Prekopa's theorem and Kiselman's minimum principle for plurisubharmonic functions.

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Abstract

Prekopa's theorem says that the integral of a log-concave function with respect to a subset of the variables is still log-concave as a function of the remaining ones. We prove a complex version of this theorem which generalizes Kiselman's minimum principle for plurisubharmonic functions.

§1 Introduction

If $\varphi(x, y)$ is a convex function in $\mathbb{R}^n_x \times \mathbb{R}^m_y$, then the so called *marginal function* of φ

$$\varphi^*(x) = \inf_y \varphi(x, y)$$

is again convex. This fact, known as the minimum principle for convex functions, is the analytic counterpart of the geometrically obvious fact that the projections of convex sets are convex. (To see this, note that

$$\{(x, t); \varphi^*(x) < t\} = \pi_x\{(x, y, t); \varphi(x, y) < t\}.)$$

Prekopa [P] found the following stronger version of the minimum principle.

Theorem 1.1. *Let $\varphi(x, y)$ be convex in $\mathbb{R}^n_x \times \mathbb{R}^m_y$ and define $\tilde{\varphi}(x)$ by*

$$e^{-\tilde{\varphi}(x)} = \int e^{-\varphi(x,y)} dy.$$

Then $\tilde{\varphi}$ is convex.

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Different proofs and generalizations of Prekopa's theorem can be found in [B1], [B2], and [B-L]. Prekopa's theorem implies the minimum principle if we replace φ by $p\varphi + |y|^2$, let p tend to $+\infty$ and use the fact that L^p-norms tend to L^∞-norms as $p \to \infty$.

In another direction, the minimum principle was generalized to plurisubharmonic functions by Kiselman [K1] who proved the following theorem.

Theorem 1.2. *Let $\varphi(z,w)$ be a plurisubharmonic function in $U_z \times V_w \subseteq \mathbb{C}^n_z \times \mathbb{C}^m_w$, where V_w is pseudoconvex.*

1. *Assume that V is a connected Reinhard domain and that φ is independent of arg (w_j), $j = 1, \dots, m$. Then $\varphi^*(z) = \inf_w \varphi(z,w)$ is plurisubharmonic in U_z.*

2. *Assume V is a tube domain*

$$V = X + i\mathbb{R}^m$$

and that φ is independent of $\Im(w_j)$, $j = 1, \dots, m$. Then $\varphi^*(z) = \inf_w \varphi(z,w)$ is plurisubharmonic in U_z.

(The general version of Kiselman's theorem deals with functions defined in certain subdomains of $\mathbb{C}^{n+m}$; see [K1] for the precise formulation.) It is very easy to see that Kiselman's theorem would be false without some extra assumption of the kind that φ only depends on $\Re w_j$ or $|w_j|$; the example in $\mathbb{C}^2$

$$\varphi(z,w) = |z - \bar{w}|^2 - |z|^2$$

is given in [K1]. Kiselman's theorem has important applications in the study of plurisubharmonic functions, in particular to the study of Lelong numbers (see e.g. [K1], [K2], [H]).

The aim of this note is to prove the following theorem which generalizes both Prekopa's and Kiselman's theorems.

Theorem 1.3. *Let $\varphi(z,w)$ be a plurisubharmonic function in $U_z \times V_w \subseteq \mathbb{C}^n_z \times \mathbb{C}^m_w$, where V_w is pseudoconvex.*

1. *Assume that V is a connected Reinhard domain and that φ is independent of arg (w_j), $j = 1, \dots, m$. Define $\tilde{\varphi}$ by*

$$e^{-\tilde{\varphi}(z)} = \int_V e^{-\varphi(z,w)} d\lambda(w) \tag{1}$$

Then $\tilde{\varphi}$ is plurisubharmonic in U. Moreover, the same conclusion holds if we only assume that V and $\varphi(z,w)$ are invariant under the maps $w \to e^{i\theta} w$ for any $\theta \in \mathbb{R}$, provided V contains the origin.

2. *Assume V is a connected tube domain*

$$V = X + i\mathbb{R}^m$$

and that φ is independent of $\Im(w_j)$, $j = 1, \ldots, m$. Define $\tilde{\varphi}$ by

$$e^{-\tilde{\varphi}(z)} = \int_X e^{-\varphi(z,\Re w)} d\lambda(\Re w).$$

Then $\tilde{\varphi}$ is plurisubharmonic in U.

Again, replacing φ by $p\varphi$ for $p \gg 0$, and letting p tend to infinity, we see that Theorem 1.3 implies Kiselman's theorem. Hence Theorem 1 would also be false without some kind of extra assumption on φ, like invariance with respect to translations or rotations. (For this, and for a general discussion of transforms like (1), see also Malliavin-Vauthier [M-V]). Note however that the second invariance assumption in part 1 of the theorem is weaker than the one in Kiselman's theorem.

Just like in the case of the minimum principle of Kiselman there is a version of Theorem 1.3 for subdomains of $\mathbb{C}^{n+m}$ that are not necessarily products. For a general pseudoconvex domain Ω in $\mathbb{C}^{n+m}$ we let

$$\Omega(z) = \{w \in \mathbb{C}^m; (z,w) \in \Omega\}$$

be its vertical section. We shall be concerned with the case when all the sections $\Omega(z)$ have the same invariance properties with respect to translations in the imaginary directions or rotations as before, so that each $\Omega(z)$ is either a circular domain or a tube domain. In case $\Omega(z)$ is a tube domain, we write

$$\Omega(z) = X(z) + i\mathbb{R}^n.$$

We then have:

Theorem 1.4. *Under the assumptions in the previous paragraph, Theorem 1.3 still holds if in the definition of $\tilde{\varphi}$ we replace V by $\Omega(z)$, and X by $X(z)$ respectively. In particular, taking φ constant, it follows that*

$$\log 1/|\Omega(z)|$$

and

$$\log 1/|X(z)|$$

are both plurisubharmonic functions (where $|E|$ is the Lebesgue measure of E).

Our proof of Theorem 1.3 is based on Hörmander's L^2-estimates for the $\bar{\partial}$-operator, and on a partial converse in one variable to these estimates. It can therefore hardly be called elementary, and it is not at all unlikely that Theorem 1.3 can be proved, for instance, along the lines in [B-L] or [K1]. On the other hand, given Hörmander's theorem, we get a quite simple proof of Theorem 1.3, in particular, of Prekopa's and Kiselman's results. Possibly similar techniques could also be used to prove convexity of other functions of interest in Stochastics (see [B2] and the references there for the examples of such problems).

Finally, I would like to thank Christer Borell for drawing my attention to Prekopa's theorem. Thanks also to the referee for a careful reading and constructive criticism of the manuscript.

§2 L^2-estimate for $\bar\partial$

We shall use the following classical L^2-estimate for the $\bar\partial$-equation of Hörmander (cf [H] Lemma 4.4.1).

Theorem 2.1. *Let Ω be a pseudoconvex domain in $\mathbb{C}^n$ and let φ be a strictly plurisubharmonic function of class C^2 in Ω. Let $c > 0$ be a lower bound for the smallest eigenvalue of the complex Hessian $(\varphi_{j,\bar k})$ Then, if f is a $\bar\partial$-closed $(0,1)$-form in Ω, we can solve the equation*

$$\bar\partial u = f$$

with the estimate

$$\int_\Omega |u|^2 e^{-\varphi} \leq 2 \int_\Omega |f|^2/c\, e^{-\varphi}.$$

In one variable, Theorem 2.1 admits a partial converse.

Proposition 2.2. *Let φ be a continuous function in $\Omega \subseteq \mathbb{C}$ and assume that for any $f \in C_c^\infty(\Omega)$ and for any $k \in N$, we can solve the equation*

$$\frac{\partial u}{\partial \bar z} = f$$

with the estimate

$$\int_\Omega |u|^2 e^{-k\varphi} \leq C \int_\Omega |f|^2 e^{-k\varphi}$$

with C independent of k. Then φ is subharmonic.

Proof. We may clearly asume Ω is a disc. It is then easy to see that if h is any harmonic function, $(\varphi - h)$ will have the property described in the Proposition if φ has. (Write $h = \Re H$, H holomorphic and replace f by $e^H f$). It therefore suffices to show that φ satisfies the maximum principle. So assume that Δ is a disc in Ω, that $\varphi < 0$ near $\partial\Delta$, but that $\varphi > 0$ in some open set inside Δ. Choose

f with compact support in that part of Δ where $\varphi > 0$. Taking k large enough , we get a solution to $\partial u / \partial \bar{z} = f$ with $|u| \leq \epsilon$ on $\partial\Delta$. This contradicts

$$\int_{\Delta} f \, d\bar{z} \wedge dz = \int_{\Delta} \bar{\partial} u \wedge dz = \int_{\partial\Delta} u \, dz.$$

$\square$

Finally, in the proof of the second part of Theorem 1.1 we shall have use for the following proposition.

Proposition 2.3. *Let U be a pseudoconvex domain in $\mathbb{C}^n$, and let*

$$V = X + i\mathbb{R}^m$$

be a tube domain. Let φ be plurisubharmonic in $U_z \times V_w$, of class C^2, independent of $\Im w$, and assume $c > 0$ is a lower bound for the complex Hessian of φ. Let f be a $\bar{\partial}$-closed $(0,1)$-form in U_z. Then there is a solution u to $\bar{\partial} u = f$ in U with

$$\int_{U \times X} |u|^2 e^{-\varphi(z, \Re w)} \leq 2 \int_{U \times X} |f|^2 / c \, e^{-\varphi(z, \Re w)}.$$

Proof. For each $R > 0$ we can find a solution u_R to $\bar{\partial} u_R = f$ in

$$U \times V \cap \{|\Im w| < R\} =: W_R$$

with

$$\frac{1}{\sigma_m R^m} \int_{W_R} |u_R|^2 e^{-\varphi} \leq 2 \int_{U \times X} |f|^2 / c \, e^{-\varphi},$$

where σ_m is the volume of the m-dimensional unit ball. Take a positive test function $\chi_R(\Im w) \leq 1$, with support in $|\Im w| < R - \sqrt{R}$, such that $\chi_R = 1$ for $|\Im w| < R - 2\sqrt{R}$, satisfying $|d\chi_R| \leq C/\sqrt{R}$. Put $a_R = \int \chi_R$, and let

$$\tilde{u}_R = u_R * \chi_R / a_R,$$

where the convolution is taken w.r.t. $\Im w$. Then $\bar{\partial} \tilde{u}_R = f$ in $|\Im w| < \sqrt{R}$ and

$$\int |\tilde{u}_R(z, \Re w, \Im w|^2 e^{-\varphi} d\lambda(z, \Re w) \leq 2(\sigma_m R^m / a_R) \int_{U \times X} |f|^2 / c \, e^{-\varphi} d\lambda(z, \Re w)$$

uniformly for $|\Im w| < \sqrt{R}$. Note that $(\sigma_m R^m / a_R)$ tends to 1 as R tends to infinity. Furthermore,

$$\sqrt{R} \frac{\partial \tilde{u}_R}{\partial \Im w}$$

satisfies a similar estimate. Regarding $\Im w \to \tilde{u}_R$ as a map with values in

$$L^2(U \times X, e^{-\varphi}),$$

we get from Arzela-Ascoli theorem that there is a sequence $R_j \to \infty$, such that $\tilde{u}_{R_j} \to u_0$, uniformly on compacts in $\mathbb{R}^n$. Since $\partial\tilde{u}_R/\partial\Im w$ tends to zero uniformly, u_0 is independent of $\Im w$, and since u_0 is also holomorphic in w, we in fact have that u_0 is independent of w. Hence $u_0 = u_0(z)$ solves $\bar{\partial} u_0 = f$ in U and

$$\int_{U\times X} |u_0|^2 e^{-\varphi(z,\Re w)} \leq 2 \int_{U\times X} |f|^2/c\, e^{-\varphi(z,\Re w)}$$

□

§3 Proof of Theorem 1.3

We start by proving the first part of the theorem. We may assume $n = 1$, and we may also assume that φ is smooth and that the smallest eigenvalue of the complex Hessian of φ is bounded from below by a positive constant c (otherwise approximate φ by a decreasing sequence of functions satisfying these assumptions and pass to the limit). We shall prove that if $\tilde{\varphi}$ is defined by

$$e^{-\tilde{\varphi}(z)} = \int e^{-\varphi(z,w)} d\lambda(w),$$

then $\tilde{\varphi}$ satisfies the hypothesis of Proposition 2.2. Note first that

$$\begin{aligned} e^{-k\tilde{\varphi}(z)} &= \prod_{j=1}^{k} \int_V e^{-\varphi(z,w_j)} d\lambda(w_j) \\ &= \int_{V^k} e^{-\sum_1^k \varphi(z,w_j)} d\lambda(w_1, \ldots, w_k). \end{aligned}$$

Take $f \in C_c^\infty(U)$ and consider

$$F = f(z) d\bar{z}$$

as a $(0,1)$-form in $\Omega_k = U_z \times (V_w)^k$ independent of w_j. Solve $\bar{\partial} u = F$ in Ω_k with

$$\begin{aligned} &\int_{\Omega_k} |u|^2 e^{-\sum_1^k \varphi(z,w_j)} \leq \\ &\leq 2 \int_{\Omega_k} |f|^2/c\, e^{-\sum_1^k \varphi(z,w_j)}. \end{aligned}$$

(This is possible by Theorem 2.1). Since f is independent of w_j, the integral in the right-hand side equals

$$\int |f|^2 e^{-k\tilde{\varphi}(z)}.$$

If we now assume that V is a connected Reinhard domain and that φ is invariant under rotation in any of the coordinates, then the uniquely determined L^2-minimal solution to $\bar{\partial}u = F$ must have the same invariance property. Since u is also holomorphic in w_j, u actually is independent of w_j. The same conclusion will also hold if we only assume that V and φ are invariant under the diagonal action of $e^{i\theta}$, provided that $0 \in V$.

In each case we get

$$\int |u|^2 e^{-k\tilde{\varphi}(z)} \leq 2\int |f|^2/c\, e^{-k\tilde{\varphi}}.$$

By Proposition 2.2, $\tilde{\varphi}$ is subharmonic, so the proof of the first part of Theorem 1.3 is complete. We next turn to the second part of the theorem. We thus assume that

$$V = X + i\mathbb{R}^m$$

is a tube domain, and that φ is independent of $\Im w$. Again, we may assume $n = 1$ and that φ is strictly plurisubharmonic with c a positive lower bound for the complex Hessian of φ. Take $f \in C_c^\infty(U)$ and consider

$$F = f(z)d\bar{z}.$$

We have

$$e^{-k\tilde{\varphi}(z)} = \int_{X^k} e^{-\sum_1^k \varphi(z,\Re w_j)} d\lambda(\Re w_1, \dots, \Re w_m).$$

By Proposition 2.3 we can solve $\bar{\partial}u = f$ in U with

$$\int_U |u|^2 e^{-k\tilde{\varphi}} \leq 2\int_U |f|^2/c\, e^{-k\tilde{\varphi}}.$$

Again Proposition 2.2 implies that $\tilde{\varphi}$ is subharmonic and the proof is complete.

Finally we shall sketch the proof of Theorem 1.4. Exhausting our domain Ω by strongly pseudoconvex subdomains with the same invariance property as Ω we may assume that Ω has a plurisubharmonic defining function, ψ, which is defined in a neighbourhood of $\bar{\Omega}$. Moreover, ψ can also be chosen invariant with respect to translations or rotations, and φ can also be taken to be defined and plurisubharmonic in a neighbourhood of $\bar{\Omega}$. Since the theorem is local in z it is enough to consider $z \in U$, where U is chosen so small that $(U \times \mathbb{C}^m) \cap \bar{\Omega} \subset U \times V$, where V is pseudoconvex, and ψ and φ are both defined in $U \times V$. Put $\psi_0 = \max(\psi, 0)$, and apply Theorem 1.3 with φ replaced by $\varphi + N\psi_0$. Letting N tend to infinity we obtain Theorem 1.3.

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