of $c_{\tilde{A}}$ implies $c_B = c_{\tilde{A}}$. Hence $c := c_{\tilde{A}} \in A_n$. Moreover, by (2.4) we know $0 \le c \le b^{n-1}$. Let $d \in \sqrt{b^{n-1}\tilde{A}}$ with $c^n = db^{n-1}$. Then $d \in A_n$, as we have just seen. Hence $c \in \sqrt{bA_n}$ which shows $(A_n, A_n^+) \in \mathrm{CRR}_n$. □

The next consequence of (2.4) will turn out to be useful in the context of geometry.

Corollary 2.7. *Let $(A, P) \in \mathrm{CRR}_n$ and let $0 \le a \le b$. Given $1 < q \in \mathbb{Q}$ there are $k, l \in \mathbb{N}$ and $c \in \sqrt{(b)}$ with*

$$a^k = cb^l \text{ and } q = \frac{k}{l}.$$

Proof. The claim is obvious for $n = 1$. So assume $n \ge 2$. We first show by induction on t that for all $t \in \mathbb{N}$ there exists $d_t \in A$ satisfying

$$a^{n^t + tn^{t-1}} = d_t \cdot b^{tn^{t-1}} \text{ and } 0 \le d_t \le b^{n^t}.$$

Since $0 \le a \le b$ there is $c_1 \in A$ with

$$a^n = c_1 \cdot b \text{ and } 0 \le c_1 \le b^{n-1}.$$

Let $d_1 := ac_1$. Then $a^{n+1} = d_1 b$. Moreover, $0 \le a \le b$ and $0 \le c_1 \le b^{n-1}$ imply $0 \le d_1 = ac_1 \le bc_1 \le b^n$. Next assume that we already have found d_t for some $t \in \mathbb{N}$. Since $0 \le d_t \le b^{n^t}$ there is $c_{t+1} \in A$ with

$$d_t^n = c_{t+1} b^{n^t} \text{ and } 0 \le c_{t+1} \le b^{n^{t+1} - n^t}.$$

Let $d_{t+1} := c_{t+1} a^{n^t}$. Then $0 \le d_{t+1} \le b^{n^{t+1}}$ and

$$a^{n^{t+1} + (t+1)n^t} = (a^{n^t + tn^{t-1}})^n a^{n^t} = d_t^n a^{n^t} b^{tn^t} = d_{t+1} b^{(t+1)n^t}$$

which shows that the elements d_t exist. Now the claim follows from

$$\lim_{t \to \infty} \frac{n^t + tn^{t-1}}{tn^{t-1}} = 1. \qquad \square$$

Let $\varphi : (A, P) \longrightarrow (\tilde{A}, \tilde{P})$ be a completely real extension. Then φ induces a morphism

$$\mathrm{Spec}(\varphi) : \mathrm{Spec}\,\tilde{A} \longrightarrow \mathrm{Spec}(A, P).$$

As mentioned in the introduction we are interested in those extensions which preserve the geometric structure of $\mathrm{Spec}(A, P)$. Therefore one might wonder whether there always exists a completely real extension $\varphi : (A, P) \longrightarrow (\tilde{A}, \tilde{P})$ such that $\mathrm{Spec}\,(\varphi)$ is an isomorphism of spectral spaces. In the next step we show that this fails.

Example 2.8. Let $A = \mathbb{R}[X, Y]$, $P \subset A$ the saturated hull of $\sum A^2$, i.e., P is the set of positive semidefinite polynomials on $\mathbb{R}^2$. Let $\varphi : (A, P) \longrightarrow (\tilde{A}, \tilde{P})$ be a completely real extension. We claim that the induced morphism

$$\mathrm{Spec}\,(\varphi) : \mathrm{Spec}\,\tilde{A} \longrightarrow \mathrm{Spec}(A, P)$$

is not an isomorphism. To this end it is sufficient to show that whenever $\mathrm{Spec}(\varphi)$ is bijective then $\mathrm{Spec}(\varphi)$ is not an isomorphisme. Since $(\tilde{A}, \tilde{P})$ is completely real there exist $n \in \mathbb{N}$ and $c \in \tilde{A}$ with

$$\varphi(X)^{2n} = c \cdot (\varphi(X)^2 + \varphi(Y)^2).$$

In A we have the prime ideals

$$\wp_1 := (X^2 + Y^2 - X^{2n+1}) \subset \wp_2 := (X, Y).$$

Let $\tilde{\wp}_i := \mathrm{Spec}(\varphi)^{-1}(\wp_i)$ and $f = 1 - c \cdot \varphi(X)$. Then $f \notin \tilde{\wp}_2$ since $\varphi(X) \in \tilde{\wp}_2$. On the other hand

$$(\varphi(X)^2 + \varphi(Y)^2) \cdot f = \varphi(X^2 + Y^2 - X^{2n+1}) \in \tilde{\wp}_1.$$

From $X^2 + Y^2 \notin \wp_1$ we conclude $\varphi(X)^2 + \varphi(Y)^2 \notin \tilde{\wp}_1$. Thus $f \in \tilde{\wp}_1$ and $f \notin \tilde{\wp}_2$. Hence $\wp_1 \subset \wp_2$ but $\tilde{\wp}_1 \not\subset \tilde{\wp}_2$ which shows that $\mathrm{Spec}(\varphi)$ is not an isomorphism. □

Remark 2.9. Let (A, P) be as in (2.8). Then we actually have the following stronger fact: given any ring homomorphism $\varphi : A \longrightarrow \tilde{A}$, then the induced map φ^* between the Zariski-spectra is not an isomorphism between $\mathrm{Spec}\tilde{A}$ and $\mathrm{Spec}(A, P)$. Namely, assume that we have found $\varphi : A \longrightarrow \tilde{A}$ such that φ induces a bijection

$$\varphi^* : \mathrm{Spec}\,\tilde{A} \longrightarrow \mathrm{Spec}(A, P).$$

Since $(X, Y) \subset A$ is the unique P-convex prime ideal which contains $X^2 + Y^2$ it follows that

$$\varphi(X) \in \sqrt{(\varphi(X)^2 + \varphi(Y)^2)} \subset \tilde{A}.$$

Now the same argument as in (2.8) proves that φ^* is not an isomorphism. □

Thus, given any partially ordered ring (A, P) we can at most expect that there is a completely real extension $\varphi : (A, P) \longrightarrow (\tilde{A}, \tilde{P})$ such that the induced map

$$\mathrm{Spec}(\varphi) : \mathrm{Spec}\,\tilde{A} \longrightarrow \mathrm{Spec}(A, P)$$

is bijective. In order to get some control on the topology on $\mathrm{Spec}\,\tilde{A}$ one is forced to consider the real spectra as well. So we end up with the fact, that from the geometric point of view at most those completely real extensions $\varphi : (A, P) \longrightarrow (\tilde{A}, \tilde{P})$ are of interest which satisfy:

$$\mathrm{Spec}(\varphi) : \mathrm{Spec}\,\tilde{A} \longrightarrow \mathrm{Spec}(A, P) \text{ is bijective}$$
$$\mathrm{Sper}(\varphi) : \mathrm{Sper}(\tilde{A}, \tilde{P}) \longrightarrow \mathrm{Sper}(A, P) \text{ is an isomorphism of spectral spaces.}$$

In Sec. 5 we will see that such extensions indeed exist.

3. REAL HULL FUNCTORS AND MONOREFLECTORS

In this section we will be concerned with certain functors $\mathcal{F} : \mathrm{PO/N} \longrightarrow \mathrm{CRR}$. Given any functor $\mathcal{F} : \mathrm{PO/N} \longrightarrow \mathrm{PO/N}$ and $(A, P) \in \mathrm{PO/N}$ we set

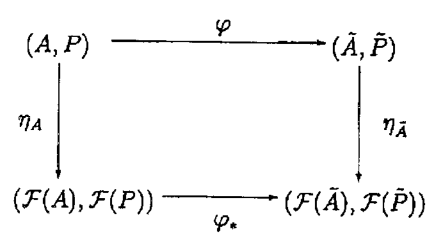
$$(\mathcal{F}(A), \mathcal{F}(P)) := \mathcal{F}((A, P))$$

and we let $\mathcal{E} : \mathrm{CRR} \longrightarrow \mathrm{PO/N}$ denote the inclusion functor. Now let us call a covariant functor $\mathcal{F} : \mathrm{PO/N} \longrightarrow \mathrm{CRR}$ a real hull functor if $\mathcal{F}$ satisfies the following properties.

(1) There exists a functorial morphism $\eta : \mathrm{Id}_{\mathrm{PO/N}} \longrightarrow \mathcal{E}\mathcal{F}$ which is a pointwise monomorphism. Hence, for all $(A, P) \in \mathrm{PO/N}$ there exists a monomorphism

$$\eta_A : (A, P) \longrightarrow (\mathcal{F}(A), \mathcal{F}(P))$$

such that given any morphism $\varphi : (A, P) \longrightarrow (\tilde{A}, \tilde{P})$, then the following diagram commutes:



(2) The functorial morphism $\eta : \mathrm{Id}_{\mathrm{PO/N}} \longrightarrow \mathcal{E}\mathcal{F}$ induces a functorial isomorphism $\mathcal{F} \longrightarrow \mathcal{F} \circ \mathcal{F}$.

The next result shows that real hull functors are related with the subcategories $\mathrm{CRR}_n \subset \mathrm{PO/N}$ introduced in Sec. 2.

Proposition 3.1. *Let* $\mathcal{F} : \mathrm{PO/N} \longrightarrow \mathrm{CRR}$ *be a real hull functor. Then there exists* $n \in \mathbb{N}$ *such that* $\mathrm{Im}(\mathcal{F}) \subset \mathrm{CRR}_n$.

Proof. Given $(B, P_P) \in \mathrm{PO/N}$ let us denote by

$$\eta_B : (B, P_B) \longrightarrow (\mathcal{F}(B), \mathcal{F}(P_B))$$

the morphism induced by the natural transformation $\mathrm{Id}_{\mathrm{PO/N}} \longrightarrow \mathcal{EF}$. Now assume by way of contradiction that for every $n \in \mathbb{N}$ there exists $(B_n, P_{B_n}) \in$ PO/N such that $(\mathcal{F}(B_n), \mathcal{F}(P_{B_n})) \notin \mathrm{CRR}_n$. Let $(A_n, P_n) := (\mathcal{F}(B_n), \mathcal{F}(P_{B_n}))$. Then there exist $0 \leq a_n \leq b_n \in A_n$ such that

$$a_n^n \notin b_n \cdot \sqrt{(b_n)} \subset A_n. \tag{1}$$

Let

$$(A, P) = \prod_{n \in \mathbb{N}} (A_n, P_n) \in \mathrm{PO/N}$$

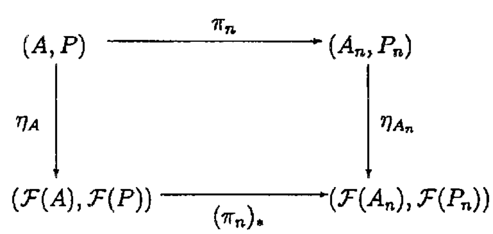
and let $a = (a_n), b = (b_n) \in A$. Then we get in (A, P) the relation $0 \leq a \leq b$. Since $(\mathcal{F}(A), \mathcal{F}(P))$ is completely real and since η_A is injective, there exists $k \in \mathbb{N}$ such that

$$0 \leq \eta_A(a)^k \in (\eta_A(b)) \cdot \mathcal{F}(A). \tag{2}$$

Let $n \geq k + 1$ and let

$$\pi_n : (A, P) \longrightarrow (A_n, P_n)$$

be the canonical projection. We have the following commutative diagram:



Since η_{A_n} is an isomorphism, we have the homomorphism $\varphi = \eta_{A_n}^{-1} \circ (\pi_n)_*$. By (2) there exists $c \in \mathcal{F}(A)$ with

$$\eta_A(a)^k = c \cdot \eta_A(b).$$

Hence we get from the diagram above

$$a_n^k = \pi_n(a^k) = \varphi(\eta_A(a)^k) = \varphi(c \cdot \eta_A(b)) = \varphi(c) \cdot b_n.$$

Consequently,

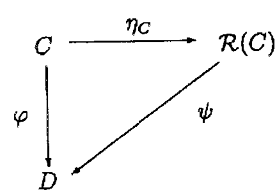
$$a_n^{k+1} \in b_n \cdot \sqrt{(b_n)} \subset A_n$$

which contradicts (1) as $n \geq k+1$. □

In order to investigate real hull functors which are in a certain sense "minimal" we need a further notion. Let $\mathcal{C}$ be a category and let $\mathcal{D}$ be a subcategory of $\mathcal{C}$. Then $\mathcal{D}$ is called a reflective subcategory of $\mathcal{C}$ if there exists a left adjoint $\mathcal{R} : \mathcal{C} \longrightarrow \mathcal{D}$ of the inclusion functor $\mathcal{E}_{\mathcal{D}} : \mathcal{D} \longrightarrow \mathcal{C}$. In this situation $\mathcal{R}$ is called a reflector. Therefore a reflection satisfies the following universal property. Let $C \in \mathcal{C}$ and $D \in \mathcal{D}$. Let us denote by η_C the morphism

$$\eta_C : C \longrightarrow \mathcal{R}(C)$$

which is induced by the functorial morphism $\mathrm{Id}_{\mathcal{C}} \longrightarrow \mathcal{E}\mathcal{R}$. Given any morphism $\varphi : C \longrightarrow D$, there exists a unique morphism $\psi : \mathcal{R}(C) \longrightarrow D$ making the diagram



commutative (see [8,6]). If in addition for all $C \in \mathcal{C}$ the morphism $\eta_C : C \longrightarrow \mathcal{R}(C)$ is a monomorphism, then $\mathcal{R}$ is called a monoreflector and $\mathcal{D}$ a monoreflective subcategory of $\mathcal{C}$.

Lemma 3.2. *Let $\mathcal{C}$ be a full subcategory of* CRR *which is closed under isomorphisms and let $\mathcal{R} : \mathrm{PO/N} \longrightarrow \mathcal{C}$ be a monoreflector. Then $\mathcal{R}$ is a real hull functor.*

Proof. We have to check the properties (1) and (2) of real hull functors. Since $\mathcal{R}$ is left adjoint for the inclusion functor $\mathcal{E} : \mathcal{C} \longrightarrow \mathrm{PO/N}$, there is a functorial morphism $\eta : \mathrm{Id}_{\mathrm{PO/N}} \longrightarrow \mathcal{E}\mathcal{R}$. Now (1) follows from the assumption that $\mathcal{R}$ is a monoreflector and (2) from the assumption that $\mathcal{C}$ is a full subcategory. □

Let us draw a first consequence from this fact.

Corollary 3.3. *Let $\mathcal{D} \subset \mathrm{CRR}$ be a full subcategory which is closed under isomorphisms and assume that $\mathcal{D}$ is a monoreflective subcategory of* PO/N. *Then there exists $n \in \mathbb{N}$ with $\mathcal{D} \subset \mathrm{CRR}_n$.*

Proof. Let $\mathcal{R} : \mathrm{PO/N} \longrightarrow \mathcal{D}$ be a monoreflector. By (3.2), $\mathcal{R}$ is a real hull functor. Now apply (3.1). □

Note that we have the infinite strictly increasing sequence

$$\mathrm{CRR}_1 \subsetneq \mathrm{CRR}_2 \subsetneq \cdots \subsetneq \mathrm{CRR}_n \subsetneq \cdots \subsetneq \mathrm{CRR}$$

of subcategories of CRR. Consequently (3.3) implies

Corollary 3.4. *The subcategory* $\mathrm{CRR} \subset \mathrm{PO/N}$ *is not monoreflective.*

The last results lead to the question whether the subcategories $\mathrm{CRR}_n \subset \mathrm{PO/N}$ are monoreflective. Later on we will see that there is a positive answer.

4. THE COMPLETELY REAL HULL OF EXPONENT 1

It is the goal of this section to show that CRR_1 is a monoreflective subcategory of PO/N. We first state some preliminary results.

Lemma 4.1. *Let (A, P) be completely real of exponent* 1. *Then dim $A = 0$.*

Proof. Let $\wp \in \operatorname{Spec} A$ and let $a \in A \setminus \wp$. Since $0 < a^2 < 2a^2$ and since $(A, P) \in \mathrm{CRR}_1$, there exists $b \in \sqrt{(a)}$ with $a^2 = 2ba^2$. Hence we find $c \in A$ and $n \in \mathbb{N}$ such that $b^n = ca$. From this we get $a^{2n}(1 - 2^n ca) = 0$. By assumption $a \notin \wp$. Hence $1 - 2^n ca \in \wp$ which shows that $a + \wp$ is a unit in $A/\wp$. □

Proposition 4.2. *Let $(A, P) \in \mathrm{PO/N}$. Then the following statements are equivalent:*

(1) *(A, P) is completely real of exponent 1.*
(2) $\operatorname{Spec} A = \operatorname{Spec}(A, P)$ *and A is von Neumann regular.*

Proof. The implication (1) ⇒ (2) follows from the last result. For the converse direction let $0 \leq a \leq b \in A$. By assumption there is $x \in A$ with $b^2 x = b$ and $x^2 b = x$. It is sufficient to show $a = axb$. Let $\wp \in \operatorname{Spec} A$. If $b \in \wp$, then $a \in \wp$, as $\wp$ is P-convex. Hence $a(1 - xb) \in \wp$. If $b \notin \wp$, then $b(bx - 1) = 0$ implies $bx - 1 \in \wp$ and consequently $a(bx - 1) \in \wp$. Hence $a = axb$, as A is reduced. □

The last result shows that a "completely real hull" of a partially ordered ring (A, P) can be regarded as a "von Neumann regular hull" of A with respect to the proconstructible subspace $\mathrm{Spec}(A, P)$. The construction of a corresponding hull with respect to the whole space $\mathrm{Spec}\, A$ can be found in.[9] In addition the methods developed there can be easily generalized to the situation we are concerned with. But for convenience we will use a slightly different approach.

Given $(A, P) \in \mathrm{PO/N}$, let

$$\mathcal{F}(A, P) = \prod_{\wp \in \mathrm{Spec}(A,P)} k(\wp).$$

Given $a \in \mathcal{F}(A, P)$ and $\wp \in \mathrm{Spec}(A, P)$, we let $a(\wp)$ denote the image of a with respect to the projection $\pi_\wp : \mathcal{F}(A, P) \longrightarrow k(\wp)$. In [4, Proposition 2.3.6] it is shown that

$$\sqrt{(0)} = \bigcap_{\wp \in \mathrm{Spec}(A,P)} \wp.$$

Thus, since A is reduced, the canonical homomorphism

$$A \longrightarrow \mathcal{F}(A, P) : a \mapsto \prod_{\wp \in \mathrm{Spec}(A,P)} a(\wp)$$

is injective. Hence we may consider A as a subring of $\mathcal{F}(A, P)$. Given $x \in \mathcal{F}(A, P)$, we set

$$Z(x) = \{\wp \in \mathrm{Spec}(A, P) \mid x(\wp) = 0\}$$

and define $x^{(-1)} \in \mathcal{F}(A, P)$ as follows:

$$x^{(-1)}(\wp) = \begin{cases} x(\wp)^{-1} & \text{if} \quad \wp \notin Z(x) \\ 0 & \text{if} \quad \wp \in Z(x) \end{cases}$$

We call $x^{(-1)}$ the formal inverse of x. Finally we set

$$A_1 := A[a^{(-1)} \mid a \in A]$$

and denote by $P_1 \subset A_1$ the saturated hull of the partial order generated by P.

The following facts are simple consequences of the construction of (A_1, P_1).

Lemma 4.3. *Let* $(A, P) \in \mathrm{PO/N}$, $\wp_1 \in \operatorname{Spec} A_1$, $\wp := \wp_1 \cap A$ *and* $a, b \in A$ *with* $0 \le a \le b$. *Then the following statements hold:*

(1) $b^{(-1)} \in (b)$.
(2) $a = ab^{(-1)}b$
(3) $\wp \in \operatorname{Spec}(A, P)$.
(4) $A_1/\wp_1 = k(\wp)$.
(5) P_1 *and* P *induce the same partial order on* $k(\wp)$.
(6) *The inclusion* $\eta_A : A \longrightarrow A_1$ *is an epimorphism in the category of rings.*

Lemma 4.4. *Let* $(A, P) \in \mathrm{PO/N}$. *Then the inclusion* $\eta_A : A \hookrightarrow A_1$ *induces the following homomorphisms with respect to the constructible topology:*

$$\operatorname{Spec}(\eta_A): \quad \operatorname{Spec} A_1 \longrightarrow \operatorname{Spec}(A, P);$$
$$\operatorname{Sper}(\eta_A): \quad \operatorname{Sper}(A_1, P_1) \longrightarrow \operatorname{Sper}(A, P).$$

Proof. Let $\pi := \operatorname{Spec}(\eta_A)$. By (4.1b) (3) we know $\operatorname{Im}(\pi) \subset \operatorname{Spec}(A, P)$. Since $A_1 \subset \mathcal{F}(A, P)$ it is clear that π is surjective. Since $\eta_A : A \hookrightarrow A_1$ is an epimorphism in the category of rings, π is injective and therefore a homomorphism with respect to the constructible topology. From (4.3)(4) and (5) we now see that

$$\operatorname{Sper}(\eta_A) : \operatorname{Sper}(A_1, P_1) \longrightarrow \operatorname{Sper}(A, P)$$

is bijective as well. □

Corollary 4.5. *Let* $(A, P) \in \mathrm{PO/N}$. *Then* (A_1, P_1) *is completely real of exponent* 1.

Proof. By (4.3) (4), $\dim A_1 = 0$. Hence A_1 is von Neumann regular and (4.4) together with (4.3) (5) shows that (A_1, P_1) is completely real. Now apply (4.2). □

In the next steps we will show that $\eta_A : (A, P) \longrightarrow (A_1, P_1)$ is the monoreflection we are looking for.

Lemma 4.6. *Let* (A, P) *be completely real of exponent* 1. *Then* $A = A_1$.

Proof. $\mathcal{F}(A,P)=\mathcal{F}(A_1,P_1)$, by (4.3) and (4.4). Let $x\in A$. It remains to show $x^{(-1)}\in A$. By (4.2), A is von Neumann regular. Hence there is $y\in A$ with $x^2y=x$ and $y^2x=y$ which means $y=x^{(-1)}$. □

Let $\varphi:(A,P)\longrightarrow(\tilde{A},\tilde{P})$ be a morphism in PO/N. Let

$$\wp\in\operatorname{Spec}(\tilde{A},\tilde{P}) \text{ and } \wp_0=\operatorname{Spec}(\varphi)(\wp).$$

Then φ induces a canonical homomorphism $\psi_\wp:k(\wp_0)\longrightarrow k(\wp)$. Let us denote by $\pi_{\wp_0}:A_1\longrightarrow k(\wp_0)$ the restriction of the canonical projection $\mathcal{F}(A,P)\longrightarrow k(\wp_0)$. Then we get a homomorphism

$$\varphi_\wp:=\psi_\wp\circ\pi_{\wp_0}:A_1\longrightarrow k(\wp).$$

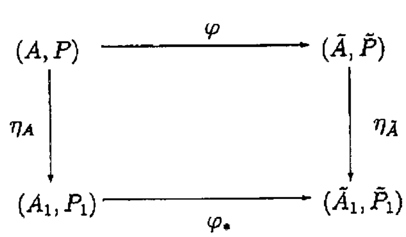
Putting these maps together we obtain a homomorphism

$$\varphi_*=\prod_{\wp\in\operatorname{Spec}(\tilde{A},\tilde{P})}\varphi_\wp:A_1\longrightarrow\mathcal{F}(\tilde{A},\tilde{P}).$$

From the construction of the rings $A_1,\tilde{A}_1$ we see $\operatorname{Im}(\varphi_*)\subset\tilde{A}_1$. Thus we have found a ring homomorphism

$$\varphi_*:A_1\longrightarrow\tilde{A}_1.$$

Moreover, since φ is order-preserving, the definition of the partial orders $P_1,\tilde{P}_1$ shows that φ_* is order-preserving as well. Finally note that the following diagram commutes:



Now consider

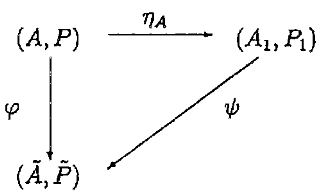
$$\mathcal{R}_1:\text{PO/N}\longrightarrow\text{CRR}_1:(A,P)\longrightarrow(A_1P_1).$$

Together with (4.6) the facts just mentioned show that $\mathcal{R}_1$ is a real hull functor.

For the sake of notational simplicity we will still write A_1 instead of $\mathcal{R}(A, P)$ or $(\mathcal{R}(A), \mathcal{R}(P))$. We are now prepared to show that $\mathcal{R}_1$ is a monoreflection.

Theorem 4.7. *The functor* $\mathcal{R}_1 : \mathrm{PO/N} \longrightarrow \mathrm{CRR}_1$ *is a monoreflector.*

Proof. We have to show that for any morphism $\varphi : (A, P) \longrightarrow (\tilde{A}, \tilde{P}) \in \mathrm{CRR}_1$ there exists a unique morphism $\psi : (A_1 P_1) \longrightarrow (\tilde{A}, \tilde{P})$ such that the following diagram commutes:



Since $\mathcal{R}_1$ is a real hull functor, (4.6) implies the existence of ψ. By (4.3) η_A is an epimorphism in the category of rings. Hence ψ is unique. $\square$

We conclude this section with an alternative description of the ring A_1 which is of interest from the geometric point of view.

We call $x \in \mathcal{F}(A, P)$ piecewise rational if there are elements $a_1, b_1, \ldots, a_n, b_n$ in A and a finite covering

$$\mathrm{Spec}(A, P) = C_1 \cup \cdots \cup C_n$$

by constructible sets $C_i \subset \mathrm{Spec}(A, P)$ with $C_i \subset D(b_i)$ such that for all $i \in \{1, \ldots, n\}$ and all $\wp \in C_i$ we have

$$x(\wp) = \frac{a_i(\wp)}{b_i(\wp)}.$$

Then the piecewise rational elements of $\mathcal{F}(A, P)$ form a subring containing A. Note that the elements of A_1 are piecewise rational. We even have

Proposition 4.8. *Let* $(A, P) \in \mathrm{PO/N}$. *Then* $A_1 \subset \mathcal{F}(A, P)$ *is the subring of the piecewise rational elements.*

Proof. Obviously it is sufficient to show that for every constructible subset $C \subset \mathrm{Spec}(A, P)$ the element $1_C \in \mathcal{F}(A, P)$ defined by

$$1_C(\wp) = \begin{cases} 1 & \text{if} \quad \wp \in C \\ 0 & \text{if} \quad \wp \notin C \end{cases}$$

belongs to A_1. First let

$$C = V(a) = \{\wp \in \operatorname{Spec}(A, P) \mid a \in \wp\}, a \in A.$$

Since A_1 is von Neumann regular, there is an idempotent $e \in A_1$ such that the ideal $(a) \subset A_1$ is generated by e. Hence $1_C = 1 - e \in A_1$. Now let $C \subset \operatorname{Spec}(A, P)$ be an arbitrary constructible subset. Then C is an element of the Boolean lattice generated by the sets $V(a)$, $a \in A$. Hence $1_C \in A_1$. □

5. COMPLETELY REAL HULLS OF EXPONENT n

Let $n \in \mathbb{N}$ be a fixed natural number with $n > 1$. In this section we will show that CRR_n is a monoreflective subcategory of PO/N. To this end we first associate with (A, P) a certain partially ordered subring $(A_n, P_n) \subset (A_1, P_1)$.

Let $(A, P) \in$ PO/N. Given any intermediate ring

$$A \subset B \subset A_1$$

we set

$$\operatorname{Frac}(B) = \{(a, b) \in B \times B \mid 0 \leq a \leq b\},$$

where "$\leq$" denotes the partial order relation induced by P_1 We define the ring A_n inductively. Let $(A^1, P^1) := (A, P)$. If (A^k, P^k) has already been defined let

$$A^{k+1} := A^k[a^n \cdot b^{(-1)} \mid (a, b) \in \operatorname{Frac}(A^k)] \subset A_1.$$

Furthermore we set $P^{k+1} = A^{k+1} \cap P_1$. Now let

$$(A_n, P_n) := \bigcup_{k \in \mathbb{N}} (A^k, P^k)$$

Finally we let $\eta_A^n : A_n \longrightarrow A_1$ denote the inclusion.

Lemma 5.1. *Given* $(A, P) \in \mathrm{PO/N}$*, then* (A_n, P_n) *is a completely real ring of exponent* n.

Proof. Let $0 \leq a \leq b \in A_n$. Then there is some $k \in \mathbb{N}$ with $a, b \in A^k$. From the construction of (A_n, P_n) we get $c := a^n \cdot b^{(-1)} \in A^{k+1} \subset A_n$. Note that $a^n = c \cdot b$. Hence it remains to show $c \in \sqrt{bA_n}$. By (4.5) (A_1, P_1) is completely real of exponent 1. In particular, $(A_1, P_1) \in \mathrm{CRR}_n$. Hence we know by (2.4) that in (A_1, P_1) we have $0 \leq c \leq b^{n-1}$. Consequently

$$d := c^n \cdot (b^{(-1)})^{n-1} \in A^{k+2} \subset A_n$$

which shows $c^n = d \cdot b^{n-1} \in bA_n$. □

Therefore we have an assignment

$$\mathcal{R}_n : \mathrm{PO/N} \longrightarrow \mathrm{CRR}_n : (A, P) \longrightarrow (A_n, P_n).$$

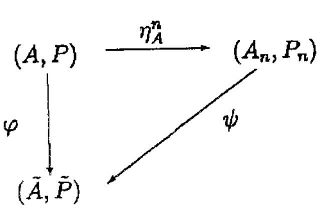
As before we will still write (A_n, P_n) rather than $\mathcal{R}_n(A, P)$ or $(\mathcal{R}_n(A), \mathcal{R}_n(P))$.

Proposition 5.2. $\mathcal{R}_n : \mathrm{PO/N} \longrightarrow \mathrm{CRR}_n$ *is a monoreflector.*

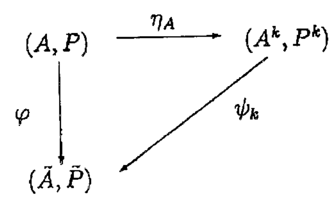
Proof. Let $(A, P) \in \mathrm{PO/N}$. We claim that

$$\eta : (A, P) \longrightarrow (A_n, P_n)$$

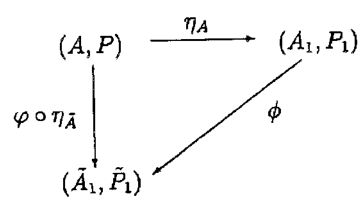
is a monoreflection of (A, P) in CRR_n. Thus, given $(\tilde{A}, \tilde{P}) \in \mathrm{CRR}_n$ and $\varphi : (A, P) \longrightarrow (\tilde{A}, \tilde{P})$ we have to show that there exists a unique morphism $\psi : (A_n, P_n) \longrightarrow (\tilde{A}, \tilde{P})$ such that the following diagram commutes:



To this end it is sufficient to prove that for all $k \in \mathbb{N}$ there exists a unique morphism $\psi_k : (A^k, P^k) \longrightarrow (\tilde{A}, \tilde{P})$ such that the following diagram commutes:



By (4.7) there exists a unique morphism $\phi : (A_1, P_1) \longrightarrow (\tilde{A}_1, \tilde{P}_1)$ such that the diagram



commutes. Let ψ_k be the restriction of ϕ to $A^k \subset A_1$. We prove by induction on k that ψ_k is the morphism we are looking for. Since $(A^1, P^1) = (A, P)$ we may assume that this has been proven for some $k - 1 \geq 1$. So let $(a, b) \in \mathrm{Frac}(A^{k-1})$. Then $0 \leq \psi_k(a) \leq \psi_k(b)$ as ψ_k is order-preserving. By (1.4) we know that there are unique elements

$$c \in \sqrt{bA_1} \text{ and } \tilde{c} \in \sqrt{\psi_k(b)\tilde{A}_1}$$

with $a^n = c \cdot b$ and $\psi_k(a)^n = \tilde{c} \cdot \psi_k(b)$. Hence necessarily

$$(*) \quad \psi_k(c) = \tilde{c}.$$

By assumption $(\tilde{A}, \tilde{P})$ is completely real of exponent n. Hence $\tilde{c} \in \tilde{A}$ by (1.4). Moreover, we know $c = a^n \cdot b^{(-1)}$. Hence the definition of A^k shows $\psi_k(A^k) \subset \tilde{A}$ and finally $(*)$ implies the uniqueness of ψ_k. □

So far we have seen that we have the infinite strictly increasing sequence

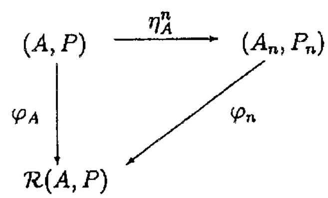
$$\mathrm{CRR}_1 \subsetneq \mathrm{CRR}_2 \subsetneq \cdots \subsetneq \mathrm{CRR}_n \subsetneq \cdots \subsetneq \mathrm{PO/N}$$

of the monoreflective subcategories $\mathrm{CRR}_n \subset \mathrm{PO/N}$. Thus, given $(A, P) \in \mathrm{PO/N}$ we obtain the infinite sequence

$$(A, P) \subset \cdots \subset (A_n, P_n) \subset \cdots \subset (A_2, P_2) \subset (A_1, P_1)$$

of completely real extensions of (A, P). In general, the intersection of the partially ordered rings (A_n, P_n) is not completely real. In particular, so far we have not found a unique "minimal" completely real extension of (A, P). In the next step we show that in the context of monoreflective subcategories such an extension does not exist.

Let $\mathcal{D} \subset$ CRR be a full subcategory. Assume that $\mathcal{D}$ is a monoreflective subcategory of PO/N with associated monoreflector $\mathcal{R} : \text{PO/N} \longrightarrow \mathcal{D}$. By (3.3) there exists $n_0 \in \mathbb{N}$ with $\mathcal{D} \subset \text{CRR}_{n_0}$. Hence $\mathcal{D} \subset \text{CRR}_n$ for all $n \geq n_0$. Let $(A, P) \in$ PO/N and let $\varphi_A : (A, P) \longrightarrow \mathcal{R}(A, P)$ be its monoreflection in $\mathcal{D}$. Let $n \geq n_0$. Since $\mathcal{R}_n$ is a monoreflector as well, there exists a unique morphism $\varphi_n : (A_n, P_n) \longrightarrow \mathcal{R}(A, P)$ such that the following diagram commutes:



Hence we have the following sharpening of (3.3):

Corollary 5.3. *Let $\mathcal{D} \subset$ CRR be a full subcategory. Assume that $\mathcal{R}$: PO/N $\longrightarrow \mathcal{D}$ is a monoreflector. Then there is some $n_o \in \mathbb{N}$ such that for all $n \geq n_0$ there is a unique A-embedding $\varphi_n : (A_n, P_n) \longrightarrow \mathcal{R}(A, P)$.*

This result shows that the rings (A_n, P_n) are indeed distinguished completely real extensions of (A, P).

In the next step we consider the space Spec A_n.

Corollary 5.4. *Let $(A, P) \in$ PO/N and let $n \geq 2$. Then the inclusion η_A^n : $A \longrightarrow \mathcal{A}_n$ induces a bijection*

$$\text{Spec}\, \mathcal{A}_n \longrightarrow \text{Spec}(A, P).$$

Proof. Since $\mathcal{R}_1$ and $\mathcal{R}_n$ are monoreflectors, $\mathcal{R}_1 \circ \mathcal{R}_n : \text{PO/N} \longrightarrow \text{CRR}_1$ is a monoreflector as well. Hence $\mathcal{R}_1(A_n, P_n) \cong (A_1, P_1)$. Now the claim follows from (4.4). □

Let $(A, P) \in$ PO/N. Given $\wp \in \text{Spec}(A, P)$ we let $\tilde{\wp} \subset A_n$ denote the prime ideal lying above $\wp$. By the construction of the ring A_n we know

$$A_n/\tilde{\wp} \subset k(\wp) := \mathrm{quot}(A/\wp).$$

We shall give a slightly more precise description. Let $P(\wp) \subset k(\wp)$ denote the partial order induced by P. Given a total order $Q \subset k(\wp)$ we let $C(Q) \subset k(\wp)$ denote the convex hull of $A/\wp$ in $(k(\wp), Q)$. Now let

$$(*) \quad H(A,\wp) := \bigcap_{Q \in \mathrm{Sper}(k(\wp),P(\wp))} C(Q).$$

Then $H(A,\wp)$ is a relative real holomorphy ring of $k(\wp)$ (see [1,3]). Using the compactness of $\mathrm{Sper}(k(\wp), P(\wp))$ one readily verifies

$$H(A,\wp) = \{x \in k(\wp) \mid \exists\, a \in A/\wp : \; a \pm x \in P(\wp)\}.$$

Moreover, from $(*)$ and the construction of the ring A_n one easily sees.

Corollary 5.5. *Let* $(A,P) \in \mathrm{PO/N}$ *and* $n \geq 2$. *Given* $\wp \in \mathrm{Spec}(A,P)$ *then* $A_n/\tilde{\wp} \subset H(A,\wp)$.

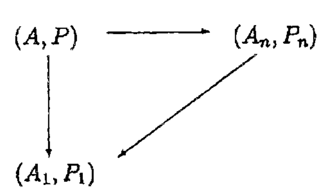
In the final step we consider the real spectra $\mathrm{Sper}\,(A_n, P_n)$ and $\mathrm{Sper}\,(A,P)$. The next result is crucial for applications of the completely real extensions constructed so far in the framework of geometry.

Theorem 5.6. *Given* $(A,P) \in \mathrm{PO/N}$ *and* $n \geq 2$, *the inclusion* $\eta_A^n : A \longrightarrow A_n$ *induces an isomorphism*

$$\mathrm{Sper}(\eta_A^n) : \mathrm{Sper}(A_n, P_n) \longrightarrow \mathrm{Sper}(A,P)$$

of spectral spaces.

Proof. Let $\pi_n := \mathrm{Spec}\,(\eta_A^n)$. We first show that π_n is bijective. Let us consider the commutative diagram



with the obvious homomorphisms. From this we obtain the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Sper}(A_1, P_1) & \xrightarrow{\pi_1} & \mathrm{Sper}(A_n, P_n) \\ {\scriptstyle \pi_A}\downarrow & \swarrow {\scriptstyle \pi_n} & \\ \mathrm{Sper}(A, P) & & \end{array}$$

Since $\mathcal{R}_1(A_{nx}) \cong A_1$, π_A and π_1 are bijective by (4.4). From this we see that π_n is bijective as well. Let $\mathcal{C}(A,P)$ denote the ring of continuous semialgebraic functions on Spec(A,P). Since $(\mathcal{C}(A,P), \mathcal{C}(A,P)^2)$ is completely real of exponent 2 there exists a unique morphism

$$e_n : (A_n, P_n) \longrightarrow (\mathcal{C}(A,P), \mathcal{C}(A,P)^2)$$

such that the following diagram commutes

$$\begin{array}{ccc} (A, P) & \longrightarrow & (A_n, P_n) \\ {\scriptstyle i}\downarrow & \swarrow {\scriptstyle e_n} & \\ (\mathcal{C}(A,P), \mathcal{C}(A,P)^2) & & \end{array}$$

where i denotes the inclusion. From this we get a commutative diagram of spectral spaces.

$$\begin{array}{ccc} \mathrm{Sper}\,\mathcal{C}(A, P) & \xrightarrow{e_n^*} & \mathrm{Sper}(A_n, P_n) \\ {\scriptstyle i^*}\downarrow & \swarrow {\scriptstyle \pi_n} & \\ \mathrm{Sper}(A, P) & & \end{array}$$

By [12, Sec. I.3] we know that i^* is an isomorphism of spectral spaces and we already have seen that π_n is bijective. Hence the commutativity of the diagram shows that π_n is an isomorphism of spectral spaces. □

6. COMPLETELY REAL RINGS AND SEMIALGEBRAIC GEOMETRY

In this final section we will briefly indicate that the rings we have constructed so far are in fact of interest for semialgebraic geometry.

A detailed study of these rings in the framework of geometry will be done in a forthcoming paper.

Throughout this section let R be a real closed field and $S \subset R^m$ a closed semialgebraic subset. We let $R[S]$ denote the ring of R-valued polynomial functions on S, $\tilde{S} \subset \operatorname{Sper} R[S]$ the constructible subset corresponding to S and

$$P[S] := \{f \in R[S] \mid \forall\, x \in S : f(x) \geq 0\}.$$

For the rest of this section we fix some $n \geq 2$. We denote by $(R_n[S], P_n[S])$ the completely real hull of exponent n of $(R[S], P[S])$ and by

$$\eta : (R[S], P[S]) \longrightarrow (R_n[S], P_n[S])$$

the canonical embedding. Since the ring $(\mathcal{C}(S), \mathcal{C}(S)^+)$ of continuous semialgebraic functions on S is completely real of exponent n there is a unique $R[S]$-embedding $R_n[S] \longrightarrow \mathcal{C}(S)$. Therefore we will regard the elements of $R_n[S]$ as semialgebraic funtions on S. By the last section we know:

(1) $\operatorname{Spec}(\eta) : \operatorname{Spec} R_n[S] \longrightarrow \operatorname{Spec}(R[S], P[S])$ is bijective.
(2) $\operatorname{Sper}(\eta) : \operatorname{Sper}(R_n[S], P_n[S]) \longrightarrow \operatorname{Sper}(R[S], P[S])$ is an isomorphism of spectral spaces.
(3) $P_n[S] = \{f \in R_n[S] \mid \forall\, x \in S : f(x) \geq 0\}$.
(4) $\dim R[S] = \dim R_n[S$.
(5) For every prime ideal $\wp \in \operatorname{Spec} R_n[S]$ we have $R_n[S]/\wp \subset k(\wp \cap R[S])$.

Given $\wp \in \operatorname{Spec}(R[S], P[S])$ we set

$$\tilde{\wp} = \operatorname{Spec}(\eta)^{-1}(\wp).$$

and

$$Z(\wp) = \{x \in S \mid \forall f \in \wp : f(x) = 0\}$$
$$Z(\tilde{\wp}) = \{x \in S \mid \forall f \in \tilde{\wp} : f(x) = 0\}$$

By (2.8) we know that in general $\operatorname{Spec}(\eta)$ is not an isomorphism. In particular we may have $Z(\wp) \neq Z(\tilde{\wp})$. In a first step, we therefore will have a closer look at the sets $Z(\tilde{\wp})$.

A point $x \in Z(\wp) \subset S$ is called a central point of $Z(\wp)$ if x admits a generalization $\alpha \in \tilde{S} \subset \operatorname{Sper} R[S]$ with $\operatorname{supp}(\alpha) = \wp$. We let $Z(\wp)_c \subset Z(\wp)$ denote the subset of the central points.

Proposition 6.1. *Let $\wp \in \operatorname{Spec}(R[S], P[S])$. Then the following statements hold:*

(1) $Z(\wp)_c \subset Z(\tilde{\wp}) \subset Z(\wp)$.
(2) $Z(\tilde{\wp})$ *is Zariski-dense in* $Z(\wp)$.
(3) $\tilde{\wp} = \{f \in R_n[S] \mid \forall x \in Z(\wp)_c : f(x) = 0\}$.

Proof. Obviously $Z(\tilde{\wp}) \subset Z(\wp)$. So let $x \in Z(\wp)_c$. Then there is some $\alpha \in \tilde{S}$ with $\wp = \operatorname{supp}(\alpha)$ which specializes to x. By (5.6) we know that

$$\operatorname{Sper}(\eta) : \operatorname{Sper}(R_n[S], P_n[S]) \longrightarrow \operatorname{Sper}(R[S], P[S])$$

is an isomorphism of spectral spaces. Let $\tilde{\alpha}, \tilde{x} \in \operatorname{Sper}(R_n[S], P_n[S])$ be the preimages of α, x. Since $\operatorname{Sper}(\eta)$ is an isomorphism we have $\tilde{\alpha} \subset \tilde{x}$. Now $\tilde{\wp} = \tilde{\alpha} \cap -\tilde{\alpha}$ implies $x \in Z(\tilde{\wp})$ which proves (1). Statement (2) now follows from (1) and the fact that $Z(\wp)_c$ is Zariski-dense in $Z(\wp)$. It remains to prove (3). Let $I \subset R_n[S]$ be the vanishing ideal of $Z(\wp)_c$. By (1) we have $\tilde{\wp} \subset I$. Conversely, let $\tilde{\wp}'$ be a minimal prime divisor of I. From (2) we infer $I \cap R[S] = \wp$ which implies $\tilde{\wp}' \cap R[S] = \wp$. Hence $\tilde{\wp} = \tilde{\wp}'$ as $\operatorname{Spec}(\eta)$ is bijective. □

Let us draw a first consequence of the last result. A point $x \in S$ is called central if $x \in Z(\wp)_c$ for some minimal prime ideal $\wp$ of $(R[S], P[S])$ and we say that S is central if all points of S are central. Finally we call S irreducible if $R[S]$ is an integral domain. In this situation we let $R(S)$ denote the quotient field of $R[S]$.

Corollary 6.2. *Assume that S is irreducible and central. Then $R_n[S]$ is an integral domain.*

Proof. Let $\tilde{\wp} \subset R_n[S]$ be the prime ideal lying above (0). Then $S_c \subset Z(\tilde{\wp})$ by (6.1) (1). Hence $Z(\tilde{\wp}) = S$ as S is central. Now the claim follows from (6.1) (3). □

In example (6.5) we will see that the converse of the last result does not hold. But let us first show that statement (1) of (6.1) above is sharp in the sense that we may have $Z(\tilde{\wp}) = Z(\wp)_c$ as well as $Z(\tilde{\wp}) = Z(\wp)$.

Example 6.3. Let $S = R^3$ and let $n \geq 2$. We consider the prime ideal $\wp \subset R[X, Y, Z]$ which is generated by

$$h = Z(X^2 + Y^2) - X^3.$$

We claim $Z(\tilde{\wp}) = Z(\wp)_c$. Since $0 \leq X^2 \leq X^2 + Y^2$ there exist by (2.7) $k, l \in \mathbb{N}$ and $g \in \sqrt{(X^2 + Y^2)} \subset R_n[S]$ with

$$2k = 3l \text{ and } X^{2k} = g \cdot (X^2 + Y^2)^l.$$

Now consider the function $f = Z^l - g \in R_n[S]$. Then

$$f \cdot (X^2 + Y^2)^l = Z^l(X^2 + Y^2)^l - X^{3l} \in \tilde{\wp}.$$

Hence $f \in \tilde{\wp}$. Next let $(x, y, z) \in Z(\wp) \setminus Z(\wp)_c$. Then $g(x, y, z) = 0$ as $g \in \sqrt{(X^2 + Y^2)}$. Hence $f(x, y, z) = z^l \neq 0$ which shows $Z(\tilde{\wp}) = Z(\wp)_c$. □

Example 6.4. Again let $S = R^3$. This time we consider the prime ideal $\wp \subset R[X, Y, Z]$ generated by

$$f = X^3 + ZX^2 - Y^2.$$

We claim $Z(\tilde{\wp}) = Z(\wp)$. First note that $Z(\wp)_c \cap \mathrm{Sing}(Z(\wp))$ is Zariski-dense in the singular locus $\mathrm{Sing}(Z(\wp))$ of the algebraic set $Z(\wp)$. Let $\wp_0 \subset R[X, Y, Z]$ be the prime ideal generated by X and Y. Then $Z(\wp_0) = \mathrm{Sing}(Z(\wp))$. Since $Z(\wp)_c \cap \mathrm{Sing}(Z(\wp))$ is Zariski-dense in $\mathrm{Sing}(Z(\wp))$ there are $\alpha \in Z(\wp)$ and $\beta \in Z(\wp_0)$ such that α specializes to β, $\mathrm{supp}(\alpha) = \wp$ and $\mathrm{supp}(\beta) = \wp_0$. Let $\tilde{\alpha}, \tilde{\beta} \in \mathrm{Sper}\, R_n[S]$ be the preimages of α, β with respect to $\mathrm{Sper}(\eta)$. Since $\mathrm{Sper}(\eta)$ is an isomorphism we get

$$\tilde{\wp} = \mathrm{supp}\,(\tilde{\alpha}) \subset \mathrm{supp}\,(\tilde{\beta}) = \tilde{\wp}_0.$$

Hence $Z(\tilde{\wp}) = Z(\wp)$. □

Next we show that in general the converse direction of (6.2) is false.

Example 6.5. Let

$$S = \{(x, y, z) \in R^3 \mid x^3 + zx^2 - y^2 = 0\}.$$

Then S is irreducible but not central. Let $\tilde{\wp} \subset \mathcal{R}_n[S]$ be the prime ideal lying above $(0) \subset R[S]$. As in the last example we get $Z(\tilde{\wp}) = S$. Hence $\tilde{\wp} = (0)$, i.e., $R_n[S]$ is an integral domain. □

We call $x \in S$ an isolated point if $\{x\}$ is open in S with respect to the strong topology. The next result shows another feature of the completely real hull $(R_n[S], P_n[S]$ of $(R[S], P[S])$.

Proposition 6.6. *Let $x \in S$ and let $\mathfrak{m}_x \subset R[S]$ be its maximal ideal. Then the following statements are equivalent:*

(1) *x is an isolated point of S.*
(2) *$\tilde{\mathfrak{m}}_x$ is a minimal prime ideal of $R_n[S]$.*

Proof. First assume that x is an isolated point of S. Then there is a function $f \in \mathfrak{m}_x$ with $f(y) \geq 1$ for all $y \in S \setminus \{x\}$. Thus

$$0 \leq f \leq f^{n+1}$$

with respect to $P_n[S]$. Hence there is some $g \in R_n[S]$ with $f^n = gf^{n+1}$. Now let $\tilde{\wp} \subset R_n[S]$ with $\tilde{\wp} \neq \tilde{\mathfrak{m}}_x$. Then $f \notin \tilde{\wp}$ by (6.1) (3) and the choice of f. Hence

$$0 = f^n(1 - g \cdot f) \in \tilde{\wp}$$

implies $1 - g \cdot f \in \tilde{\wp}$. But $1 - g \cdot f \notin \tilde{\mathfrak{m}}_x$. Hence $\tilde{\wp} \subsetneq \tilde{\mathfrak{m}}_x$ which shows that $\tilde{\mathfrak{m}}_x$ is a minimal prime ideal. Conversely, assume that $x \in S$ is not isolated. Then there exists a prime ideal $\wp \subset R[S]$ with $\dim \wp \geq 1$ and $x \in Z(\wp)_c$. But then $x \in Z(\tilde{\wp})$ by (6.1) (1). Hence $\wp \subsetneq \tilde{\mathfrak{m}}_x$. □

Let us denote by $\mathrm{Iso}(S) \subset S$ the subset of the isolated points of S. We conclude this paper with a closer look at semialgebraic sets of dimension 1.

Corollary 6.7. *Let S be an irreducible semialgebraic set of dimension 1. Then*

$$\mathrm{card}(\mathrm{Min} - \mathrm{Spec}\, R_n[S]) = 1 + \mathrm{card}(\mathrm{Iso}(S)).$$

Proof. By the last result every $x \in \mathrm{Iso}(S)$ corresponds to a minimal prime ideal of $R_n[S]$. So let $\tilde{\wp} \in \mathrm{Min} - \mathrm{Spec}\, R_n[S]$ be given such that $\tilde{\wp} \cap R[S]$ is not a maximal ideal. Then $\dim Z(\tilde{\wp}) = 1$. But then $\tilde{\wp} \cap R[S] = (0)$ as S is an irreducible of dimension 1. □

As an immediate consequence of the last result we see that a point $x \in S$ of a 1-dimensional semialgebraic set S is central iff its maximal ideal $\mathfrak{m}_x$ has height 1.

Corollary 6.8. *Let S be an irreducible semialgebraic set of dimension* 1. *Then the following statements are equivalent:*

(1) *$R_n[S]$ is an integral domain.*
(2) *S is central.*
(3) $\mathrm{Spec}(\eta) : \mathrm{Spec}\, R_n[S] \longrightarrow \mathrm{Spec}\,(R[S], P[S])$ *is an isomorphism.*

Proof. The equivalence of (1) and (2) is an immediate consequence of (6.7) and obviously (3) implies (1). So assume that $R_n[S]$ is an integral domain. Then we may consider $R_n[S]$ as a subring of the quotient field $R(S)$ of $R[S]$. Let $0 \neq f \in R_n[S]$. Then there are $g, h \in R[S]$ with $f = \frac{g}{h}$. This shows that

$$V(f) = \{\tilde{\wp} \in \text{ Sper } R_n[S] \mid f \in \tilde{\wp}\}$$

is finite. Hence Spec (η) is a closed morphism. Now (3) follows as Spec(η) is bijective. □

In view of (6.8) the next result completely describes the difference between the topologies on Spec $R[S]$ and Spec $R_n[S]$ in the case of 1-dimensional semialgebraic sets.

Corollary 6.9. *Let S be an irreducible semialgebraic set of dimension 1 and let k be the number of isolated points of S. Then*

$$R_n[S] \cong \prod_{i=1}^{k} R \times R_n[S_c].$$

Proof. Let $\tilde{\wp} \subset R_n[S]$ be the prime ideal lying above (0). By the (6.7) and its proof we have

$$(*) \quad R_n[S] \cong \prod_{i=1}^{k} R \times R_n[S]/\tilde{\wp} \text{ and } Z(\tilde{\wp}) = S_c.$$

So it remains to show $R_n[S]/\tilde{\wp} \cong R_n[S_c]$. First note that $R[S] = R[S_c]$ as S is irreducible. Therefore we have a morphism

$$\varphi : (R_n[S], P_n[S]) \longrightarrow (R_n[S_c], P_n[S_c]).$$

From $Z(\tilde{\wp}) = S_c$ we infer $\ker(\varphi) = \tilde{\wp}$. Let $f \in P[S_c]$. By $(*)$ there exists $g \in R_n[S]$ with $g(x) = 0$ for all $x \in \text{Iso}(S)$ and $g|S_c = f$. Hence $g \in P_n[S]$ and $f + \tilde{\wp} = g + \tilde{\wp} \in P_n[S]/\tilde{\wp}$. Hence

$$(R[S_c], P[S_c]) \subset (R_n[S]/\tilde{\wp}, P_n[S]/\tilde{\wp})$$

is a completely real extension of exponent n. Now the claim follows from the fact that $(R_n[S_c], P_n[S_c])$ is a monoreflection of $(R[S_c], P[S_c])$ in CRR_n. □

REFERENCES

1. Becker, E. The Real Holomorphy Ring and Sums of 2*n*th Powers. In: Colliot-Thélène, J.L., Coste, M., Mathé, L., Roy, M.F., Eds.; Géométrie Algébrique Réelle et Formes Quadratiques, Rennes 1981. Lect. Notes Math. 959, 139–181, Berlin Heidelberg, New York, 1982
2. Bochnak, J.; Coste, M.; Roy, M.F. Géométrie algébrique réelle, Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge Bd 12.
3. Berr, R. On Real Holomorphy Rings, In *Real Analytic and Algebraic Geometry, Trento 1992*; Broglia, F. et al., Eds.; Walter de Gruyter: Berlin, 1995.
4. Brumfiel, G.W. Partially Ordered Rings and Sesmi-algebraic Geometry, London Mathematical Society Lecture Notes Series, 1979, *37*.
5. Gillman, L.; Jerison, M. *Rings of Continuous Functions*; New York, Heidelberg, Berlin, 1976.
6. Mac Lane, S. *Categories for the Working Mathematician*; New York, Heidelberg, Berlin, 1971.
7. Larson, S. Convexity Conditions on *f*-Rings. Can. J. Math. **1986**, *38*, 48–64.
8. Mitchell, B. *Theory of Categories*; New York and London, 1965.
9. Olivier, J.-P. Anneau Absoluement Plats Universels et Épimorphismes d'anneaux, Sém. d'algèbre Commutative, P. Samuel, Paris 1967/68.
10. Schülting, H.W. On Real Places of a Field and their Holomorphy Ring. Comm. Alg. **1982**, *10*, 1239–1284.
11. Schwartz, N. *Real Closed Spaces*; Habilitationsschrift: Munich, 1984.
12. Schwartz, N. The Basic Theory of Real Closed Spaces. Amer. Math. Soc. **1989**, *397*.

Received November 1998
Revised February 2001