ON THE CAUCHY PROBLEM FOR NON-EFFECTIVELY HYPERBOLIC OPERATORS, THE GEVREY 5 WELL-POSEDNESS

By

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Abstract. In this paper, we prove that for non-effectively hyperbolic operators with smooth double characteristics exhibiting a Jordan block of size 4 on the double manifold, the Cauchy problem is well-posed in the Gevrey 5 class, beyond the generic Gevrey class 2 (see, e.g., [5]). Moreover, we show that this value is optimal, due to certain geometric constraints on the Hamiltonian flow of the principal symbol. These results, together with results already proved, give a complete picture of the well-posedness of the Cauchy problem around hyperbolic double characteristics.

1 Introduction

Let

$$P(x,D) = D_0^2 + \sum_{|\alpha|\le 2, \alpha_0<2} a_\alpha(x) D^\alpha = P_2 + P_1 + P_0$$

be a second order differential operator, defined in an open neighborhood of the origin of $\mathbb{R}^{n+1}$, hyperbolic with respect to the x_0 direction and having principal symbol $p(x,\xi)$ where $x = (x_0, x_1, \dots, x_n)$, $\xi = (\xi_0, \xi_1, \dots, \xi_n)$. Let $\bar\rho \in T^*\mathbb{R}^{n+1} \setminus \{0\}$ be a double characteristic of p, that is, $p(\bar\rho) = 0$, $dp(\bar\rho) = 0$. We are interested in the (microlocal) Cauchy problem around $\bar\rho$ and the behavior of null bicharacteristics of p. Since $\bar\rho$ is a singular point of the Hamilton vector field H_p of p, to investigate the behavior of null bicharacteristics, we consider the linearization of H_p at $\bar\rho$, which is called the Hamilton map $F_p(\bar\rho)$ of p at $\bar\rho$ defined as (see e.g. [10], [6])

$$\frac{1}{2} Q_{\bar\rho}(X,Y) = \sigma(X, F_p(\bar\rho)Y), \quad X, Y \in T^*\mathbb{R}^{n+1},$$

where $Q_{\bar\rho}$ is the polar form of the Hesse matrix of p at $\bar\rho$ and $\sigma = \sum_{j=0}^n d\xi_j \wedge dx_j$ is the canonical symplectic two form on $T^*\mathbb{R}^{n+1}$. It is well-known that all eigenvalues of $F_p(\bar\rho)$ are on the imaginary axis, with the possible exception of a pair of non-zero

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real eigenvalues ([10], [6]). When all the eigenvalues of $F_p(\bar\rho)$ are on the imaginary axis, p is called non-effectively hyperbolic at $\bar\rho$. Our aim in this paper is to study the Cauchy problem around non-effectively double characteristics.

We would like to stress the fact that beyond the usual demand of algebraic hyperbolicity, i.e. real roots of the principal symbol or, more generally, of some localized entity connecting the principal symbol with some relevant lower order terms, there may be other conditions one has to impose in order to get the well-posedness of the Cauchy problem in the full range of the Gevrey classes: such new necessary conditions depend on the nature of the Hamilton system, as is made clear below. We now state more precisely our assumptions. We assume in the following that p vanishes exactly to order 2 on a C^∞ submanifold Σ on which σ has constant rank and p is non-effectively hyperbolic; that is, we assume that $\Sigma = \{(x,\xi) : p(x,\xi) = 0, dp(x,\xi) = 0\}$ is a C^∞ manifold and

$$\mathrm{Sp}(F_p(\rho)) \subset i\mathbb{R}, \qquad \rho \in \Sigma, \tag{1.1}$$

$$\mathrm{dim} T_\rho\Sigma = \mathrm{dim}\mathrm{Ker} F_p(\rho), \qquad \rho \in \Sigma, \tag{1.2}$$

$$\mathrm{rank}\sigma = \text{constant} \qquad \text{on } \Sigma, \tag{1.3}$$

where $\mathrm{Sp}(F_p(\rho))$ denotes the spectrum of $F_p(\rho)$. We denote by $P_{sub}(\rho)$ the subprincipal symbol of P. According to the spectral structure of $F_p(\rho)$, two different cases may arise:

$$\mathrm{Ker}F_p^2(\rho) \cap \mathrm{Im}F_p^2(\rho) = \{0\}, \quad \mathrm{Ker}F_p^2(\rho) \cap \mathrm{Im}F_p^2(\rho) \neq \{0\}.$$

In the first case, under suitable conditions on lower order terms (the Levi condition and the Ivrii-Petkov-Hörmander condition), the Cauchy problem is C^∞ well-posed ([10], [6], [8]).

In the second case, the linear algebraic properties of $F_p(\rho)$ are not enough by themselves to determine completely the behavior of the null bicharacteristics of the principal symbol. It can be readily verified that perturbing the quadratic part of the principal symbol with a suitable term vanishing to order three on the double manifold Σ may cause the Hamilton system to exhibit null bicharacteristics landing on Σ. (See model (1.5) below and [13]).

Let us recall that $f(x) \in C^\infty(\mathbb{R}^n)$ belongs to $\gamma^{(s)}(\mathbb{R}^n)$, the Gevrey space of order $s \geq 1$ if for any compact set $K \subset \mathbb{R}^n$, there exist $C > 0$, $h > 0$ such that

$$\left|\partial_x^\alpha f(x)\right| \leq C h^{|\alpha|}|\alpha|!^s, \quad x \in K \tag{1.4}$$

for every $\alpha \in \mathbb{N}^n$. In particular, $\gamma^{(1)}(\mathbb{R}^n)$ is the space of real analytic functions on $\mathbb{R}^n$.

In this paper, we prove the following result.

Theorem 1.1. *Assume that* $\mathrm{Ker}F_p^2(\rho) \cap \mathrm{Im}F_p^2(\rho) \neq \{0\}$ *and* $P_{sub}(\rho) = 0$ *everywhere on* Σ*. Then the Cauchy problem for* P *is well-posed in the Gevrey* 5 *class.*

The Gevrey index 5 in Theorem 1.1 is optimal if indeed there is a null bicharacteristic tangent to the doubly characteristic manifold. To see this, consider

$$P = -D_0^2 + 2x_1 D_0 D_n + D_1^2 + x_1^3 D_n^2. \tag{1.5}$$

This is a model (canonical) operator, first studied in [13], such that

$$\mathrm{Im}F_p^2(\rho) \cap \mathrm{Ker}F_p^2(\rho) \neq \{0\}, \quad \rho \in \Sigma,$$

and admitting a null bicharacteristic with a limit point in the doubly characteristic set. Note that $P_{sub} = 0$ identically. Sufficient conditions for Gevrey 5 energy estimates for the operator (1.5) were obtained in [2].

We say that the Cauchy problem for P is locally solvable in $\gamma^{(s)}$ at the origin if for any $\Phi = (u_0, u_1) \in (\gamma^{(s)}(\mathbb{R}^n))^2$, there exists a neighborhood U_Φ of the origin such that the Cauchy problem

$$\begin{cases} Pu = 0 & \text{in } U_\Phi \\ D_0^j u(0, x') = u_j(x'), & j = 0, 1,\ x \in U_\Phi \cap \{x_0 = 0\} \end{cases}$$

has a solution $u(x) \in C^\infty(U_\Phi)$. Then we have

Theorem 1.2. *If* $s > 5$*, the Cauchy problem for* P *is not locally solvable at the origin in the Gevrey* s *class.*

On the other hand, if there is no null bicharacteristic tangent to Σ, assuming $\mathrm{Ker}F_p^2(\rho) \cap \mathrm{Im}F_p^2(\rho) \neq \{0\}$ it has been proved in [14], [15], [4], [8] that the Cauchy problem is C^∞ well-posed. Thus we can give now a complete picture of the well-posedness of the Cauchy problem around hyperbolic double characteristics and the behavior of null bicharacteristics near the reference double characteristic.

This is summarized in the following table:

<table>
<tr><th>Eigenvalues of F_p</th><th>$\mathrm{Im}F_p^2 \cap \mathrm{Ker}F_p^2$</th><th>Behavior of null bicharacteristic w.r.t. Σ</th><th>Well-posedness</th></tr>
<tr><td>there are non-zero real eigenvalues</td><td>$\mathrm{Im}F_p^2 \cap \mathrm{Ker}F_p^2 = \{0\}$</td><td>every null bicharacteristic falling in Σ is transversal</td><td>C^∞ well-posed with any lower order term</td></tr>
<tr><td rowspan="3">there is no non-zero real eigenvalue</td><td>$\mathrm{Im}F_p^2 \cap \mathrm{Ker}F_p^2 = \{0\}$</td><td rowspan="2">there is no null bicharacteristic tangent to Σ</td><td rowspan="2">C^∞ well-posed with Levi, strict I-P-H condition</td></tr>
<tr><td rowspan="2">$\mathrm{Im}F_p^2 \cap \mathrm{Ker}F_p^2 \neq \{0\}$</td></tr>
<tr><td>there is a null bicharacteristic tangent to Σ</td><td>Gevrey 5 well-posed with Levi condition</td></tr>
</table>

2 Non effectively hyperbolic characteristics

Without restrictions, we may assume that the principal symbol $p(x, \xi)$ of P has the form

$$p(x, \xi) = -\xi_0^2 + q(x, \xi'), \tag{2.1}$$

where $q \geq 0$. As stated in Introduction, we always assume that (1.1), (1.2) and (1.3) are satisfied. We also assume that

$$\mathrm{Ker}F_p(\rho)^2 \cap \mathrm{Im}F_p(\rho)^2 \neq \{0\}, \quad \forall \rho \in \Sigma, \tag{2.2}$$

which means that the Hamilton map $F_p(\rho)$ has a Jordan block of size 4 in its Jordan canonical form at every $\rho \in \Sigma$. From the hypothesis (1.2), one can write, near every $\bar{\rho} \in \Sigma$

$$p(x, \xi) = -\xi_0^2 + \sum_{j=1}^{r} \phi_j(x, \xi')^2,$$

where $d\phi_j$ are linearly independent at $\bar{\rho}$ and Σ is given near $\bar{\rho}$ by

$$\Sigma = \{(x, \xi) : \phi_j(x, \xi) = 0, j = 0, 1, \ldots, r\}$$

and we have set $\phi_0(x, \xi) = \xi_0$. Assume (2.2); then by Theorem 21.5.3 in [7], then in suitable symplectic coordinates, the quadratic form $Q = p_\rho$ takes the form

$$Q = -\xi_0^2 + 2\xi_0\xi_1 + x_1^2 + \sum_{j=2}^{k} \mu_j(x_j^2 + \xi_j^2) + \sum_{j=k+1}^{k+\ell} \xi_j^2. \tag{2.3}$$

Examining the canonical model (2.3) one sees easily that

$$\dim \mathrm{Im}F_p^2(\rho) = 2 + 2(k-1), \quad \dim \mathrm{Im}F_p^3(\rho) = 1 + 2(k-1)$$

which are independent of ρ by our assumption. Since $\dim(\mathrm{Ker}F_p(\rho) \cap \mathrm{Im}F_p^3(\rho)) = 1$, $\dim(\mathrm{Ker}F_p^2(\rho) \cap \mathrm{Im}F_p^2(\rho)) = 2$, which is easily verified on the standard model (2.3), one can choose smooth $z_1(\rho)$, $h_j(\rho)$, $j = 1, 2$ so that

$$\begin{aligned} \langle z_1(\rho)\rangle &= \mathrm{Ker}F_p(\rho) \cap \mathrm{Im}F_p^3(\rho), \qquad && \rho \in \Sigma, \\ \mathrm{Ker}F_p^2(\rho) \cap \mathrm{Im}F_p^2(\rho) &= \langle h_1(\rho), h_2(\rho)\rangle, && \rho \in \Sigma. \end{aligned}$$

Lemma 2.1. *There are smooth sections $z_1(\rho)$, $z_2(\rho)$ such that*

$$\begin{gathered} \mathrm{Ker}F_p^2(\rho) \cap \mathrm{Im}F_p^2(\rho) = \langle z_1(\rho), z_2(\rho)\rangle, \\ F_p(\rho)z_1(\rho) = 0, \quad F_p(\rho)z_2(\rho) \neq 0. \end{gathered}$$

Proof. Let $\mathrm{Ker}F_p^2(\rho) \cap \mathrm{Im}F_p^2(\rho) = \langle h_1(\rho), h_2(\rho)\rangle$. Since $F_p(\rho)h_j(\rho)$, $j = 1, 2$ are in $\mathrm{Im}F_p^3(\rho) \cap \mathrm{Ker}F_p(\rho)$, there exist smooth $\alpha(\rho)$, $\beta(\rho)$ such that

$$\alpha(\rho)F_p(\rho)h_1(\rho) + \beta(\rho)F_p(\rho)h_2(\rho) = 0.$$

Then it is enough to choose

$$z_1(\rho) = \alpha(\rho)h_1(\rho) + \beta(\rho)h_2(\rho), \quad z_2(\rho) = \beta(\rho)h_1(\rho) - \alpha(\rho)h_2(\rho).$$ □

Note that in the canonical model (2.3), it is easy to see that

$$\mathrm{Ker}F_p^2(\rho) \cap \mathrm{Im}F_p^2(\rho) = \langle H_{\xi_0}, H_{x_1}\rangle, \quad z_2(\rho) = aH_{\xi_0} + bH_{x_1} \tag{2.4}$$

with $b \neq 0$.

Lemma 2.2. *We have*

$$\begin{gathered} w \in \langle z_1(\rho)\rangle^\sigma \Longrightarrow \sigma(w, F_p(\rho)w) \geq 0 \\ w \in \langle z_1(\rho)\rangle^\sigma, \ \sigma(w, F_p(\rho)w) = 0 \Longrightarrow w \in \mathrm{Ker}F_p(\rho)\dot{+}\langle z_2(\rho)\rangle. \end{gathered}$$

Proof. Choose symplectic coordinates with which p_ρ takes the form (2.3). Then it is easy to see that $\langle z_1(\rho)\rangle = \langle H_{\xi_0}\rangle$. Hence if $w \in \langle z_1(\rho)\rangle^\sigma$, then

$$\sigma(w, F_p(\rho)w) = Q(w) = x_1^2 + \sum_{j=2}^{k} \mu_j(x_j^2 + \xi_j^2) + \sum_{j=k+1}^{k+\ell} \xi_j^2 \geq 0$$

which is the first assertion. From $w \in \langle z_1(\rho)\rangle^\sigma$ and $\sigma(w, F_p(\rho)w) = 0$ it follows that $w \in \mathrm{Ker}F_p(\rho) + \langle H_{x_1}\rangle$. Thus (2.4) proves the second assertion. □

We summarize what we have proved in

Lemma 2.3. *Assume that* p *satisfies* (2.2). *Then there exist two smooth sections of* $T_\Sigma T^*(\mathbb{R}^{n+1})$, z_1, z_2 *such that*

$$z_1(\rho) \in \mathrm{Ker}F_p(\rho) \cap \mathrm{Im}F_p^3(\rho), \quad \rho \in \Sigma, \tag{2.5}$$

$$z_2(\rho) \in \mathrm{Ker}F_p^2(\rho) \cap \mathrm{Im}F_p^2(\rho), \quad \rho \in \Sigma, \tag{2.6}$$

$$w \in \langle z_1(\rho)\rangle^\sigma \Longrightarrow \sigma(w, F_p(\rho)w) \geq 0, \tag{2.7}$$

$$w \in \langle z_1(\rho)\rangle^\sigma,\ \sigma(w, F_p(\rho)w) = 0 \Longrightarrow w \in \mathrm{Ker}F_p(\rho)\dot{+}\langle z_2(\rho)\rangle. \tag{2.8}$$

Our aim in this section is to prove

Proposition 2.1. *Suppose that* p *vanishes exactly to order* 2 *on a* C^∞ *submanifold* Σ *of* $T^*\mathbb{R}^{n+1} \setminus \{0\}$ *on which the canonical* 2 *form has constant rank and such that* $\mathrm{Sp}(F_p(\rho)) \subset i\mathbb{R}$, $\mathrm{Ker}F_p^2(\rho) \cap \mathrm{Im}F_p^2(\rho) \neq 0$, *for* $\rho \in \Sigma$. *Then one can write, near a reference point* $\bar{\rho}$,

$$p(x,\xi) = -\xi_0^2 + \sum_{j=1}^r \phi_j(x,\xi')^2$$

where Σ *is given by* $\{\xi_0 = 0, \phi_1 = \cdots = \phi_r = 0\}$ *and*

$$\{\xi_0 + \phi_1, \phi_j\} = 0,\ j = 1, \ldots, r, \quad \{\phi_1, \phi_2\} \neq 0, \quad \textit{on } \Sigma. \tag{2.9}$$

Proof. First note that there exists smooth $\Lambda(x,\xi)$ vanishing on Σ such that $H_\Lambda(\rho) = z_1(\rho)$, $\rho \in \Sigma$, for $\mathrm{Im}F_p(\rho) = \langle H_{\xi_0}, H_{\phi_1}(\rho), \ldots, H_{\phi_r}(\rho)\rangle$. Since $\sigma(H_{x_0}, F_p(\rho)H_{x_0}) = -1$ we get from (2.8) that $\sigma(z_1, H_{x_0}) \neq 0$, for $\rho \in \Sigma$. Thus we may assume that $\Lambda(x,\xi) = \xi_0 - \sum_{j=1}^r \gamma_j(x,\xi')\phi_j(x,\xi')$ and hence

$$H_\Lambda(\rho) = H_{\xi_0} - \sum_{j=1}^r \gamma_j(\rho) H_{\phi_j}(\rho), \quad \rho \in \Sigma.$$

Thus,

$$\begin{aligned}\sigma(z, F_p z) = &- \big[\sigma(z, H_{\xi_0}) - \sum_{j=1}^r \gamma_j(\rho)\sigma(z, H_{\phi_j})\big]\big[\sigma(z, H_{\xi_0}) + \sum_{j=1}^r \gamma_j(\rho)\sigma(z, H_{\phi_j})\big] \\ &+ \sum_{j=1}^r \sigma(z, H_{\phi_j})^2 - \big(\sum_{j=1}^r \gamma_j(\rho)\sigma(z, H_{\phi_j})\big)^2.\end{aligned}$$

By (2.7), we have

$$\sum_{j=1}^r \sigma(z, H_{\phi_j})^2 - \left(\sum_{j=1}^r \gamma_j(\rho)\sigma(z, H_{\phi_j})\right)^2 \geq 0 \tag{2.10}$$

if $z \in \langle H_\Lambda(\rho)\rangle^\sigma$. Here we note that

$$\langle H_\Lambda(\rho)\rangle^\sigma / T_\rho\Sigma \ni z \mapsto \big(\sigma(z, H_{\phi_j})\big)_{j=1,\dots,r} \in \mathbb{R}^r$$

is surjective. Indeed, $\sigma(z, H_\Lambda) = 0$, $\sigma(z, H_{\phi_j}) = 0$, $1 \le j \le r$ implies that $z \in \langle H_\Lambda, H_{\phi_1}, \dots, H_{\phi_r}\rangle^\sigma = T_\rho\Sigma$. Then (2.10) shows that $\sum_{j=1}^r \gamma_j(\rho)^2 = |\gamma(\rho)| \le 1$, $\rho \in \Sigma$. We show that

$$\sum_{j=1}^r \gamma_j(\rho)^2 = 1, \quad \rho \in \Sigma. \tag{2.11}$$

As noted above, we can take z so that $(\sigma(z, H_{\phi_j}))_{1\le j\le r}$ is proportional to $\gamma(\rho) = (\gamma_1(\rho), \dots, \gamma_r(\rho))$. Then by (2.10), we have $z \in \mathrm{Ker}F_p(\rho)\dot{+}\langle z_2(\rho)\rangle$. Write $z = u + v$ with $u \in \mathrm{Ker}F_p$, $v \in \langle z_2(\rho)\rangle$, so that $\sigma(z, H_{\phi_j}) = \sigma(v, H_{\phi_j})$. This proves that $(\sigma(z_2(\rho), H_{\phi_j}(\rho)))_{1\le j\le r}$ is proportional to $\gamma(\rho)$, and hence one can write $\gamma_j(\rho) = \alpha\sigma(z_2(\rho), H_{\phi_j}(\rho))$ with some α. We insert this into

$$\sum_{j=1}^r \sigma(z_2, H_{\phi_j})^2 - \Big(\sum_{j=1}^r \gamma_j(\rho)\sigma(z_2, H_{\phi_j})\Big)^2 = 0$$

to get $\alpha = 1/\sqrt{\sum_{j=1}^r \sigma(z_2, H_{\phi_j})^2}$. Thus the assertion is proved. We still denote by $\gamma(x, \xi')$ an extension of $\gamma(\rho)$ outside Σ such that $|\gamma(x, \xi')| = 1$. Thus we can write

$$p(x, \xi) = -(\xi_0 - \langle\gamma, \phi\rangle)(\xi_0 + \langle\gamma, \phi\rangle) + |\phi|^2 - \langle\gamma, \phi\rangle^2,$$

where $\{\Lambda, \phi_j\} = 0$, $j = 1, \dots, r$ on Σ since $H_\Lambda \in \mathrm{Im}F_p$. Let us now set $\psi_1(x, \xi') = \sum_{j=1}^r \gamma_j(x, \xi')\phi_j(x, \xi')$ and, taking a smooth orthonormal basis

$$\gamma(x, \xi'), e_2(x, \xi'), \dots, e_r(x, \xi'),$$

where $e_j(x, \xi') = e_{j1}(x, \xi'), \dots, e_{jr}(x, \xi')$ and define

$$\psi_j(x, \xi') = \sum_{h=1}^r e_{jh}(x, \xi')\phi_h(x, \xi')$$

so that $\sum_{j=1}^r \psi_j(x, \xi')^2 = \sum_{j=1}^r \phi_j(x, \xi')^2$. Switching the notation to $\{\phi_j\}$, we can thus write

$$p(x, \xi) = -(\xi_0 + \phi_1(x, \xi'))(\xi_0 - \phi_1(x, \xi')) + \sum_{j=2}^r \phi_j(x, \xi')^2,$$

where $\{\xi_0 + \phi_1, \phi_j\} = 0$ on Σ for $j = 1, \dots, r$. We finally check that $\{\phi_1, \phi_k\} \ne 0$ for some k. Indeed, otherwise, we would have $\{\xi_0, \phi_j\} = 0$, $j = 1, \dots, r$; and this would contradict (2.2). Renumbering the coordinates so that $k = 2$, we have the assertion. □

3 A lemma for Weyl calculus in the Gevrey class

In this section, we introduce a class of symbols of pseuodifferential operators which used in Section 4 to derive Gevrey a priori estimates for P and for $a(x, D)$ with such symbol we prove a composition formula $e^{\pm\phi(D)}a(x, D)e^{\mp\phi(D)}$, where $\phi(D) \sim (1 + |D|)^{1/5}$.

Let

$$\bar{g} = \langle \mu\xi \rangle^{\sigma} \{|dx|^2 + \langle \xi \rangle_{\mu}^{-2}|d\xi|^2\}, \;\; \langle \xi \rangle_{\mu} = (\mu^{-2} + |\xi|^2)^{1/2}, \;\; (x, \xi) \in \mathbb{R}^{2n}$$

be a metric, where $0 < \sigma < 1$; σ eventually takes the value $4/5$. We say $b(x, \xi, \mu) \in \gamma^{(s)}S(m(\xi, \mu), \bar{g})$ if $b(x, \xi, \mu)$ satisfies the estimates

$$|\partial_x^{\alpha}\partial_{\xi}^{\beta}b(x, \xi, \mu)| \le C_{\beta}m(\xi, \mu)(\langle \mu\xi \rangle^{-\sigma/2}\langle \xi \rangle_{\mu})^{-|\beta|}A^{|\alpha|}|\alpha|!^{s/2}(|\alpha|^{s/2} + \langle \mu\xi \rangle^{\sigma/2})^{|\alpha|}$$

for every α, $\beta \in \mathbb{N}^n$. We assume that $b(x, \xi, \mu)$ is independent of x for $|x| \ge M$ with M large.

Lemma 3.1. *Let $s \ge 4$. Assume that*

$$|\partial_x^{\alpha}\partial_{\xi}^{\beta}f(x, \xi, \mu)| \le C_{\beta}m(\xi, \mu)(\langle \mu\xi \rangle^{-\sigma/2}\langle \xi \rangle_{\mu})^{-|\beta|}A^{|\alpha|}|\alpha|!^{s}$$

for every α, $\beta \in \mathbb{N}^n$. Then

$$w(x, \xi, \mu) = \sqrt{f(x, \xi, \mu)^2 + \langle \mu\xi \rangle^{-\sigma}} \in \gamma^{(s)}S(m(\xi, \mu), \bar{g}).$$

Proof. It is enough to prove the following lemma.

Lemma 3.2. *Let $f(x)$ be in the Gevrey $s \ge 4$ space, i.e., assume that*

$$|f^{(j)}(x)| \le C_1C_2^j j!^s, \quad j \ge 1 \tag{3.1}$$

with C_1, C_2 positive constants. Let $w(x) = \sqrt{f(x)^2 + B^{-2}}$, with B positive. Then there exists a positive constant A_1 such that

$$|w^{(j)}(x)| \le w(x)A_1^j j!^{s/2}(j^{s/2} + B)^j, \quad j \ge 1. \tag{3.2}$$

Proof. Note that

$$w(x)w'(x) = f(x)f'(x) = F(x), \tag{3.3}$$

where we may assume that

$$|F^{(j)}(x)| \le A_2^{j+1}j!^s, \quad j \ge 1.$$

Assume that the inequalities (3.2) hold for $j \leq n$ and let us study $w^{(n+1)}(x)$. Using (3.3), we have:

$$ww^{(n+1)} = -\sum_{j=1}^{n} \binom{n}{j} w^{(j)} w^{(n+1-j)} + F^{(n)}, \tag{3.4}$$

which gives

$$\begin{aligned}|ww^{(n+1)}| \leq &\sum_{j=1}^{n} \binom{n}{j} w A_1^j j!^{s/2} (j^{s/2}+B)^j w A_1^{n+1-j} (n+1-j)!^{s/2} \\ &\times ((n+1-j)^{s/2}+B)^{n+1-j} + A_2^{n+1} n!^s \\ \leq & w^2 A_1^{n+1} \Big[\sum_{j=1}^{n} \binom{n}{j} j!^{s/2} (n+1-j)!^{s/2} (j^{s/2}+B)^j \\ &\times ((n+1-j)^{s/2}+B)^{n+1-j} + w^{-2}(A_2/A_1)^{n+1} n!^s\Big].\end{aligned}$$

Note that

$$(j^{s/2}+B)^j((n+1-j)^{s/2}+B)^{n+1-j} \leq ((n+1)^{s/2}+B)^{n+1}$$

and $w^{-2} \leq B^2$ and

$$\begin{aligned}((n+1)^{s/2}+B)^{n+1}(n+1)!^{s/2} &\geq \frac{(n+1)n}{2}(n+1)^{(n-1)s/2} B^2 (n+1)!^{s/2} \\ &\geq \frac{(n+1)n}{2} B^2 n!^s.\end{aligned}$$

Thus

$$\begin{aligned}|w^{(n+1)}| \leq & w A_1^{n+1}((n+1)^{s/2}+B)^{n+1}(n+1)!^{s/2} \\ &\times \Big[\sum_{j=1}^{n} \binom{n}{j} \binom{n+1}{j}^{-s/2} + \Big(\frac{A_2}{A_1}\Big)^{n+1} \frac{2}{(n+1)n}\Big].\end{aligned}$$

We now check that

$$\sum_{j=1}^{n} \binom{n}{j} \binom{n+1}{j}^{-s/2} + \Big(\frac{A_2}{A_1}\Big)^{n+1} \frac{2}{(n+1)n} \leq 1$$

if $A_2/A_1 \leq 1$. Indeed,

$$\begin{aligned}\sum_{j=1}^{n} \binom{n}{j} \binom{n+1}{j}^{-s/2} &= \sum_{j=1}^{n} \frac{n+1-j}{n} \binom{n+1}{j}^{1-s/2} \leq n \binom{n+1}{1}^{1-s/2} \\ &= \frac{n}{(n+1)^{s/2-1}} \leq \frac{n}{n+1}.\end{aligned}$$

□

Lemma 3.3. *Let* $a_i(x,\xi,\mu) \in \gamma^{(s)}S(m_i(\xi,\mu),\bar{g})$, $i=1,2$. *Then*

$$a_1(x,\xi,\mu)a_2(x,\xi,\mu) \in \gamma^{(s)}S(m_1(\xi,\mu)m_2(\xi,\mu),\bar{g}).$$

Proof. It is enough to note that

$$\sum_{\alpha'\le\alpha}\binom{\alpha}{\alpha'}A_1^{|\alpha'|}|\alpha'|!^{s/2}A_2^{|\alpha-\alpha'|}|\alpha-\alpha'|!^{s/2} \le A_1^{1+|\alpha|}(A_1-A_2)^{-1}|\alpha|!^{s/2}$$

for $A_1 > A_2$ and

$$(|\alpha'|^{s/2}+\langle\mu\xi\rangle^{\sigma/2})^{|\alpha'|}(|\alpha-\alpha'|^{s/2}+\langle\mu\xi\rangle^{\sigma/2})^{|\alpha-\alpha'|} \le (|\alpha|^{s/2}+\langle\mu\xi\rangle^{\sigma})^{|\alpha|}. \qquad \square$$

It is obvious that $b(x,\xi,\mu) \in \gamma^{(s)}S(m(\xi,\mu),\bar{g})$ if

$$|\partial_x^\alpha\partial_\xi^\beta b(x,\xi,\mu)| \le C_\beta m(\xi,\mu)\langle\xi\rangle_\mu^{-|\beta|}A^{|\alpha|}|\alpha|!^s, \quad \forall\alpha,\beta.$$

Let us consider

$$e^{\phi(D,\mu)}b^w(x,D,\mu)e^{-\phi(D,\mu)}.$$

Since

$$e^{\phi(D,\mu)}b^w(x,D,\mu)u = \int e^{i[x\xi-z\xi+(z-y)\eta]}e^{\phi(\xi,\mu)}b\Big(\frac{z+y}{2},\eta,\mu\Big)u(y)dyd\eta dzd\xi,$$

inserting

$$e^{-\phi(D,\mu)}u(y) = \int e^{iy\zeta-\phi(\zeta,\mu)}\hat{u}(\zeta)d\zeta,$$

we have

$$e^{\phi(D,\mu)}b^w(x,D,\mu)e^{-\phi(D,\mu)}u = \int e^{ix\zeta}I(x,\zeta,\mu)\hat{u}(\zeta)d\zeta,$$

where

$$I = \int e^{i[x\xi-z\xi+(z-y)\eta+y\zeta-x\zeta]}e^{\phi(\xi,\mu)}b\Big(\frac{z+y}{2},\eta,\mu\Big)e^{-\phi(\zeta,\mu)}dyd\eta dzd\xi.$$

Let us consider $I(x,\zeta,\mu)$. Making the change of variables

$$\tilde{z} = (y+z)/2, \quad \tilde{y} = (y-z)/2,$$

we have

$$\begin{aligned}
I =& 2^n\int e^{i[-\tilde{z}(\xi-\zeta)+\tilde{y}(\xi-2\eta+\zeta)+x(\xi-\zeta)]}e^{\phi(\xi,\mu)}b(\tilde{z},\eta,\mu)e^{-\phi(\zeta,\mu)}d\tilde{y}d\eta d\tilde{z}d\xi\\
=& 2^n\int e^{i\tilde{y}(\xi-2\eta+\zeta)}d\tilde{y}\int e^{-i(\tilde{z}-x)(\xi-\zeta)}e^{\phi(\xi,\mu)}b(\tilde{z},\eta,\mu)e^{-\phi(\zeta,\mu)}d\eta d\tilde{z}d\xi\\
=& 2^n\int e^{-2i(\tilde{z}-x)(\eta-\zeta)}e^{\phi(2\eta-\zeta,\mu)}b(\tilde{z},\eta,\mu)e^{-\phi(\zeta,\mu)}d\eta d\tilde{z}\\
=& \int e^{-i\tilde{z}\eta}e^{\phi(\sqrt{2}\eta+\zeta,\mu)-\phi(\zeta,\mu)}b\Big(x+\frac{\tilde{z}}{\sqrt{2}},\zeta+\frac{\eta}{\sqrt{2}},\mu\Big)d\eta d\tilde{z}.
\end{aligned}$$

Thus we conclude

$$\begin{aligned}e^{\phi(D,\mu)}b^w(x,D,\mu)e^{-\phi(D,\mu)}u =& \int e^{i(x-y)\xi}a(x,\xi,\mu)u(y)dyd\xi\\ =& \int e^{i(x-y)\xi}a(x,\xi,\mu)u(y)dyd\xi\\ =& a(x,D,\mu)u,\end{aligned}$$

with

$$a(x,\xi,\mu) = \int e^{-iy\eta}e^{\phi(\xi+\sqrt{2}\eta,\mu)-\phi(\xi,\mu)}b(x+y/\sqrt{2},\xi+\eta/\sqrt{2},\mu)dyd\eta.$$

Here we remark that with

$$q(x,\xi,\mu) = \int e^{iz\zeta}p(x+z/\sqrt{2},\xi+\zeta/\sqrt{2},\mu)dzd\zeta,$$

we have

$$q^w(x,D,\mu) = p(x,D,\mu). \tag{3.5}$$

Indeed,

$$\begin{aligned}q^w(x,D,\mu)u &= \int e^{i(x-y)\xi}q\Big(\frac{x+y}{2},\xi,\mu\Big)u(y)dyd\xi\\ &= \int e^{i(x\xi-y\xi+z\zeta)}p\Big(\frac{x+y}{2}+\frac{z}{\sqrt{2}},\xi+\frac{\zeta}{\sqrt{2}},\mu\Big)u(y)dyd\xi dzd\zeta\\ &= \int e^{i[(x-y-z)\xi+z\zeta]}p\Big(\frac{x+y+z}{2},\zeta,\mu\Big)u(y)dyd\xi dzd\zeta\\ &= \int e^{iz\zeta}p(x,\zeta,\mu)u(x-z)dzd\zeta\\ &= \int e^{i(x-z)\zeta}p(x,\zeta,\mu)u(z)dzd\zeta = p(x,D,\mu)u.\end{aligned}$$

From (3.5) it follows that

$$c^w(x,D,\mu) = a(x,D,\mu)$$

with

$$c(x,\xi,\mu) = \int e^{iz\zeta}a(x+z/\sqrt{2},\xi+\zeta/\sqrt{2},\mu)dzd\zeta.$$

Insert the expression of $a(x,\xi,\mu)$ to get

$$\begin{aligned}c(x,\xi,\mu) = \int e^{i(z\zeta-y\eta)}&e^{\phi(\sqrt{2}\eta+\xi+\frac{\zeta}{\sqrt{2}},\mu)-\phi(\xi+\frac{\zeta}{\sqrt{2}},\mu)}\\ &\times b\Big(x+\frac{z+y}{\sqrt{2}},\xi+\frac{\eta+\zeta}{\sqrt{2}},\mu\Big)dyd\eta dzd\zeta.\end{aligned}$$

Now change variables

$$\tilde{z} = (z+y)/\sqrt{2}, \quad \tilde{y} = (y-z)/\sqrt{2}, \quad \tilde{\zeta} = (\zeta+\eta)/\sqrt{2}, \quad \tilde{\eta} = (\eta-\zeta)/\sqrt{2}$$

to get

$$\begin{aligned} c(x,\xi,\mu) &= \int e^{-i(\tilde{z}\tilde{\eta}+\tilde{y}\tilde{\zeta})} e^{\phi(\frac{3\tilde{\zeta}}{2}+\xi+\frac{\tilde{\eta}}{2},\mu)-\phi(\xi+\frac{\tilde{\zeta}}{2}-\frac{\tilde{\eta}}{2},\mu)} b(x+\tilde{z},\xi+\tilde{\zeta},\mu) d\tilde{y}d\tilde{\eta}d\tilde{z}d\tilde{\zeta} \\ &= \int e^{-i\tilde{z}\tilde{\eta}} e^{\phi(\xi+\frac{\tilde{\eta}}{2},\mu)-\phi(\xi-\frac{\tilde{\eta}}{2},\mu)} b(x+\tilde{z},\xi,\mu) d\tilde{z}d\tilde{\eta}. \end{aligned}$$

We obtain

Lemma 3.4. $e^{\phi(D,\mu)} b^w(x,D,\mu) e^{-\phi(D,\mu)} = c^w(x,D,\mu)$, *where*

$$c(x,\xi,\mu) = \int e^{-iy\eta} e^{\phi(\xi+\frac{\eta}{2},\mu)-\phi(\xi-\frac{\eta}{2},\mu)} b(x+y,\xi,\mu) dy d\eta.$$

Let $\kappa = 1/s$. As for $\phi(\xi,\mu)$, we assume that

$$\begin{cases} \phi(\xi,\mu) \in S(\langle\mu\xi\rangle^{\kappa}, |dx|^2 + \langle\xi\rangle_{\mu}^{-2}|d\xi|^2), \\ \phi(\xi+\eta,\mu) - \phi(\xi-\eta,\mu) \le C\langle\mu\eta\rangle^{\kappa}. \end{cases} \tag{3.6}$$

Then the following proposition holds.

Proposition 3.1. *Let* $\sigma+\kappa \le 1$ *and* $b(x,\xi,\mu) \in \gamma^{(1/\kappa)} S(m(x,\xi'),\bar{g})$, $1/\kappa \ge 4$. *Assume* (3.6). *Let* $e^{\phi(D,\mu)} b^w(x,D,\mu) e^{-\phi(D,\mu)} = c^w(x,D,\mu)$. *Then*

$$c(x,\xi,\mu) = \sum_{j=0}^{N-1} c_j(x,\xi,\mu) + R_N(x,\xi,\mu),$$

where

$$\begin{aligned} c_j &= \sum_{|\alpha|=j} \frac{1}{\alpha!} \partial_\eta^\alpha e^{\phi(\xi+\frac{\eta}{2},\mu)-\phi(\xi-\frac{\eta}{2},\mu)}\Big|_{\eta=0} b_{(\alpha)}(x,\xi,\mu) \\ &\in \mu^j S(m(\xi,\mu)\langle\mu\xi\rangle^{-j(1-\kappa-\sigma/2)},\bar{g}), \\ R_N(x,\xi,\mu) &\in \mu^N S(m(\xi,\mu)\langle\mu\xi\rangle^{-N(1-\kappa-\sigma/2)+n\sigma/2},\bar{g}). \end{aligned}$$

Proof. Recall that

$$c(x,\xi,\mu) = \int e^{-iy\eta} e^{\phi(\xi+\frac{\eta}{2},\mu)-\phi(\xi-\frac{\eta}{2},\mu)} b(x+y,\xi,\mu) dy d\eta. \tag{3.7}$$

Write

$$\begin{aligned} & b(x+y,\xi,\mu) \\ &= \sum_{|\alpha|<N} \frac{1}{\alpha!} b_{(\alpha)}(x,\xi,\mu)(iy)^\alpha + \sum_{|\alpha|=N} \frac{N}{\alpha!}(iy)^\alpha \int_0^1 (1-s)^{N-1} b_{(\alpha)}(x+sy,\xi,\mu) ds \end{aligned}$$

and insert this into (3.7) to get

$$c(x,\xi,\mu)=\sum_{|\alpha|<N}\frac{1}{\alpha!}\int e^{-iy\eta}e^{\phi(\xi+\frac{\eta}{2},\mu)-\phi(\xi-\frac{\eta}{2},\mu)}b_{(\alpha)}(x,\xi,\mu)(iy)^{\alpha}dyd\eta$$
$$+\sum_{|\alpha|=N}\frac{N}{\alpha!}\int e^{-iy\eta}e^{\phi(\xi+\frac{\eta}{2},\mu)-\phi(\xi-\frac{\eta}{2},\mu)}(iy)^{\alpha}dyd\eta$$
$$\times\int_0^1(1-s)^{N-1}b_{(\alpha)}(x+sy,\xi,\mu)ds.$$

The first term in the right-hand side is

$$\sum_{|\alpha|<N}\frac{1}{\alpha!}\partial_\eta^\alpha e^{\phi(\xi+\frac{\eta}{2},\mu)-\phi(\xi-\frac{\eta}{2},\mu)}\Big|_{\eta=0}b_{(\alpha)}(x,\xi,\mu) \tag{3.8}$$

because $e^{-iy\eta}(iy)^\alpha=(-\partial_\eta)^\alpha e^{-iy\eta}$. Set

$$\partial_\eta^\alpha e^{\phi(\xi+\frac{\eta}{2},\mu)-\phi(\xi-\frac{\eta}{2},\mu)}=\sum_{\beta+\gamma=\alpha}\frac{\alpha!}{\beta!\gamma!}\partial_\eta^\beta e^{\phi(\xi+\frac{\eta}{2},\mu)}\partial_\eta^\gamma e^{-\phi(\xi-\frac{\eta}{2},\mu)}$$
$$=\Big(\frac{1}{2}\Big)^{|\alpha|}\sum_{\beta+\gamma=\alpha}\frac{\alpha!}{\beta!\gamma!}\partial_\xi^\beta e^{\phi(\xi+\frac{\eta}{2},\mu)}(-\partial_\xi)^\gamma e^{-\phi(\xi-\frac{\eta}{2},\mu)}$$
$$=\Big(\frac{1}{2}\Big)^{|\alpha|}\alpha! H_\alpha(\xi,\eta,\mu).$$

Then the second term in the right-hand side yields

$$\Big(\frac{1}{2}\Big)^{N}\sum_{|\alpha|=N}N\int e^{-iy\eta}H_\alpha(\xi,\eta,\mu)dyd\eta\int_0^1(1-s)^{N-1}b_{(\alpha)}(x+sy,\xi,\mu)ds$$
$$=\Big(\frac{1}{2}\Big)^{N}\sum_{|\alpha|=N}N\int\int_0^1 e^{ix\eta}(1-s)^{N-1}H_\alpha(\xi,s\eta,\mu)d\eta ds\int e^{-iy\eta}b_{(\alpha)}(y,\xi,\mu)dy.$$

With $B_\alpha(\eta,\xi,\mu)=\int e^{-iy\eta}b_{(\alpha)}(y,\xi,\mu)dy$, one has

$$\Big(\frac{1}{2}\Big)^{N}\sum_{|\alpha|=N}N\int\int_0^1 e^{ix\eta}(1-s)^{N-1}H_\alpha(\xi,s\eta,\mu)B_\alpha(\eta,\xi,\mu)d\eta ds. \tag{3.9}$$

It is easy to see that

$$\partial_\eta^\alpha e^{\phi(\xi+\frac{\eta}{2},\mu)-\phi(\xi-\frac{\eta}{2},\mu)}\Big|_{\eta=0}$$

is a linear combination of terms of the form

$$\partial_\xi^{\alpha_1}\phi(\xi,\mu)\cdots\partial_\xi^{\alpha_s}\phi(\xi,\mu),\quad \sum\alpha_j=\alpha,\ |\alpha_j|\geq 1,$$

which are in $S(\langle\mu\xi\rangle^{j\kappa}\langle\xi\rangle_\mu^{-j}, |dx|^2 + \langle\xi\rangle_\mu^{-2}|d\xi|^2)$. Thus it is clear that

$$\begin{aligned} c_j = &\sum_{|\alpha|=j} \frac{1}{\alpha!}\partial_\eta^\alpha e^{\phi(\xi+\frac{\eta}{2},\mu)-\phi(\xi-\frac{\eta}{2},\mu)}\Big|_{\eta=0} b_{(\alpha)}(x,\xi,\mu) \\ &\in \mu^j S(m(\xi,\mu)\langle\mu\xi\rangle^{-j(1-\kappa-\sigma/2)}, \bar{g}). \end{aligned}$$

We turn to consider

$$R_N = \Big(\frac{1}{2}\Big)^N \sum_{|\alpha|=N} N \int\int_0^1 e^{ix\eta}(1-s)^{N-1} H_\alpha(\xi, s\eta, \mu) B_\alpha(\eta,\xi,\mu) d\eta ds.$$

Lemma 3.5. *One can choose M so that*

$$\begin{aligned} |\partial_\xi^\delta B_\alpha(\eta,\xi,\mu)| \le & C_{\alpha,\delta} m(\xi,\mu)(\langle\mu\xi\rangle^{-\sigma/2}\langle\xi\rangle_\mu)^{-|\delta|} e^{-c|\eta|^\kappa}, \quad |\eta| \ge M\langle\mu\xi\rangle^\sigma \\ |\partial_\xi^\delta B_\alpha(\eta,\xi,\mu)| \le & C_{\alpha,\delta} m(\xi,\mu)(\langle\mu\xi\rangle^{-\sigma/2}\langle\xi\rangle_\mu)^{-|\delta|} \\ & \times \langle\mu\xi\rangle^{|\alpha|\sigma/2} e^{-c(\langle\mu\xi\rangle^{-\sigma/2}|\eta|)^\kappa}, \quad |\eta| \le M\langle\mu\xi\rangle^\sigma \end{aligned}$$

with some $c > 0$.

Proof. Recall that

$$\eta^\nu \partial_\xi^\delta B_\alpha(\eta,\xi,\mu) = \int e^{-iy\eta}\partial_\xi^\delta b_{(\alpha+\nu)}(y,\xi,\mu) dy$$

hence one gets

$$\begin{aligned} |\partial_\xi^\delta B_\alpha(\eta,\xi,\mu)| \le & C_\delta m(\xi,\mu)(\langle\mu\xi\rangle^{-\sigma/2}\langle\xi\rangle_\mu)^{-|\delta|} \\ & \times A^{|\alpha+\nu|}|\alpha+\nu|!^{s/2}(|\alpha+\nu|^{s/2} + \langle\mu\xi\rangle^{\sigma/2})^{|\alpha+\nu|}|\eta|^{-|\nu|} \\ \le & C_\delta m(\xi,\mu)(\langle\mu\xi\rangle^{-\sigma/2}\langle\xi\rangle_\mu)^{-|\delta|} C_1 C_2^{|\nu|}|\nu|!^{s/2} \\ & \times (|\nu|^{s/2} + \langle\mu\xi\rangle^{\sigma/2})^{|\alpha+\nu|}|\eta|^{-|\nu|}, \end{aligned}$$

where $c_i = c_i(|\alpha|)$. We minimize $C_2^{|\nu|}|\nu|!^{s/2}(|\nu|^{s/2} + \langle\mu\xi\rangle^{\sigma/2})^{|\nu|}|\eta|^{-|\nu|}$. Note that

$$C_2^{|\nu|}|\nu|!^{s/2}(|\nu|^{s/2} + \langle\mu\xi\rangle^{\sigma/2})^{|\nu|} \le (2C_2)^{|\nu|}(|\nu|^s + \langle\mu\xi\rangle^\sigma)^{|\nu|}.$$

Choose ν so that $|\nu| = [e^{-s}(2C_2)^{-1}|\eta| - \langle\mu\xi\rangle^\sigma]^{1/s}$, assuming that

$$|\eta| \ge 4C_2 e^s \langle\mu\xi\rangle^\sigma.$$

Then we see that we have

$$(2C_2)^{|\nu|}(|\nu|^s + \langle\mu\xi\rangle^\sigma)^{|\nu|}|\eta|^{-|\nu|} \le e^{-c|\eta|^{1/s}}$$

with some $c > 0$, so we may conclude that

$$\begin{aligned}|\partial_\xi^\delta B_\alpha(\eta,\xi,\mu)| \le& C_{\alpha,\delta} m(\xi,\mu)(\langle\mu\xi\rangle^{-\sigma/2}\langle\xi\rangle_\mu)^{-|\delta|}(C|\eta|^{1/2}+\langle\mu\xi\rangle^{\sigma/2})^{|\alpha|}e^{-c|\eta|^\kappa}\\ \le& C_{\alpha,\delta} m(\xi,\mu)(\langle\mu\xi\rangle^{-\sigma/2}\langle\xi\rangle_\mu)^{-|\delta|}e^{-c'|\eta|^\kappa}\end{aligned}$$

with some $c' > 0$.

We now consider the case $|\eta| \le 4C_2 e^s\langle\mu\xi\rangle^\sigma$. Note that

$$C_2^{|\nu|}|\nu|!^{s/2}(|\nu|^{s/2}+\langle\mu\xi\rangle^{\sigma/2})^{|\nu|} \le (2C_2)^{|\nu|}(|\nu|^s\langle\mu\xi\rangle^{\sigma/2})^{|\nu|}.$$

Choose ν so that $|\nu| = e^{-1}(2C_2)^{-1/s}(|\eta|\langle\mu\xi\rangle^{-\sigma/2})^{1/s}$; then

$$(2C_2)^{|\nu|}(|\nu|^s\langle\mu\xi\rangle^{\sigma/2})^{|\nu|}|\eta|^{-|\nu|} \le e^{-c(|\eta|\langle\mu\xi\rangle^{-\sigma/2})^{1/s}}$$

with some $c > 0$. Therefore,

$$\begin{aligned}|\partial_\xi^\delta B_\alpha(\eta,\xi,\mu)| \le& C_{\alpha,\delta} m(\xi,\mu)(\langle\mu\xi\rangle^{-\sigma/2}\langle\xi\rangle_\mu)^{-|\delta|}\\ &\times (C(|\eta|\langle\mu\xi\rangle^{-\sigma/2})^{1/2}+\langle\mu\xi\rangle^{\sigma/2})^{|\alpha|}e^{-c(|\eta|\langle\mu\xi\rangle^{-\sigma/2})^\kappa}\\ \le& C_{\alpha,\delta} m(\xi,\mu)(\langle\mu\xi\rangle^{-\sigma/2}\langle\xi\rangle_\mu)^{-|\delta|}\langle\mu\xi\rangle^{\sigma|\alpha|/2}e^{-c'(|\eta|\langle\mu\xi\rangle^{-\sigma/2})^\kappa}\end{aligned}$$

with some $c' > 0$. □

Note that $H_\alpha(\xi,\eta,\mu)$ is a linear combination of terms of the form

$$\begin{aligned}&\partial_\xi^{\beta_1}\phi(\xi+\eta/2,\mu)\cdots\partial_\xi^{\beta_s}\phi(\xi+\eta/2,\mu)\partial_\xi^{\gamma_1}\phi(\xi-\eta/2,\mu)\cdots\partial_\xi^{\gamma_t}\phi(\xi-\eta/2,\mu)\\ &\quad\times e^{\phi(\xi+\eta/2,\mu)-\phi(\xi-\eta/2,\mu)} = h_{\beta_1,\dots,\beta_s,\gamma_1,\dots,\gamma_t}(\xi,\eta,\mu)e^{\phi(\xi+\eta/2,\mu)-\phi(\xi-\eta/2,\mu)},\end{aligned}$$

where $\sum\beta_j = \beta$, $\sum\gamma_j = \gamma$ and $|\beta_j| \ge 1$, $|\gamma_j| \ge 1$, $\beta+\gamma = \alpha$. It is easy to verify that

$$|\partial_\xi^\delta h_{\beta_1,\dots,\beta_s,\gamma_1,\dots,\gamma_t}(\xi,\eta,\mu)| \le C_\delta \mu^{|\alpha|}\langle\mu\xi\rangle^{-|\alpha|(1-\kappa)}\langle\xi\rangle_\mu^{-|\delta|}\langle\mu\eta\rangle^{|\alpha|+|\delta|}. \tag{3.10}$$

On the other hand, noting that

$$\begin{aligned}&\partial_\xi^\alpha\phi(\xi+\eta/2,\mu)-\partial_\xi^\alpha\phi(\xi-\eta/2,\mu)\\ &\qquad = \sum_{k=1}^n \frac{1}{2}\eta_k[\partial_\xi^\alpha\partial_{\xi_k}\phi(\xi+(\theta\eta)/2,\mu)+\partial_\xi^\alpha\partial_{\xi_k}\phi(\xi-(\theta\eta)/2,\mu)],\end{aligned} \tag{3.11}$$

we see that

$$|\partial_\xi^\delta e^{\phi(\xi+\eta/2,\mu)-\phi(\xi-\eta/2,\mu)}| \le C_\delta\langle\xi\rangle_\mu^{-|\delta|}\langle\mu\eta\rangle^{|\delta|(3-\kappa)}e^{C\langle\mu\eta\rangle^\kappa}. \tag{3.12}$$

From Lemma 3.5, (3.10) and (3.12), it follows that for $|\eta| \ge M\langle\mu\xi\rangle^\sigma$, $|\alpha| = N$,

$$|\partial_\xi^\delta(H_\alpha(\xi,\eta,\mu)B_\alpha(\eta,\xi,\mu))| \le C_\delta\mu^N m(\xi,\mu)(\langle\mu\xi\rangle^{-\sigma/2}\langle\xi\rangle_\mu)^{-|\delta|}\langle\mu\xi\rangle^{-N(1-\kappa)}e^{-c''\langle\mu\eta\rangle^\kappa}$$

with some $c'' > 0$. We consider the case $|\eta| \leq M\langle \mu\xi\rangle^{\sigma}$. Since $\sigma < 1$, taking μ small, we have $\langle \mu(\xi+\theta\eta)\rangle \sim \langle \mu\xi\rangle$ and $\langle \xi+\theta\eta\rangle_{\mu} \sim \langle \xi\rangle_{\mu}$, where $|\theta| \leq 1$; hence

$$|\partial_{\xi}^{\delta} h_{\beta_1,\ldots,\beta_s,\gamma_1,\ldots,\gamma_t}(\xi,\eta,\mu)| \leq C_{\delta}\mu^{|\alpha|}\langle \mu\xi\rangle^{-|\alpha|(1-\kappa)}\langle \xi\rangle_{\mu}^{-|\delta|}.$$

On the other hand, since $\kappa+\sigma \leq 1$, one can see from (3.11) and $\langle \mu\eta\rangle \leq C\langle \mu\xi\rangle^{\sigma}$ that

$$|\phi(\xi+\eta/2,\mu) - \phi(\xi-\eta/2,\mu)| \leq C\langle \mu\eta\rangle\langle \mu\xi\rangle^{\kappa-1} \leq C'.$$

Then we obtain

$$|\partial_{\xi}^{\delta} e^{\phi(\xi+\eta/2,\mu)-\phi(\xi-\eta/2,\mu)}| \leq C_{\delta}\langle \xi\rangle_{\mu}^{-|\delta|},$$

again using (3.11). Thus we conclude

$$\begin{aligned}&|\partial_{\xi}^{\delta}(H_{\alpha}(\xi,\eta,\mu)B_{\alpha}(\eta,\xi,\mu))| \\ &\qquad \leq C_{\delta}\mu^{N} m(\xi,\mu)(\langle \mu\xi\rangle^{-\sigma/2}\langle \xi\rangle_{\mu})^{-|\delta|}\langle \mu\xi\rangle^{N\sigma/2-N(1-\kappa)} e^{-c''(|\eta|\langle \mu\xi\rangle^{-\sigma/2})^{\kappa}}\end{aligned}$$

with some $c'' > 0$. Since

$$|\eta^{\beta} e^{-c''(|\eta|\langle \mu\xi\rangle^{-\sigma/2})^{\kappa}}| \leq C_{\beta}\langle \mu\xi\rangle^{\sigma|\beta|/2} e^{-c'''(|\eta|\langle \mu\xi\rangle^{-\sigma/2})^{\kappa}},$$

we have

$$\begin{aligned}&|\partial_x^{\alpha}\partial_{\xi}^{\delta} R_N(x,\xi,\mu)| \\ &\qquad \leq C_{\alpha,\delta}\mu^{N} m(\xi,\mu)\langle \mu\xi\rangle^{-N(1-\kappa-\sigma/2)}\langle \mu\xi\rangle^{\sigma n/2}\langle \mu\xi\rangle^{\sigma|\alpha|/2}(\langle \mu\xi\rangle^{\sigma/2}\langle \xi\rangle_{\mu}^{-1})^{|\delta|},\end{aligned}$$

because $\int e^{-c'''(|\eta|\langle \mu\xi\rangle^{-\sigma/2})^{\kappa}} d\eta \leq C\langle \mu\xi\rangle^{n\sigma/2}$. That is,

$$R_N \in \mu^{N} S(m(\xi,\mu)\langle \mu\xi\rangle^{-N(1-\kappa-\sigma/2)+n\sigma/2}, \bar{g}).$$

This ends the proof. □

4 A priori estimate in the Gevrey class

In Subsection 4.1, we prepare a precise energy identity for pseudodifferential operators which are obtained by conjugation with operators of the type $e^{\pm\phi(D)}$. In Subsections 4.2 and 4.3, we derive a priori estimates in the Gevrey class assuming that (2.9) is globally satisfied; this gives the proof of Theorem 1.1. (The final step based on standard functional analytical arguments is omitted.) Since routine arguments of partition of unity allow us to reduce our estimate to a global one, we also skip this standard step.

4.1 Energy equality.

Let

$$P = -M\Lambda + Q,$$

where $\Lambda = D_0 - iA - \lambda$ and $A = \langle \mu D' \rangle^\kappa$. We compute

$$2\mathrm{Im}(Pv, \Lambda v) = -2\mathrm{Im}(M\Lambda v, \Lambda v) + 2\mathrm{Im}(Qv, \Lambda v) = I_1 + I_2.$$

Now $M = D_0 - iA - m$; therefore, with $w = \Lambda v$,

$$I_1 = -2\mathrm{Im}(D_0 w, w) + 2\mathrm{Re}(Aw, w) + 2\mathrm{Im}(mw, w),$$

so that

$$I_1 = \frac{d}{dx_0}||\Lambda v||^2 + 2||A^{1/2}\Lambda v||^2 + 2(\mathrm{Im} m\Lambda v, \Lambda v). \tag{4.1}$$

We now consider $I_2 = 2\mathrm{Im}(Qv, \Lambda v)$;

$$I_2 = 2\mathrm{Im}(Qv, D_0 v) + 2\mathrm{Im}(Qv, -iAv) + 2\mathrm{Im}(Qv, -\lambda v) = I_{2,1} + I_{2,2} + I_{2,3}.$$

We claim that

$$\begin{aligned} I_{2,1} =& \frac{d}{dx_0}(\mathrm{Re}Qv, v) + \mathrm{Im}([D_0, \mathrm{Re}Q]v, v) \\ &+ 2\mathrm{Re}(\Lambda v, \mathrm{Im}Qv) + 2\mathrm{Re}(iAv, \mathrm{Im}Qv) + 2\mathrm{Re}(\lambda v, \mathrm{Im}Qv). \end{aligned} \tag{4.2}$$

First, it is clear that

$$(Qv, D_0 v) = -D_0(Qv, v) + (QD_0 v, v) + ([D_0, Q]v, v).$$

Then

$$\begin{aligned} &2\mathrm{Im}(Qv, D_0 v) \\ &\quad = 2\mathrm{Im}(-D_0(Qv, v) + (QD_0 v, v) + ([D_0 v, Q]v, v)) \\ &\quad = 2\mathrm{Im}(i\frac{d}{dx_0}(Qv, v)) + 2\mathrm{Im}(D_0 v, Q^* v) + 2\mathrm{Im}([D_0, Q]v, v) \\ &\quad = 2\mathrm{Re}(\frac{d}{dx_0}(Qv, v)) + 2\mathrm{Im}(D_0 v, (Q^* - Q)v) + 2\mathrm{Im}(D_0 v, Qv) + 2\mathrm{Im}([D_0, Q]v, v) \\ &\quad = 2\frac{d}{dx_0}(\mathrm{Re}Qv, v) + 2\mathrm{Im}(D_0 v, (Q^* - Q)v) + 2\mathrm{Im}(D_0 v, Qv) + 2\mathrm{Im}([D_0, Q]v, v). \end{aligned}$$

Therefore,

$$2\mathrm{Im}(Qv, D_0 v) = \frac{d}{dx_0}(\mathrm{Re}Qv, v) + \mathrm{Im}([D_0, Q]v, v) + \mathrm{Im}(D_0 v, (Q^* - Q)v).$$

Now

$$\mathsf{Im}([D_0, Q]v, v) = \mathsf{Im}([D_0, \mathsf{Re}Q]v, v) + \mathsf{Im}(i[D_0, \mathsf{Im}Q]v, v) = \mathsf{Im}([D_0, \mathsf{Re}Q]v, v)$$

and

$$\mathsf{Im}(D_0 v, (Q^* - Q)v) = \mathsf{Im}(D_0 v, -2i\mathsf{Im}Qv) = 2\mathsf{Re}(D_0 v, \mathsf{Im}Qv).$$

Thus

$$2\mathsf{Im}(Qv, D_0 v) = \frac{d}{dx_0}(\mathsf{Re}Qv, v) + \mathsf{Im}([D_0, \mathsf{Re}Q]v, v) + 2\mathsf{Re}(D_0 v, \mathsf{Im}Qv).$$

Since $D_0 = \Lambda + iA + \lambda$, we have

$$\begin{aligned} I_{2,1} =& \frac{d}{dx_0}(\mathsf{Re}Qv, v) + \mathsf{Im}([D_0, \mathsf{Re}Q]v, v) + 2\mathsf{Re}(\Lambda v, \mathsf{Im}Qv) \\ &+ 2\mathsf{Re}(iAv, \mathsf{Im}Qv) + 2\mathsf{Re}(\lambda v, \mathsf{Im}Qv), \end{aligned} \tag{4.3}$$

which is (4.2).

Let us now consider

$$I_{2,2} = 2\mathsf{Im}(Qv, -iAv) = -2\mathsf{Im}(Qv, iAv) = 2\mathsf{Re}(Qv, Av).$$

Note that

$$2\mathsf{Re}(Qv, Av) = 2[\mathsf{Re}(A\mathsf{Re}Qv, v) - \mathsf{Im}(\mathsf{Im}Qv, Av)];$$

this is just a straightforward verification. Then

$$\begin{aligned} I_{2,2} &= 2\mathsf{Re}(A\mathsf{Re}Qv, v) - 2\mathsf{Im}(\mathsf{Im}Qv, Av) \\ &= 2\mathsf{Re}(A\mathsf{Re}Qv, v) + 2\mathsf{Im}(Av, \mathsf{Im}Qv). \end{aligned} \tag{4.4}$$

The last summand in (4.4) cancels out the fourth term in (4.3), i.e.,

$$2\mathsf{Re}(iAv, \mathsf{Im}Qv) = -2\mathsf{Im}(Av, \mathsf{Im}Qv).$$

We now consider

$$I_{2,3} = 2\mathsf{Im}(Qv, -\lambda v) = -2\mathsf{Re}(\mathsf{Im}Qv, \lambda v) - 2\mathsf{Im}(\mathsf{Re}Qv, \lambda v), \tag{4.5}$$

and note that the first summand in (4.5) cancels out the fifth term in (4.3). So finally

$$\begin{aligned} I_2 =& \frac{d}{dx_0}(\mathsf{Re}Qv, v) + \mathsf{Im}([D_0, \mathsf{Re}Q]v, v) + 2\mathsf{Re}(\Lambda v, \mathsf{Im}Qv) \\ &+ 2\mathsf{Re}(A\mathsf{Re}Qv, v) - 2\mathsf{Im}(\mathsf{Re}Qv, \lambda v). \end{aligned} \tag{4.6}$$

We now observe that

$$\begin{aligned}2\mathsf{Im}(\mathsf{Re}Qv,\lambda v) &= 2\mathsf{Im}(\mathsf{Re}Qv,\mathsf{Re}\lambda v) - 2\mathsf{Re}(\mathsf{Re}Qv,\mathsf{Im}\lambda v)\\ &= \Big(\frac{1}{i}[\mathsf{Re}\lambda,\mathsf{Re}Q]v,v\Big) - 2\mathsf{Re}(\mathsf{Re}Qv,\mathsf{Im}\lambda v)\\ &= \mathsf{Re}\Big(\frac{1}{i}[\mathsf{Re}\lambda,\mathsf{Re}Q]v,v\Big) - 2\mathsf{Re}(\mathsf{Re}Qv,\mathsf{Im}\lambda v),\end{aligned}$$

since $\frac{1}{i}[\mathsf{Re}\lambda,\mathsf{Re}Q]$ is self-adjoint. Going back to (4.6), we have

$$\begin{aligned}I_2 =&\frac{d}{dx_0}(\mathsf{Re}Qv,v) + \mathsf{Im}([D_0-\mathsf{Re}\lambda,\mathsf{Re}Q]v,v) + 2\mathsf{Re}(\Lambda v,\mathsf{Im}Qv)\\ &+ 2\mathsf{Re}(A\mathsf{Re}Qv,v) + 2\mathsf{Re}(\mathsf{Re}Qv,\mathsf{Im}\lambda v).\end{aligned}$$

Let us now add I_1 and summarize

Proposition 4.1. *We have*

$$\begin{aligned}2\mathsf{Im}(Pv,\Lambda v) =&\frac{d}{dx_0}(\|\Lambda v\|^2 + (\mathsf{Re}Qv,v)) + 2\|A^{1/2}\Lambda v\| + 2\mathsf{Re}(A\mathsf{Re}Qv,v)\\ &+ 2(\mathsf{Im}m\Lambda v,\Lambda v) + 2\mathsf{Re}(\Lambda v,\mathsf{Im}Qv)\\ &+ \mathsf{Im}([D_0-\mathsf{Re}\lambda,\mathsf{Re}Q]v,v) + 2\mathsf{Re}(\mathsf{Re}Qv,\mathsf{Im}\lambda v).\end{aligned}$$

From (4.1), we have

$$-2\mathsf{Im}(\Lambda w,w) = \frac{d}{dx_0}\|w\|^2 + 2\|A^{1/2}w\|^2 + 2(\mathsf{Im}\lambda w,w).$$

From this, it follows that with small $\alpha>0$,

$$\alpha^{-1}\|A^{-1/2}\Lambda Aw\|^2 \geq \frac{d}{dx_0}\|Aw\|^2 + (2-\alpha)\|A^{3/2}w\|^2 + 2(\mathsf{Im}\lambda Aw,Aw). \tag{4.7}$$

4.2 Energy inequality (Preparations). We start with

$$p = -(\xi_0+\phi_1)(\xi_0-\phi_1) + \sum_{j=2}^{r}\phi_j^2,$$

where we assume that our assumptions are satisfied globally;

$$\{\xi_0+\phi_1,\phi_j\} = \sum_{k=1}^{r} C_{jk}\phi_k,\quad j=1,\ldots,r,\quad P_{sub} = \sum_{k=0}^{r} C_k\phi_k \tag{4.8}$$

where $C_{jk}(x,\xi')$, $C_k(x,\xi')$ are homogeneous of degree 0, $\phi_0=\xi_0$ and

$$|\{\phi_1,\phi_2\}| \geq c|\xi'| \tag{4.9}$$

with some $c > 0$. We now make a dilation of the variable: $x_0 \to \mu x_0$ so that

$$\mu^2 p(x,\xi,\mu) = -(\xi_0 - \phi_1(\mu x_0, x', \mu\xi'))(\xi_0 + \phi_1(\mu x_0, x', \mu\xi')) + \sum_{j=2}^{r} \phi_j(\mu x_0, x', \mu\xi')^2.$$

For simplicity, we write $\phi_j(\mu x_0, x', \mu\xi')$ as $\phi_j(x,\xi',\mu)$ or sometimes $\phi_j(x,\xi')$. Then (4.8) and (4.9) become

$$\{\xi_0 + \phi_1, \phi_j\} = \mu \sum_{k=1}^{r} C_{jk}\phi_k,\ |\{\phi_1,\phi_2\}| \ge c\mu|\mu\xi'|,\ P_{sub} = \mu\sum_{k=0}^{r} C_k\phi_k \tag{4.10}$$

for $j = 1,\dots,r$. Let g_0 be the standard $S_{1,0}$ metric with small parameter $\mu > 0$:

$$g_0 = \mu^2 dx_0^2 + |dx'|^2 + \langle\xi'\rangle_\mu^{-2}|d\xi'|^2, \quad \langle\xi'\rangle_\mu = (\mu^{-2} + |\xi'|^2)^{1/2}.$$

Let $\chi(s) = 0$ near 0 and identically 1 outside $|s| \ge 2$. We use $\chi(\mu|\xi'|)$ as a cutoff function and consider $\phi(x,\xi',\mu)\chi(\mu|\xi'|)$.

Lemma 4.1. *Let $\phi(x,\xi')$ be homogeneous of degree k in ξ'. Then*

$$\phi(\mu x_0, x', \mu\xi')\chi(\mu|\xi'|) \in S(\langle\mu\xi'\rangle^k, g_0).$$

Proof. Note that

$$\phi(x,\xi')\chi(|\xi'|) \in S(\langle\xi'\rangle^k, |dx|^2 + \langle\xi'\rangle^{-2}|d\xi'|^2) \quad \text{and} \quad a(\mu x_0, x', \mu\xi') \in S(\langle\mu\xi'\rangle^k, g_0)$$

if $a(x,\xi') \in S(\langle\xi'\rangle^k, |dx|^2 + \langle\xi'\rangle^{-2}|d\xi'|^2)$. □

Let us set

$$w = \sqrt{\langle\mu\xi'\rangle^{-2}\phi_1^2 + \langle\mu\xi'\rangle^{-4/5}}, \quad \Phi = \sqrt{1 - aw} \tag{4.11}$$

with some constant $a > 0$ so that $1 - aw \ge a_1 > 0$. We introduce the metric

$$g = w^{-2}g_0 = w^{-2}\big\{\mu^2 dx_0^2 + |dx'|^2 + \langle\xi'\rangle_\mu^{-2}|d\xi'|^2\big\},$$

which is used throughout this section. Note that

$$w \in S(w,g), \quad \phi_1 \in S(\langle\mu\xi'\rangle w, g).$$

We rewrite p as

$$p = -(\xi_0 - \phi_1\Phi)(\xi_0 + \phi_1\Phi) + \sum_{j=2}^{r}\phi_j^2 + aw\phi_1^2 \tag{4.12}$$

because $1 - \Phi^2 = aw$. Note also that

$$\xi_0 + \phi_1\Phi = \xi_0 + \phi_1 + \phi_1(\Phi - 1) = \xi_0 + \phi_1 - \phi_1\psi,$$

where

$$\psi = 1 - \Phi = \frac{aw}{1 + \sqrt{1 - aw}} \in S(w,g).$$

Lemma 4.2. *We have*

$$\phi_1\psi \in S(w^2\langle\mu\xi'\rangle, g), \quad \phi_1(1+\psi) \in S(w\langle\mu\xi'\rangle, g),$$
$$\partial_x^\alpha(\phi_1\Phi) \in S(\langle\mu\xi'\rangle, g), \quad |\alpha| = 2.$$

Proof.

The first two assertions are clear. To check the third, it suffices to note that $\partial_x^\alpha\Phi \in S(1,g)$ for $|\alpha| = 1$. □

To simplify notation, in what follows we set

$$\kappa = 1/5, \quad \delta = 4/5$$

and assume that $a = 1$ without restrictions. Since $w \geq \langle\mu\xi'\rangle^{-\delta/2}$, we have $w^{-1/2} \leq \langle\mu\xi'\rangle^{\delta/4} = \langle\mu\xi'\rangle^{1/5} = \langle\mu\xi'\rangle^{\kappa}$, so that $w^{-1/2} \in S(\langle\mu\xi'\rangle^{\kappa}, g)$.

Thanks to Lemma 4.2 and (4.10), we have

$$(\xi_0 - \phi_1\Phi)\#(\xi_0 + \phi_1\Phi) = \xi_0^2 - \phi_1^2\Phi^2 + \mu\sum_{j=1}^{r} c_j\phi_j + \mu^2 S(1,g)$$

with $c_j \in S(1,g)$. Noting that $P = (p + P_{sub})^w + \mu^2 S(1,g_0)$ and (4.10), we write

$$P = -M\Lambda + \left[\sum_{j=2}^{r}\phi_j^2 + w\phi_1^2 + R\right]^w + \mu^2 S(1,g),$$

where $M = (\xi_0 - \phi_1\Phi)^w$, $\Lambda = (\xi_0 + \phi_1\Phi)^w$ and $R = \mu\sum_{j=0}^{r} c_j\phi_j$, $c_j \in S(1,g)$. We rewrite $c_0\phi_0$ as

$$c_0\xi_0 = c_0\#(\xi_0 + \phi_1\Phi) - c_0\phi_1\Phi + \mu^2 S(\langle\mu\xi'\rangle^{2\kappa}, \bar{g}),$$

which gives

$$P = -M\Lambda + C\Lambda + Q, \quad Q = \left[\sum_{j=2}^{r}\phi_j^2 + w\phi_1^2 + R\right]^w + \mu^2 S(\langle\mu\xi'\rangle^{2\kappa}, \bar{g}), \tag{4.13}$$

where $C \in \mu S(1,g)$ and $R = \sum_{j=1}^{r} c_j\phi_j$, $c_j \in \mu S(1,g)$.

Lemma 4.3. *Let $a \in S(m_1, g)$ and $b \in S(m_2, g_0)$. Then (cf., e.g., Theorem 18.5.5 [7]) we have*

(i) $a\#a - a^2 \in \mu^2 S(m_1^2 w^{-4}\langle\mu\xi'\rangle^{-2}, g)$

(ii) $a\#b - b\#a - \frac{1}{i}\{a,b\} \in \mu^3 S(m_1 m_2 w^{-3}\langle\mu\xi'\rangle^{-3}, g)$

(iii) $a\#b + b\#a - 2ab \in \mu^2 S(m_1 m_2 w^{-2}\langle\mu\xi'\rangle^{-2}, g)$.

Corollary 4.1. *Let $a \in S(m_1, g)$ and $b \in S(m_2, g_0)$ be real. Then*

$$([ab]^w u, u) = \mathsf{Re}(b^w u, a^w u) + (T^w u, u)$$

with $T \in \mu^2 S(m_1 m_2 w^{-2} \langle \mu\xi' \rangle^{-2}, g)$.

Lemma 4.4. *Let $a \in \mu S(1, g)$. Then*

$$\mathsf{Re}([a\phi_1^2 w]^w u, u) \le C\mu \mathsf{Re}([\phi_1^2 w]^w u, u) + C\mu^3 \|\langle \mu D' \rangle^{\kappa} u\|^2,$$
$$\mathsf{Re}([a\phi_j^2]^w u, u) \le C\mu \mathsf{Re}([\phi_j^2]^w u, u) + C\mu^3 \|\langle \mu D' \rangle^{2\kappa} u\|^2, \quad j \ge 2.$$

Let $a \in \mu S(\langle \mu\xi' \rangle^{\kappa}, g)$. Then

$$\mathsf{Re}([a\phi_1^2 w]^w u, u) \le C\mu \mathsf{Re}([\langle \mu\xi' \rangle^{\kappa} \phi_1^2 w]^w u, u) + C\mu^3 \|\langle \mu D' \rangle^{3\kappa/2} u\|^2,$$
$$\mathsf{Re}([a\phi_j^2]^w u, u) \le C\mu \mathsf{Re}([\phi_j^2 \langle \mu\xi' \rangle^{\kappa}]^w u, u) + C\mu^3 \|\langle \mu D' \rangle^{1/2} u\|^2, \quad j \ge 2.$$

Let $a \in \mu S(1, g)$ then

$$\|[a\langle \mu\xi' \rangle^{\kappa/2} \phi_j]^w u\|^2 \le C\mu^2 \mathsf{Re}([\langle \mu\xi' \rangle^{\kappa} \phi_j^2]^w u, u) + C\mu^4 \|\langle \mu D' \rangle^{1/2} u\|^2, \quad j \ge 2$$
$$\|[a\langle \mu\xi' \rangle^{\kappa/2} \sqrt{w}\phi_1]^w u\|^2 \le C\mu^2 \mathsf{Re}([\langle \mu\xi' \rangle^{\kappa} w\phi_1^2]^w u, u) + C\mu^4 \|\langle \mu D' \rangle^{3\kappa/2} u\|^2.$$

Proof. We first prove the third and fourth assertions. Since $\mathsf{Re}(P^w u, u) = ([\mathsf{Re} P]^w u, u)$, we may assume that a is real. Consider

$$\begin{aligned} C\mu\langle \mu\xi' \rangle^{\kappa} \phi_1^2 w - a\phi_1^2 w =& C\mu\langle \mu\xi' \rangle^{\kappa} \phi_1^2 w\{1 - C^{-1}\mu^{-1} a \langle \mu\xi' \rangle^{-\kappa}\} \\ =& C\mu\langle \mu\xi' \rangle^{\kappa} \phi_1^2 w\psi^2 \\ =& C\mu\langle \mu\xi' \rangle^{\kappa/2} \phi_1 \sqrt{w}\psi \# \langle \mu\xi' \rangle^{\kappa/2} \phi_1 \sqrt{w}\psi + \mu^3 S(\langle \mu\xi' \rangle^{3\kappa}, g), \end{aligned}$$

where $\psi = \sqrt{1 - C^{-1}\mu^{-1} a \langle \mu\xi' \rangle^{-\kappa}} \in S(1, g)$. Then

$$C\mu \mathsf{Re}([\langle \mu\xi' \rangle^{\kappa} \phi_1^2 w]^w u, u) - \mathsf{Re}([a\phi_1^2 w]^w u, u) \ge -C\mu^3 \|\langle \mu D' \rangle^{3\kappa/2} u\|^2,$$

which proves the third assertion. From Corollary 4.1, we have

$$([a\phi_j^2]^w u, u) = \mathsf{Re}([\mu^{1/2} \langle \mu\xi' \rangle^{\kappa/2} \phi_j]^w u, [\mu^{-1/2} \langle \mu\xi' \rangle^{-\kappa/2} a\phi_j]^w u) + (T^w u, u)$$

with $T \in \mu^3 S(\langle \mu\xi' \rangle^{\kappa} w^{-2}, g) \subset \mu^3 S(\langle \mu\xi' \rangle, g)$. Since one can write

$$\mu^{-1/2} \langle \mu\xi' \rangle^{-\kappa/2} a\phi_j = \mu^{-1/2} a \langle \mu\xi' \rangle^{-\kappa} \# \langle \mu\xi' \rangle^{\kappa/2} \phi_j + \mu^{3/2} S(\langle \mu\xi' \rangle^{\kappa/2} w^{-1}, g)$$

and $S(\langle \mu\xi' \rangle^{\kappa/2} w^{-1}, g) \subset S(\langle \mu\xi' \rangle^{1/2}, g)$, we have

$$\|[\mu^{-1/2} \langle \mu\xi' \rangle^{-\kappa/2} a\phi_j]^w u\|^2 \le C\mu \|[\langle \mu\xi' \rangle^{\kappa/2} \phi_j]^w u\|^2 + C\mu^3 \|\langle \mu D' \rangle^{1/2} u\|^2.$$

This proves the fourth assertion. The first two assertions are proved repeating the same arguments.

We now turn to the last two assertions. Observe that

$$a\langle\mu\xi'\rangle^{\kappa/2}\phi_j = a\#\langle\mu\xi'\rangle^{\kappa/2}\phi_j + \mu^2 S(\langle\mu\xi'\rangle^{1/2}, g)$$

and hence

$$\|[a\langle\mu\xi'\rangle^{\kappa/2}\phi_j]^w u\|^2 \le C\mu^2\|[\langle\mu\xi'\rangle^{\kappa/2}\phi_j]^w u\|^2 + C\mu^4\|\langle\mu D'\rangle^{1/2}u\|^2.$$

Since $\langle\mu\xi'\rangle^{\kappa/2}\phi_j\#\langle\mu\xi'\rangle^{\kappa/2}\phi_j = \langle\mu\xi'\rangle^{\kappa}\phi_j^2 + \mu^2 S(\langle\mu\xi'\rangle^{\kappa}, g)$, we get the fifth assertion. Finally, note that

$$a\sqrt{w}\langle\mu\xi'\rangle^{\kappa/2}\phi_1\# a\sqrt{w}\langle\mu\xi'\rangle^{\kappa/2}\phi_1 = a^2 w\langle\mu\xi'\rangle^{\kappa}\phi_1^2 + \mu^4 S(\langle\mu\xi'\rangle^{3\kappa}, g),$$

which, together with the fourth assertion, proves the last assertion. □

Lemma 4.5. *For some $c > 0$,*

$$\begin{aligned} &c([\mu\langle\mu\xi'\rangle^{1+\kappa}\sqrt{w}]^w u, u) \\ &\qquad \le \mathsf{Re}([\langle\mu\xi'\rangle^{\kappa}\phi_1^2 w]^w u, u) + \mathsf{Re}([\langle\mu\xi'\rangle^{\kappa}\phi_2^2]^w u, u) + C\mu\|\langle\mu D'\rangle^{3\kappa/2}u\|^2 \end{aligned}$$

Proof. Put $A = \langle\mu\xi'\rangle^{\kappa/2}\phi_1\sqrt{w}$, $B = \langle\mu\xi'\rangle^{\kappa/2}\phi_2$ and note that

$$A \in S(\langle\mu\xi'\rangle^{1+\kappa/2}w^{3/2}, g), \quad B \in S(\langle\mu\xi\rangle^{1+\kappa/2}, g_0).$$

Note that $|([A^w, B^w]u, u)| \le ([A\#A]^w u, u) + ([B\#B]^w u, u)$ and recall

$$[A^w, B^w] = \frac{1}{i}\{A, B\}^w + \mu^3 S(\langle\mu\xi'\rangle^{-1+\kappa}w^{-3/2}, g).$$

Therefore,

$$i[A^w, B^w] = \{A, B\}^w + \mu^3 S(\langle\mu\xi'\rangle^{-\kappa}, g).$$

Let us consider the first term on the right-hand side

$$\begin{aligned} \{A, B\} =&\{\phi_1\sqrt{w}, \langle\mu\xi'\rangle^{\kappa/2}\}\phi_2\langle\mu\xi'\rangle^{\kappa/2} + \{\langle\mu\xi'\rangle^{\kappa/2}, \phi_2\}\sqrt{w}\phi_1\langle\mu\xi'\rangle^{\kappa/2} \\ &+ \{\sqrt{w}, \phi_2\}\phi_1\langle\mu\xi'\rangle^{\kappa} + \{\phi_1, \phi_2\}\sqrt{w}\langle\mu\xi'\rangle^{\kappa} \\ =&K_1 + K_2 + K_3 + K_4. \end{aligned}$$

Since $\{\phi_1\sqrt{w}, \langle\mu\xi'\rangle^{\kappa/2}\} \in \mu S(\langle\mu\xi'\rangle^{\kappa/2}\sqrt{w}, g)$, $\{\langle\mu\xi'\rangle^{\kappa/2}, \phi_2\} \in \mu S(\langle\mu\xi'\rangle^{\kappa/2}, g_0)$, the first and the second terms can be written as

$$T_1\# B + R_1, \quad T_2\# A + R_2$$

where $T_i \in \mu S(\langle \mu\xi' \rangle^{\kappa/2}, g)$ and $R_i \in \mu^2 S(\langle \mu\xi' \rangle^{2\kappa}, g)$. Then we have

$$\mathsf{Re}(K_1 u, u) \geq -C\mu\big[\|\langle \mu D' \rangle^{\kappa/2} u\|^2 + \|B^w u\|^2\big] - C\mu^2 \|\langle \mu D' \rangle^{\kappa} u\|^2$$
$$\mathsf{Re}(K_2 u, u) \geq -C\mu\big[\|\langle \mu D' \rangle^{\kappa/2} u\|^2 + \|A^w u\|^2\big] - C\mu^2 \|\langle \mu D' \rangle^{\kappa} u\|^2.$$

Consider K_3. Note that

$$\{\sqrt{w}, \phi_2\} = \frac{1}{4} w^{-3/2}\big(2\{\phi_1, \phi_2\}\phi_1 \langle \mu\xi' \rangle^{-2} + \{\langle \mu\xi' \rangle^{-2}, \phi_2\}\phi_1^2 + \{\langle \mu\xi' \rangle^{-\delta}, \phi_2\}\big).$$

Since $w^{-5/2}\{\phi_1, \phi_2\}\langle \mu\xi' \rangle^{-2} \in \mu S(1, g)$ and $w^{-5/2}\{\langle \mu\xi' \rangle^{-2}, \phi_2\}\phi_1 \in \mu S(1, g)$ and $w^{-3/2}\{\langle \mu\xi' \rangle^{-\delta}, \phi_2\} \in \mu S(w^{-3/2}\langle \mu\xi' \rangle^{-\delta}, g)$, one can write

$$K_3 = a\phi_1^2 w \langle \mu\xi' \rangle^{\kappa} + b\phi_1$$

with $a \in \mu S(1, g)$, $b \in \mu S(1, g)$. We first consider $\mathsf{Re}([b\phi_1]^w u, u)$. Note that

$$2\mathsf{Re}([b\phi_1]^w u, u) \geq -\mu^{-1}\|[\langle \mu\xi' \rangle^{-\kappa/2} \# b\phi_1]^w u\|^2 - \mu \|\langle \mu D' \rangle^{\kappa/2} u\|^2$$

and $\langle \mu\xi' \rangle^{-\kappa/2} \# b\phi_1 = b\langle \mu\xi' \rangle^{-\kappa/2}\phi_1 + \mu^2 S(\langle \mu\xi' \rangle^{-\kappa/2}, g)$. Remarking that

$$b\langle \mu\xi' \rangle^{-\kappa/2}\phi_1 \# b\langle \mu\xi' \rangle^{-\kappa/2}\phi_1 = b^2 \langle \mu\xi' \rangle^{-\kappa}\phi_1^2 + \mu^4 S(\langle \mu\xi' \rangle^{3\kappa}, g)$$

and $b^2\langle \mu\xi' \rangle^{-\kappa}\phi_1^2 = a\langle \mu\xi' \rangle^{\kappa} w \phi_1^2$ with $a \in \mu^2 S(1, g)$ and applying Lemma 4.4, we conclude that

$$2\mathsf{Re}([b\phi_1]^w u, u) \geq -C\mu \mathsf{Re}([\langle \mu\xi' \rangle^{\kappa} w \phi_1^2]^w u, u) - C\mu \|\langle \mu D' \rangle^{3\kappa/2} u\|^2.$$

From Lemma 4.4 again we see that

$$\mathsf{Re}(K_3^w u, u) \geq -C\mu \mathsf{Re}([\phi_1^2 w \langle \mu\xi' \rangle^{\kappa}]^w u, u) - C\mu \|\langle \mu D' \rangle^{3\kappa/2} u\|^2.$$

We turn to K_4. We may assume that

$$C\mu\langle \mu\xi' \rangle \geq \{\phi_1, \phi_2\} \geq c\mu \langle \mu\xi' \rangle$$

with some $C > 0$, $c > 0$. Let us write

$$\begin{aligned} CK_4 - \mu\sqrt{w}\langle \mu\xi' \rangle^{1+\kappa} &= C\sqrt{w}\langle \mu\xi' \rangle^{\kappa}\{\phi_1, \phi_2\}\big[1 - C^{-1}\mu\{\phi_1, \phi_2\}^{-1}\langle \mu\xi' \rangle\big] \\ &= \psi \# \psi + \mu^3 S(\langle \mu\xi' \rangle^{3\kappa}, g) \end{aligned}$$

with

$$\begin{aligned} \psi =& \sqrt{C} w^{1/4} \langle \mu\xi' \rangle^{\kappa/2} \{\phi_1, \phi_2\}^{1/2} \sqrt{1 - C^{-1}\mu\{\phi_1, \phi_2\}^{-1}\langle \mu\xi' \rangle} \\ &\in \sqrt{\mu} S(w^{1/4}\langle \mu\xi' \rangle^{1/2+\kappa/2}, g). \end{aligned}$$

This shows that

$$C\mathsf{Re}(K_4^w u, u) \geq \mathsf{Re}([\mu\sqrt{w}\langle\mu\xi'\rangle^{1+\kappa}]^w u, u) - C\mu^3\|\langle\mu D'\rangle^{3\kappa/2}u\|^2.$$

The above estimates show

$$\begin{aligned}\mathsf{Re}(\{A, B\}^w u, u) \geq & c\mathsf{Re}([\mu\sqrt{w}\langle\mu\xi'\rangle^{1+\kappa}]^w u, u)\\ & - C\mu\big[\|A^w u\|^2 + \|B^w u\|^2 + \|\langle\mu D'\rangle^{3\kappa/2}u\|^2\big]\\ & - C\mu\mathsf{Re}([\phi_1^2 w\langle\mu\xi'\rangle^{\kappa}]^w u, u).\end{aligned}$$

Let us turn to $A\#A$, $B\#B$. Since $A \in S(w^{3/2}\langle\mu\xi'\rangle^{1+\kappa/2}, g)$, $B \in S(\langle\mu\xi'\rangle^{1+\kappa/2}, g_0)$, we have

$$\begin{aligned}A\#A &= w\phi_1^2\langle\mu\xi'\rangle^{\kappa} + \mu^2 S(\langle\mu\xi'\rangle^{3\kappa}, g),\\ B\#B &= \phi_2^2\langle\mu\xi'\rangle^{\kappa} + \mu^2 S(\langle\mu\xi'\rangle^{\kappa}, g_0),\end{aligned}$$

and hence

$$\begin{aligned}\|A^w u\|^2 &= ([A\#A]^w u, u) \leq \mathsf{Re}([\phi_1^2 w\langle\mu\xi'\rangle^{\kappa}]^w u, u) + C\mu^2\|\langle\mu D'\rangle^{3\kappa/2}u\|^2,\\ \|B^w u\|^2 &= ([B\#B]^w u, u, u) \leq \mathsf{Re}([\phi_2^2\langle\mu\xi'\rangle^{\kappa}]^w u, u) + C\mu^2\|\langle\mu D'\rangle^{\kappa/2}u\|^2.\end{aligned}$$

These prove the assertion. □

Corollary 4.2. *We have*

$$\begin{aligned}&\mu\|\langle\mu D'\rangle^{1/2}u\|^2\\ &\quad\leq C\mathsf{Re}([\phi_1^2 w\langle\mu\xi'\rangle^{\kappa}]^w u, u) + C\mathsf{Re}([\phi_2^2\langle\mu\xi'\rangle^{\kappa}]^w u, u) + C\mu\|\langle\mu D'\rangle^{3\kappa/2}u\|^2.\end{aligned}$$

Proof. Writing

$$\begin{aligned}C\mu\langle\mu\xi'\rangle^{1+\kappa}\sqrt{w} - \mu\langle\mu\xi'\rangle &= C\mu\langle\mu\xi'\rangle^{1+\kappa}\sqrt{w}\big[1 - C^{-1}w^{-1/2}\langle\mu\xi'\rangle^{-\kappa}\big]\\ &= \psi\#\psi + \mu^3 S(\langle\mu\xi'\rangle^{3\kappa}, g)\end{aligned}$$

with

$$\psi = \sqrt{C\mu}w^{1/4}\langle\mu\xi'\rangle^{1/2+\kappa/2}\sqrt{1 - C^{-1}w^{-1/2}\langle\mu\xi'\rangle^{-\kappa}} \in \sqrt{\mu}S(w^{1/4}\langle\mu\xi'\rangle^{1/2+\kappa/2}, g),$$

we have

$$C\mu\mathsf{Re}([\langle\mu\xi'\rangle^{1+\kappa}\sqrt{w}]^w u, u) - \mu\|\langle\mu D'\rangle^{1/2}u\|^2 \geq -C\mu^3\|\langle\mu\xi'\rangle^{3\kappa/2}u\|^2$$

which proves the assertion. □

Lemma 4.6. *Let $j \neq 1$ and $a \in \mu S(1, g)$. Then*

$$\begin{aligned}&\mathsf{Re}([a\phi_1\phi_j]^w u, u)\\&\quad \le C\mu\mathsf{Re}([\phi_j^2\langle\mu\xi'\rangle^\kappa]^w u, u) + C\mu\mathsf{Re}([\phi_1^2 w\langle\mu\xi'\rangle^\kappa]^w u, u) + C\mu^3\|\langle\mu D'\rangle^{3\kappa/2}u\|^2.\end{aligned}$$

Proof. We may assume that a is real. Consider

$$a\phi_1\phi_j = \mathsf{Re}(\mu^{1/2}\langle\mu\xi'\rangle^{\kappa/2}\phi_j\#\mu^{-1/2}a\langle\mu\xi'\rangle^{-\kappa/2}\phi_1) + \mu^3 S(\langle\mu\xi'\rangle^{2\kappa}, g).$$

Now $\mu^{1/2}\langle\mu\xi'\rangle^{\kappa/2}\phi_j\#\mu^{1/2}\langle\mu\xi'\rangle^{\kappa/2}\phi_j = \mu\langle\mu\xi'\rangle^\kappa\phi_j^2 + \mu^3 S(\langle\mu\xi'\rangle^\kappa, g_0)$ and

$$\begin{aligned}&\mu^{-1/2}a\langle\mu\xi'\rangle^{-\kappa/2}\phi_1\#\mu^{-1/2}a\langle\mu\xi'\rangle^{-\kappa/2}\phi_1\\&\qquad = (\mu^{-1}a^2w^{-1}\langle\mu\xi'\rangle^{-2\kappa})\phi_1^2 w\langle\mu\xi'\rangle^\kappa + \mu^3 S(\langle\mu\xi'\rangle^{3\kappa}, g).\end{aligned}$$

Noting that $\mu^{-1}a^2w^{-1}\langle\mu\xi'\rangle^{-2\kappa} \in \mu S(1, g)$, we have by Lemma 4.4,

$$\begin{aligned}&\mathsf{Re}([a\phi_1\phi_j]^w u, u)\\&\quad \le \frac{1}{2}\{\mu\|[\phi_j\langle\mu\xi'\rangle^{\kappa/2}]^w u\|^2 + \mu^{-1}\|[a\langle\mu\xi'\rangle^{-\kappa/2}\phi_1]^w u\|^2\} + C\mu^3\|\langle\mu D'\rangle^\kappa u\|^2\\&\quad \le C\mu\mathsf{Re}([\langle\mu\xi'\rangle^\kappa\phi_j^2]^w u, u) + C\mu\mathsf{Re}([\phi_1^2 w\langle\mu\xi'\rangle^\kappa]^w u, u) + C\mu^3\|\langle\mu D'\rangle^{3\kappa/2}u\|^2,\end{aligned}$$

which is the assertion of the lemma. □

Lemma 4.7. *Let $a \in \mu S(\langle\mu\xi'\rangle w, g)$. Then for $j \neq 1$, we have*

$$\begin{aligned}\mathsf{Re}([a\phi_j]^w u, u) \le& C\mu^{1/2}\mathsf{Re}([\phi_j^2\langle\mu\xi'\rangle^\kappa]^w u, u) + C\mu^{1/2}\mathsf{Re}([\phi_1^2 w\langle\mu\xi'\rangle^\kappa]^w u, u)\\&+ C\mu^{1/2}\mathsf{Re}([\phi_2^2\langle\mu\xi'\rangle^\kappa]^w u, u) + C\mu^{3/2}\|\langle\mu D'\rangle^{3\kappa/2}u\|^2.\end{aligned}$$

Proof. Writing

$$a\phi_j = \mathsf{Re}(\mu^{1/4}\langle\mu\xi'\rangle^{\kappa/2}\phi_j\#\mu^{-1/4}\langle\mu\xi'\rangle^{-\kappa/2}a) + \mu^3 S(\langle\mu\xi'\rangle^{2\kappa}, g),$$

we have

$$\begin{aligned}&\mathsf{Re}([a\phi_j]^w u, u)\\&\quad \le \mu^{1/2}\|[\langle\mu\xi'\rangle^{\kappa/2}\phi_j]^w u\|^2 + \mu^{-1/2}\|[\langle\mu\xi'\rangle^{-\kappa/2}a]^w u\|^2 + C\mu^3\|\langle\mu D'\rangle^\kappa u\|^2.\end{aligned}$$

Note that $\langle\mu\xi'\rangle^{-\kappa/2}a\#\langle\mu\xi'\rangle^{-\kappa/2}a = \langle\mu\xi'\rangle^{-\kappa}a^2 + \mu^4 S(\langle\mu\xi'\rangle^{3\kappa}, g)$ and write

$$\begin{aligned}\langle\mu\xi'\rangle^{-\kappa}a^2 &= (w^{-2}a^2\langle\mu\xi'\rangle^{-2})w^2\langle\mu\xi'\rangle^{2-\kappa} = b(\langle\mu\xi'\rangle^{-2}\phi_1^2 + \langle\mu\xi'\rangle^{-\delta})\langle\mu\xi'\rangle^{2-\kappa}\\&= b\langle\mu\xi'\rangle^{-2\kappa}w^{-1}(\phi_1^2 w\langle\mu\xi'\rangle^\kappa) + b\langle\mu\xi'\rangle^{2-\delta-\kappa}\end{aligned}$$

where $b = w^{-2}a^2\langle\mu\xi'\rangle^{-2} \in \mu^2 S(1,g)$. Since $2-\delta-\kappa = 1$ we obtain from Lemma 4.4

$$\begin{aligned}&\mu^{-1/2}\|[\langle\mu\xi'\rangle^{-\kappa/2}a]^w u\|^2\\ &\quad\le C\mu^{3/2}\mathsf{Re}([\phi_1^2 w\langle\mu\xi'\rangle^{\kappa}]^w u,u) + C\mu^{5/2}\|\langle\mu D'\rangle^{3\kappa/2}u\|^2 + C\mu^{3/2}\|\langle\mu D'\rangle^{1/2}u\|^2.\end{aligned}$$

Then, applying Corollary 4.2, we get the assertion. □

We now estimate $\{\xi_0+\phi_1\Phi,\phi_j^2\}$, $\{\xi_0+\phi_1\Phi, w\phi_1^2\}$. Recall that

$$\{\xi_0+\phi_1\Phi,\phi_j^2\} = \{\xi_0+\phi_1,\phi_j^2\} - \{\phi_1\psi,\phi_j^2\}.$$

From the assumption, we have

$$\{\xi_0+\phi_1,\phi_j^2\} = 2\{\xi_0+\phi_1,\phi_j\}\phi_j = \sum_{k=1}^{r} C_{jk}\phi_k\phi_j,$$

where $C_{jk}\in\mu S(1,g_0)$. Note that for $j,k\ge 2$,

$$\mathsf{Re}([C_{jk}\phi_k\phi_j]^w u,u) \le C\mu\mathsf{Re}([\phi_k^2]^w u,u) + C\mu\mathsf{Re}([\phi_j^2]^w u,u) + C\mu^3\|u\|^2.$$

For $C_{j1}\phi_1\phi_j$, we apply Lemma 4.6 to get

$$\begin{aligned}&\mathsf{Re}([C_{j1}\phi_1\phi_j]^w u,u)\\ &\quad\le C\mu\mathsf{Re}([\phi_j^2\langle\mu\xi'\rangle^{\kappa}]^w u,u) + C\mu\mathsf{Re}([\phi_1^2 w\langle\mu\xi'\rangle^{\kappa}]^w u,u) + C\mu^3\|\langle\mu D'\rangle^{3\kappa/2}u\|^2.\end{aligned}$$

Consider

$$\{\xi_0+\phi_1, w\phi_1^2\} = 2\{\xi_0+\phi_1,\phi_1\}\phi_1 w + \{\xi_0+\phi_1,w\}\phi_1^2.$$

For the first term of the right hand side, we remark that

$$\{\xi_0+\phi_1,\phi_1\} = \sum_{k=1}^{r} C_{1k}\phi_k$$

and apply Lemma 4.6 and Lemma 4.4. Let us study

$$\begin{aligned}\{\xi_0+\phi_1,w\} =& \frac{1}{2}w^{-1}\{\xi_0+\phi_1,\langle\mu\xi'\rangle^{-2}\phi_1^2+\langle\mu\xi'\rangle^{-\delta}\}\\ =& \frac{1}{2}w^{-1}\{\phi_1,\langle\mu\xi'\rangle^{-2}\}\phi_1^2 + w^{-1}\{\xi_0+\phi_1,\phi_1\}\phi_1\langle\mu\xi'\rangle^{-2}\\ &+\frac{1}{2}w^{-1}\{\phi_1,\langle\mu\xi'\rangle^{-\delta}\}.\end{aligned}$$

Note that $w^{-1}\{\phi_1,\langle\mu\xi'\rangle^{-2}\}\phi_1^2$, $w^{-1}\{\phi_1,\langle\mu\xi'\rangle^{-\delta}\}\in\mu S(w,g)$ and apply Lemma 4.4 to $w^{-1}\{\phi_1,\langle\mu\xi'\rangle^{-2}\}\phi_1^4$ and $w^{-1}\{\phi_1,\langle\mu\xi'\rangle^{-\delta}\}\phi_1^2$. Note that

$$w^{-1}\{\xi_0+\phi_1,\phi_1\}\phi_1^3\langle\mu\xi'\rangle^{-2} = \sum_{k=1}^{r} T_k\phi_k\phi_1$$

with $T_k \in \mu S(w,g)$ and apply Lemma 4.6 and Lemma 4.4. Let us consider for $j \geq 2$

$$\{\phi_1\psi, \phi_j^2\} = 2\{\phi_1, \phi_j\}\phi_j\psi + 2\{\psi, \phi_j\}\phi_j\phi_1.$$

Write

$$\{\phi_1, \phi_j\}\phi_j\psi = \mathsf{Re}(\langle\mu\xi'\rangle^{\kappa/2}\phi_j\#\{\phi_1,\phi_j\}\psi\langle\mu\xi'\rangle^{-\kappa/2}) + \mu^3 S(\langle\mu\xi'\rangle^{2\kappa}, g),$$

where $\langle\mu\xi'\rangle^{\kappa/2}\phi_j \in S(\langle\mu\xi'\rangle^{1+\kappa/2}, g_0)$, $\{\phi_1,\phi_j\}\psi\langle\mu\xi'\rangle^{-\kappa/2} \in \mu S(w\langle\mu\xi'\rangle^{1-\kappa/2}, g)$, and note that

$$\begin{aligned}\{\phi_1,\phi_j\}^2\psi^2\langle\mu\xi'\rangle^{-\kappa} &= (\{\phi_1,\phi_j\}^2\psi^2\langle\mu\xi'\rangle^{-2}w^{-2})w^2\langle\mu\xi'\rangle^{2-\kappa} = Tw^2\langle\mu\xi'\rangle^{2-\kappa}\\ &= T(\langle\mu\xi'\rangle^{-2}\phi_1^2 + \langle\mu\xi'\rangle^{-\delta})\langle\mu\xi'\rangle^{2-\kappa}\\ &= (Tw^{-1}\langle\mu\xi'\rangle^{-2\kappa})w\langle\mu\xi'\rangle^{\kappa}\phi_1^2 + T\langle\mu\xi'\rangle^{2-\delta-\kappa}\end{aligned}$$

with $T = \{\phi_1,\phi_j\}^2\psi^2\langle\mu\xi'\rangle^{-2}w^{-2} \in \mu^2 S(1,g)$. Hence $Tw^{-1}\langle\mu\xi'\rangle^{-2\kappa} \in \mu^2 S(1,g)$, $T\langle\mu\xi'\rangle^{2-\delta-\kappa} \in \mu^2 S(\langle\mu\xi'\rangle, g)$. Now apply Lemma 4.6 and Corollary 4.1. Finally, consider

$$\{\phi_1\psi, \phi_1^2 w\} = \{\phi_1, w\}\phi_1^2\psi + \{\psi, w\}\phi_1^3 + 2\{\psi,\phi_1\}\phi_1^2 w.$$

Note that $\{\phi_1, w\}$, $\{\psi,\phi_1\} \in \mu S(1,g)$, $\{\psi, w\}\phi_1 \in \mu S(w,g)$ and apply Lemma 4.4.

Proposition 4.2. *We have*

$$\begin{aligned}&|\mathsf{Re}(\{\xi_0+\phi_1\Phi, \phi_j^2\}^w u, u)|, |\mathsf{Re}(\{\xi_0+\phi_1\Phi, w\phi_1^2\}^w u, u)|\\ &\quad\leq C\mu\Big\{\sum_{j=2}^{r}\mathsf{Re}([\langle\mu\xi'\rangle^{\kappa}\phi_j^2]^w u, u) + \mathsf{Re}([w\langle\mu\xi'\rangle^{\kappa}\phi_1^2]^w u, u)\Big\} + C\mu^2\|\langle\mu D'\rangle^{1/2}u\|^2.\end{aligned}$$

4.3 Energy inequality (continued). Assuming $C = 0$ and $R = 0$ in (4.13), we study

$$P = -M\Lambda + Q$$

where $M = D_0 - (\phi_1\Phi)^w = D_0 - m^w$, $\Lambda = D_0 + (\phi_1\Phi)^w = D_0 - \lambda^w$ and

$$Q = \sum_{j=2}^{r}[\phi_j^2]^w + [w\phi_1^2]^w.$$

We conjugate $e^{-x_0\langle\mu D'\rangle^{\kappa}} = e^{\phi}$ to P:

$$e^{\phi}Pe^{-\phi} = -e^{\phi}Me^{-\phi}e^{\phi}\Lambda e^{-\phi} + e^{\phi}Qe^{-\phi}.$$

Denote $e^{\phi}Me^{-\phi}$, $e^{\phi}\Lambda e^{-\phi}$, $e^{\phi}Qe^{-\phi}$ once again by M, Λ, Q, respectively. Let us consider

$$M = e^{\phi}(D_0 - m^w)e^{-\phi} = D_0 - i\langle\mu D'\rangle^{\kappa} - e^{\phi}m^w e^{-\phi}.$$

Since $m \in S(w\langle \mu\xi' \rangle, g)$, we apply Proposition 3.1 for the Weyl calculus with $\sigma = 4/5 = 4\kappa$. Then we have

$$e^{\phi} m^w e^{-\phi} = -[m_0 + m_1 + m_2]^w, \quad m_0 = -\phi_1 \Phi, \tag{4.14}$$

where $m_1 \in \mu S(\langle \mu\xi' \rangle^{\kappa}, g)$ is pure imaginary and $m_2 \in \mu^2 S(\langle \mu\xi' \rangle^{-\kappa}, \bar{g})$ by Proposition 3.1 where we recall $\bar{g} = \langle \mu\xi' \rangle^{4\kappa} g_0$. Let us consider

$$\Lambda = e^{\phi}(D_0 - \lambda^w)e^{-\phi} = D_0 - i\langle \mu D' \rangle^{\kappa} - e^{\phi}\lambda^w e^{-\phi}.$$

Since $\lambda \in S(w\langle \mu\xi' \rangle, g)$, repeating the same arguments, we have

$$e^{\phi}\lambda^w e^{-\phi} = -[\lambda_0 + \lambda_1 + \lambda_2]^w, \quad \lambda_0 = \phi_1 \Phi, \tag{4.15}$$

where $\lambda_1 \in \mu S(\langle \mu\xi' \rangle^{\kappa}, g)$ is pure imaginary and $\lambda_2 \in \mu^2 S(\langle \mu\xi' \rangle^{-\kappa}, \bar{g})$. Consider $e^{\phi} Q e^{-\phi}$. We have

$$e^{\phi}[\phi_j^2]^w e^{-\phi} = [\phi_j^2 + a_j\phi_j + r_j]^w,$$

where $a_j \in \mu S(\langle \mu\xi' \rangle^{\kappa}, g)$ is pure imaginary and $r_j \in \mu^2 S(\langle \mu\xi' \rangle^{2\kappa}, \bar{g})$. We next consider

$$e^{\phi}[w\phi_1^2]^w e^{-\phi} = [w\phi_1^2 + a_1 w\phi_1 + r_1]^w,$$

where $a_1 \in \mu S(\langle \mu\xi' \rangle^{\kappa}, g)$ is pure imaginary and $r_1 \in \mu^2 S(w\langle \mu\xi' \rangle^{2\kappa}, \bar{g})$. Here we have applied Lemma 3.1 with $f(x, \xi', \mu) = \phi_1(x, \xi', \mu)\langle \mu\xi' \rangle^{-1}$ to get $w \in \gamma^{(1/\kappa)} S(1, \bar{g})$. Hence we can write

$$e^{\phi} Q e^{-\phi} = \Big[\sum_{j=2}^{r} \phi_j^2 + w\phi_1^2 + \sum_{j=2}^{r} a_j\phi_j + a_1 w\phi_1 + r \Big]^w, \tag{4.16}$$

where $a_j \in \mu S(\langle \mu\xi' \rangle^{\kappa}, g)$ and $r \in \mu^2 S(\langle \mu\xi' \rangle^{2\kappa}, \bar{g})$. Let us write

$$q = \sum_{j=2}^{r} \phi_j^2 + w\phi_1^2, \quad q_1 = \sum_{j=2}^{r} a_j\phi_j + a_1 w\phi_1.$$

Summarizing, we have

Proposition 4.3. *We can write*

$$e^{\phi} P e^{-\phi} = -M\Lambda + Q$$

with

$$M = D_0 - i\langle \mu D' \rangle^{\kappa} + [m_0 + m_1 + m_2]^w = D_0 - i\langle \mu D' \rangle^{\kappa} - m^w,$$
$$\Lambda = D_0 - i\langle \mu D' \rangle^{\kappa} + [\lambda_0 + \lambda_1 + \lambda_2]^w = D_0 - i\langle \mu D' \rangle^{\kappa} - \lambda^w,$$

where m_1, $\lambda_1 \in \mu S(\langle \mu\xi' \rangle^{\kappa}, g)$ *are pure imaginary and* m_2, $\lambda_2 \in \mu^2 S(\langle \mu\xi' \rangle^{-\kappa}, \bar{g})$. *As for* Q,

$$Q = [q + q_1 + r]^w, \quad q = \sum_{j=2}^{r} \phi_j^2 + w\phi_1^2, \quad q_1 = \sum_{j=2}^{r} a_j\phi_j + a_1 w\phi_1$$

where $a_j \in \mu S(\langle \mu\xi' \rangle^{\kappa}, g)$ *are pure imaginary and* $r \in \mu^2 S(\langle \mu\xi' \rangle^{2\kappa}, \bar{g})$.

We turn to (4.7). Since $\mathrm{Im}\lambda \in \mu S(\langle \mu\xi' \rangle^{\kappa}, g)$, one sees that

$$\alpha^{-1}\|A^{-1/2}\Lambda A u\|^2 \geq \frac{d}{dx_0}\|Au\|^2 + (2 - \alpha - C\mu)\|A^{3/2}u\|^2$$

with small $0 < \alpha < 2$. Now $A^{-1/2}\Lambda A = A^{1/2}\Lambda + A^{-1/2}[\Lambda, A]$; noting that $\lambda_0 \in S(w\langle \mu\xi' \rangle, g)$ and $A = \langle \mu D' \rangle^{\kappa}$, and hence $[\Lambda, A] \in \mu S(\langle \mu\xi' \rangle^{\kappa}, \bar{g})$, we have

$$\|A^{-1/2}[\Lambda, A]u\|^2 \leq C\mu\|A^{1/2}u\|^2.$$

Lemma 4.8. *We have*

$$\|A^{1/2}\Lambda u\|^2 \geq \frac{d}{dx_0}\|Au\|^2 + (1 - C\mu)\|A^{3/2}u\|^2. \tag{4.17}$$

Since $\mathrm{Im} m \in \mu S(\langle \mu\xi' \rangle^{\kappa}, \bar{g})$, it follows that

$$|2(\mathrm{Im} m\Lambda u, \Lambda u)| \leq C\mu\|A^{1/2}\Lambda u\|^2. \tag{4.18}$$

Let us study $2\mathrm{Re}(\Lambda u, \mathrm{Im}Qu)$. Recall that $\mathrm{Im}Q = [q_1 + r_1]^w$ with $r_1 \in \mu^2 S(\langle \mu\xi' \rangle^{2\kappa}, \bar{g})$. Then one can estimate $|2\mathrm{Re}(\Lambda u, \mathrm{Im}Qu)| \leq \mu\|A^{1/2}\Lambda u\|^2 + \mu^{-1}\|A^{-1/2}\mathrm{Im}Qu\|^2$. Note that

$$\langle \mu\xi' \rangle^{-\kappa/2}\#(q_1 + r_1) = \langle \mu\xi' \rangle^{-\kappa/2}q_1 + \mu^2 S(\langle \mu\xi' \rangle^{1/2}, \bar{g})$$

because $q_1 \in \mu S(\langle \mu\xi' \rangle^{1+\kappa}, g)$. Applying Lemma 4.4 to $\|[\langle \mu\xi' \rangle^{-\kappa/2}q_1]^w u\|^2$ we get

Lemma 4.9. *We have*

$$\begin{aligned}
&|2\mathrm{Re}(\Lambda u, \mathrm{Im}Qu)| \\
&\qquad \leq \mu\|A^{1/2}\Lambda u\|^2 + C\mu\Big\{\sum_{j=2}^{r} \mathrm{Re}([\langle \mu\xi' \rangle^{\kappa}\phi_j^2]^w u, u) \\
&\qquad\qquad + \mathrm{Re}([\langle \mu\xi' \rangle^{\kappa}w\phi_1^2]^w u, u)\Big\} + C\mu^3\|\langle \mu D' \rangle^{1/2}u\|^2.
\end{aligned}$$

Let us consider $\mathrm{Re}(\mathrm{Re}Qu, \mathrm{Im}\lambda u)$. From Proposition 4.3 we have $\mathrm{Re}Q = [q + r]^w$ with $r \in \mu^2 S(\langle \mu\xi' \rangle^{2\kappa}, \bar{g})$ and $\mathrm{Im}\lambda \in \mu S(\langle \mu\xi' \rangle^{\kappa}, \bar{g})$. It is clear that it suffices to study $\mathrm{Re}(q^w u, \mathrm{Im}\lambda u)$ modulo $\mu^3\|\langle \mu D' \rangle^{3\kappa/2}u\|^2$. Since one can write

$$\mathrm{Im}\lambda = \lambda_1 + \mu^N S(\langle \mu\xi' \rangle^{1-2N\kappa+2n\kappa}, \bar{g}), \quad \lambda_1 \in \mu S(\langle \mu\xi' \rangle^{\kappa}, g)$$

for any N, we may assume $\mathsf{Im}\lambda = \lambda_1 \in \mu S(\langle\mu\xi'\rangle^{\kappa}, g)$ modulo $\mu^3\|\langle\mu D'\rangle^{3\kappa/2}u\|^2$. Note that

$$\mathsf{Re}(\lambda_1\#q) = \lambda_1 q + \mu^3 S(\langle\mu\xi'\rangle, g)$$

because $5\kappa = 1$. Applying Lemma 4.4, we get

Lemma 4.10. *We have*

$$\begin{aligned}&|2\mathsf{Re}(\mathsf{Re}Qu, \mathsf{Im}\lambda u)|\\ &\quad\le C\mu\Big\{\sum_{j=2}^{r}\mathsf{Re}([\langle\mu\xi'\rangle^{\kappa}\phi_j^2]^w u, u) + \mathsf{Re}([\langle\mu\xi'\rangle^{\kappa}w\phi_1^2]^w u, u)\Big\} + C\mu^3\|\langle\mu D'\rangle^{1/2}u\|^2.\end{aligned}$$

We now estimate $\mathsf{Im}([D_0 - \mathsf{Re}\lambda, \mathsf{Re}Q]u, u)$. Note that one can write

$$\begin{aligned}\mathsf{Re}Q &= q + r + \mu^N S(\langle\mu\xi'\rangle^{2-2N\kappa+2n\kappa}, \bar{g}), & r &\in \mu^2 S(\langle\mu\xi'\rangle^{2\kappa}, g),\\ \mathsf{Re}\lambda &= -\lambda_0 - \lambda_2 + \mu^N S(\langle\mu\xi'\rangle^{1-2N\kappa+2n\kappa}, \bar{g}), & \lambda_2 &\in \mu^2 S(\langle\mu\xi'\rangle^{-\kappa}, g).\end{aligned}$$

This proves that $|\mathsf{Im}([D_0 - \mathsf{Re}\lambda, \mathsf{Re}Q]u, u)| = |\mathsf{Re}(\{\xi_0 - \mathsf{Re}\lambda, \mathsf{Re}Q\}^w u, u)|$ modulo $\mu^3\|\langle\mu D'\rangle^{2\kappa}u\|^2$. Note that

$$\{\xi_0 - \mathsf{Re}\lambda, \mathsf{Re}Q\} = \{\xi_0 - \lambda_0, q\} - \{\lambda_2, q\} + \mu^3 S(\langle\mu\xi'\rangle^{4\kappa}, \bar{g}).$$

Since one can write $\{\lambda_2, q\} = \sum_{j=2}^{r} a_j\langle\mu\xi'\rangle^{\kappa}\phi_j + a_1\langle\mu\xi'\rangle^{\kappa}w\phi_1$ with $a_j \in \mu^3 S(1, g)$, we have

$$\begin{aligned}&|\mathsf{Im}([D_0 - \mathsf{Re}\lambda, \mathsf{Re}Q]u, u)|\\ &\le|\mathsf{Re}(\{\xi_0 - \lambda_0, q\}^w u, u)| + C\mu^3\left\{\sum_{j=2}^{r}\|[\phi_j]^w u\|^2 + \|[\sqrt{w}\phi_1]^w u\|^2 + \|\langle\mu D'\rangle^{2\kappa}u\|^2\right\}\\ &\le|\mathsf{Re}(\{\xi_0 - \lambda_0, q\}^w u, u)|\\ &\quad + C\mu^3\left\{\sum_{j=2}^{r}\mathsf{Re}([\phi_j^2]^w u, u) + \mathsf{Re}([w\phi_1^2]^w u, u) + \|\langle\mu D'\rangle^{2\kappa}u\|^2\right\}.\end{aligned}$$

By Proposition 4.2, we obtain

Lemma 4.11. *We have*

$$\begin{aligned}&|\mathsf{Im}([D_0 - \mathsf{Re}\lambda, \mathsf{Re}Q]u, u)|\\ &\quad\le C\mu\Big\{\sum_{j=2}^{r}\mathsf{Re}([\langle\mu\xi'\rangle^{\kappa}\phi_j^2]^w u, u) + \mathsf{Re}([w\langle\mu\xi'\rangle^{\kappa}\phi_1^2]^w u, u)\Big\} + C\mu^2\|\langle\mu D'\rangle^{1/2}u\|^2.\end{aligned}$$

It remains to estimate $\mathsf{Re}(A\mathsf{Re}Qu,u)$. Clearly, it is enough to estimate $\mathsf{Re}(Aq^w u,u)$ modulo $\mu^2\|\langle\mu D'\rangle^{3\kappa/2}u\|^2$. Note that

$$\mathsf{Re}(\langle\mu\xi'\rangle^\kappa\#\phi_j^2) = \langle\mu\xi'\rangle^\kappa\phi_j^2 + \mu^2 S(\langle\mu\xi'\rangle^\kappa, g),$$
$$\mathsf{Re}(\langle\mu\xi'\rangle^\kappa\#w\phi_1^2) = \langle\mu\xi'\rangle^\kappa w\phi_1^2 + \mu^2 S(w\langle\mu\xi'\rangle^\kappa, g),$$

and hence

$$\mathsf{Re}(Aq^w u,u) \geq \sum_{j=2}^r \mathsf{Re}([\langle\mu\xi'\rangle^\kappa\phi_j^2]^w u,u) + \mathsf{Re}([\langle\mu\xi'\rangle^\kappa w\phi_1^2]^w u,u) - C\mu^2\|\langle\mu D'\rangle^{\kappa/2}u\|^2.$$

This proves

Lemma 4.12. *We have*

$$\mathsf{Re}(A\mathsf{Re}Qu,u) \geq \sum_{j=2}^r \mathsf{Re}([\langle\mu\xi'\rangle^\kappa\phi_j^2]^w u,u) + \mathsf{Re}([\langle\mu\xi'\rangle^\kappa w\phi_1^2]^w u,u) - C\mu^2\|\langle\mu D'\rangle^{3\kappa/2}u\|^2.$$

From Lemmas 4.8, 4.9, 4.10, 4.11, 4.12 and (4.18), we have

Proposition 4.4. *There exist* $\mu_0>0$, $C>0$, $c>0$ *such that*

$$\begin{aligned} C\|\langle\mu D'\rangle^{-\kappa/2}Pu\|^2 \geq &\frac{d}{dx_0}\{\|\Lambda u\|^2 + (\mathsf{Re}Qu,u) + \|\langle\mu D'\rangle^\kappa u\|^2\} + c\|\langle\mu D'\rangle^{\kappa/2}\Lambda u\|^2 \\ &+ c\Big\{\sum_{j=2}^r \mathsf{Re}([\phi_j^2\langle\mu\xi'\rangle^\kappa]^w u,u) + \mathsf{Re}([\phi_1^2 w\langle\mu\xi'\rangle^\kappa]^w u,u)\Big\} \\ &+ c\|\langle\mu D'\rangle^{3\kappa/2}u\|^2 + c\mu\|\langle\mu D'\rangle^{1/2}u\|^2 \end{aligned}$$

for $0<\mu<\mu_0$.

Here we return to (4.13). Denoting $e^\phi Ce^{-\phi}$ and $e^\phi Re^{-\phi}$ again by C and R, we have $C\in\mu S(1,\bar g)$ and

$$R = \Big[\sum_{j=1}^r c_j\phi_j + r\Big]^w, \quad r\in\mu^2 S(\langle\mu\xi'\rangle^{3\kappa},\bar g).$$

Then applying Lemma 4.4, one sees easily that Proposition 4.4 holds for $P+C\Lambda+R$.

From Lemma 4.4 it follows that

$$\sum_{j=2}^r \mathsf{Re}([\phi_j\langle\mu\xi'\rangle^\kappa]^w u,u) + \mathsf{Re}([\phi_1^2 w\langle\mu\xi'\rangle^\kappa]^w u,u) \geq -C\mu^2\|\langle\mu D'\rangle^{1/2}u\|^2 - C\mu^2\|\langle\mu D'\rangle^{3\kappa/2}u\|^2.$$

Taking into account that $\phi_j^2 = \phi_j \# \phi_j + \mu^2 S(1, g_0)$,

$$w\phi_1^2 = \sqrt{w}\phi_1 \# \sqrt{w}\phi_1 + \mu^2 S(w^{-1}, g) \subset \mu^2 S(\langle \mu\xi' \rangle^{2\kappa}, g),$$

we obtain

$$(\mathsf{Re}Q u, u) \geq -C\mu^2 \|\langle \mu D' \rangle^{\kappa} u\|^2.$$

Integrating the inequality in Proposition 4.4 from $-\infty$ to t with respect to x_0 yields

$$\begin{aligned} C&\int_{-\infty}^{t} \|\langle \mu D' \rangle^{-\kappa/2} P u\|^2 dx_0 \\ &\geq \big\{ \|\Lambda u(t,\cdot)\|^2 + c\|\langle \mu D' \rangle^{\kappa} u(t,\cdot)\|^2 \big\} \\ &\quad + c\int_{-\infty}^{t} \big\{ \|\langle \mu D' \rangle^{\kappa/2} \Lambda u\|^2 + \|\langle \mu D' \rangle^{3\kappa/2} u\|^2 + \mu \|\langle \mu D' \rangle^{1/2} u\|^2 \big\} dx_0 \end{aligned}$$

for $0 < \mu < \mu_0$. Returning to the original $P = -M\Lambda + C\Lambda + Q$ and replacing u by $e^{-x_0 \langle \mu D' \rangle^{\kappa}} u$, we obtain

Proposition 4.5. *We have*

$$\begin{aligned} C&\int_{-\infty}^{t} \|\langle \mu D' \rangle^{-\kappa/2} e^{-x_0 \langle \mu D' \rangle^{\kappa}} P u\|^2 dx_0 \\ &\geq \big\{ \|e^{-t\langle \mu D' \rangle^{\kappa}} \Lambda u(t,\cdot)\|^2 + c\|\langle \mu D' \rangle^{\kappa} e^{-t\langle \mu D' \rangle^{\kappa}} u(t,\cdot)\|^2 \big\} \\ &\quad + c\int_{-\infty}^{t} \big\{ \|\langle \mu D' \rangle^{\kappa/2} e^{-x_0\langle \mu D' \rangle^{\kappa}} \Lambda u\|^2 + \|\langle \mu D' \rangle^{3\kappa/2} e^{-x_0\langle \mu D' \rangle^{\kappa}} u\|^2 \\ &\quad + \mu \|\langle \mu D' \rangle^{1/2} e^{-x_0\langle \mu D' \rangle^{\kappa}} u\|^2 \big\} dx_0 \end{aligned}$$

for $0 < \mu < \mu_0$.

Since we have the same a priori estimate for P^*, we can apply standard duality arguments to prove Theorem 1.1.

5 Optimality of the Gevrey index

The optimality of the Gevrey index 5 in Theorem 1.2 can be proved by examining and estimating carefully the exact solution which is found in [3].

5.1 Construction of solutions.

Consider

$$P(x, D) = -D_0^2 + 2x_1 D_0 D_n + D_1^2 + x_1^3 D_n^2 \tag{5.1}$$

and apply the Fourier transform with respect to x_0 and x_n. The equation $Pu = 0$ is given by

$$u''(x_1) = (\xi_n^2 x_1^3 + 2\xi_0 \xi_n x_1 - \xi_0^2) u(x_1). \tag{5.2}$$

Let us change the independent variable x_1: set $y = \xi_n^{2/5} x_1$, where we assume once and for all that ξ_n is positive in the microlocal region where we set ourselves. Equation (5.2) transforms into

$$v''(y) = (y^3 + 2\xi_0 \xi_n^{-1/5} y - \xi_0^2 \xi_n^{-4/5}) v(y), \tag{5.3}$$

where we set $v(y) = v(y, \xi_0, \xi_n) = u(y\xi_n^{-2/5}, \xi_0, \xi_n)$. Finally, putting $\xi_0 = \frac{\zeta}{2}\xi_n^{1/5}$, we arrive at the equation

$$v''(y) = \Big(y^3 + \zeta y - \frac{\zeta^2}{4}\xi_n^{-2/5}\Big) v(y). \tag{5.4}$$

Note that

$$U(x) = \exp\Big(i\rho^5 x_n + \frac{i}{2}\zeta\rho x_0\Big) w\Big(x_1\rho^2; \zeta, -\frac{1}{4}\zeta^2\rho^{-2}\Big)$$

satisfies $PU = 0$. Taking (5.4) into account, we deal with the ordinary differential equation

$$w''(y) = (y^3 + \zeta y + \epsilon) w(y) \tag{5.5}$$

where ζ, ϵ are complex numbers and ϵ is thought of as small in the final arguments. We briefly recap, for this special situation, the general theory of subdominant solutions of the equation (5.5), following the exposition, in the book of Sibuya [16]. Theorem 6.1 in [16] states that the differential equation (5.5) has a solution

$$w(y; \zeta, \epsilon) = \mathcal{Y}(y; \zeta, \epsilon)$$

such that

(i) $\mathcal{Y}(y; \zeta, \epsilon)$ is an entire function of (y, ζ, ϵ);

(ii) $\mathcal{Y}(y; \zeta, \epsilon)$ admits an asymptotic representation

$$\mathcal{Y}(y; \zeta, \epsilon) \sim y^{-3/4}\Big[1 + \sum_{N=1}^{\infty} B_N y^{-N/2}\Big] \exp\left[-E(y; \zeta)\right]$$

uniformly on each compact set in (ζ, ϵ) space as y goes to infinity in any closed subsector of the open sector $|\arg y| < 3\pi/5$. Here

$$E(y; \zeta) = \frac{2}{5} y^{5/2} + \zeta y^{1/2}$$

and B_N are polynomials in (ζ, ϵ).

Note that if we set $\omega = \exp\left[i\frac{2\pi}{5}\right]$ and

$$\mathcal{Y}_k(y; \zeta, \epsilon) = \mathcal{Y}(\omega^{-k} y; \omega^{-2k}\zeta, \omega^{-3k}\epsilon),$$

where $k = 0, 1, 2, 3, 4$, then all the five functions $\mathcal{Y}_k(y; \zeta, \epsilon)$ solve (5.5). In particular, $\mathcal{Y}_0(y; \zeta, \epsilon) = \mathcal{Y}(y; \zeta, \epsilon)$. Let us denote

$$Y = y^{-3/4}\left[1 + \sum_{N=1}^{\infty} B_N y^{-N/2}\right] \exp\,[-E(y; \zeta)].$$

Then

(i) $\mathcal{Y}_k(y; \zeta, \epsilon)$ is an entire function of (y, ζ, ϵ);

(ii) $\mathcal{Y}_k(y; \zeta, \epsilon) \sim Y(\omega^{-k} y; \omega^{-2k}\zeta, \omega^{-3k}\epsilon)$ uniformly on each compact set in (ζ, ϵ) space as y goes to infinity in any closed subsector of the open sector $|\arg y - 2k\pi/5| < 3\pi/5$.

Let S_k denote the open sector $|\arg y - 2k\pi/5| < \pi/5$. We say that a solution of (5.5) is subdominant in the sector S_k if it tends to 0 as y tends to infinity along any direction in the sector. A solution is called dominant in the sector S_k if this solution tends to ∞ as y tends to infinity along any direction in the sector. Since

$$\mathrm{Re}[y^{5/2}] > 0 \quad \text{for} \;\; y \in S_0 \tag{5.6}$$

and $\mathrm{Re}[y^{5/2}] < 0$ for $y \in S_{-1} = S_4$ and for S_1, the solution $\mathcal{Y}_0(y; \zeta, \epsilon)$ is subdominant in S_0 and dominant in S_4 and S_1. Similarly $\mathcal{Y}_k(y; \zeta, \epsilon)$ is subdominant in S_k and dominant in S_{k-1} and S_{k+1}. Thus it is clear that $\mathcal{Y}_{k+1}$ and $\mathcal{Y}_{k+2}$ are linearly independent. Therefore, $\mathcal{Y}_k$ is a linear combination

$$\mathcal{Y}_k(y; \zeta, \epsilon) = C_k(\zeta, \epsilon)\mathcal{Y}_{k+1}(y; \zeta, \epsilon) + \tilde{C}_k(\zeta, \epsilon)\mathcal{Y}_{k+2}(y; \zeta, \epsilon).$$

The above relation, connection formula for $\mathcal{Y}_k(y; \zeta, \epsilon)$ and the coefficients C_k, $\tilde{C}_k$ are called the Stokes coefficients for $\mathcal{Y}_k(y; \zeta, \epsilon)$. We summarize in the following proposition some of the known and useful facts about the Stokes coefficients for our particular equation (5.5). Proofs can be found in Chapter 5 of [16].

Proposition 5.1. *The following results hold.*

(i) $\tilde{C}_k(\zeta, \epsilon) = -\omega$, *for all* k; ϵ *and* ζ,

(ii) $C_k(\zeta, \epsilon) = C_0(\omega^{-2k}\zeta, \omega^{-3k}\epsilon)$, *for all* k, ϵ, ζ *and* $C_0(\zeta, \epsilon)$ *is an entire function of* (ζ, ϵ);

(iii) $\partial_\zeta C_0(\zeta, \epsilon)|_{(\zeta,\epsilon)=(0,0)} \neq 0$;

(iv) $\partial_\epsilon C_0(\zeta, \epsilon)|_{(\zeta,\epsilon)=(0,0)} \neq 0$.

We also have

Proposition 5.2. *Setting*

$$S_k(\zeta, \epsilon) = \begin{bmatrix} C_k(\zeta, \epsilon) & 1 \\ -\omega & 0 \end{bmatrix}, \qquad k = 0, 1, 2, 3, 4,$$

we have

$$S_4(\zeta,\epsilon)\cdot S_3(\zeta,\epsilon)\cdot S_2(\zeta,\epsilon)\cdot S_1(\zeta,\epsilon)\cdot S_0(\zeta,\epsilon)=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}. \tag{5.7}$$

The proof of Proposition 5.2 is straightforward. Applying this proposition yields the following interesting result.

Proposition 5.3. *Let us denote* $c_k(\zeta)=C_k(\zeta,0)$. *Then* (5.7) *is equivalent to*

$$c_k(\zeta)+\omega^2 c_{k+2}(\zeta)c_{k+3}(\zeta)-\omega^3=0 \mod 5.$$

Otherwise stated,

$$c(\zeta)+\omega^2 c(\omega\zeta)c(\omega^4\zeta)-\omega^3=0, \quad \textit{for all } \zeta\in\mathbb{C},$$

where $c(\zeta)=c_0(\zeta)=C_0(\zeta,0)$.

Proof. A straightforward computation from (5.7). □

The following key lemma is proved in [3]. We repeat here the short proof.

Lemma 5.1. *The Stokes coefficient* $C_0(\zeta,0)$ *vanishes for at least one (non-zero)* ζ_0.

Proof. Suppose that $c(\zeta)\neq 0$ for all $\zeta\in\mathbb{C}$. Then it follows from Proposition 5.3 that $c(\zeta)\neq\omega^3$ for all $\zeta\in\mathbb{C}$. Since $c(\zeta)$ is an entire function, Picard's Little Theorem implies that $c(\zeta)$ is constant because $c(\zeta)$ avoids the distinct values 0 and ω^3. But this contradicts (iii) of Proposition 5.1. □

Let us now consider the equation

$$C_0\Big(\zeta,-\frac{1}{4}\zeta^2\epsilon\Big)=0.$$

Let ζ_0 be a complex number such that $c(\zeta_0)=C_0(\zeta_0,0)=0$. Let μ be the multiplicity of the root ζ_0. Since $C_0(\zeta,0)$ is holomorphic, μ is finite; and by the Weierstrass preparation theorem, we can write

$$C_0(\zeta,-\zeta^2\epsilon/4)=\gamma(\zeta,\epsilon)((\zeta-\zeta_0)^\mu+\sum_{j=1}^{\mu}a_j(\epsilon)(\zeta-\zeta_0)^{\mu-j}),$$

where $\gamma(\zeta_0,0)\neq 0$, $a_j(0)=0$ and $a_j(\epsilon)$ is holomorphic at $\epsilon=0$. Then each root $\zeta(\epsilon)$ of $C_0(\zeta,-\zeta^2\epsilon/4)=0$ admits the Puiseux expansion

$$\zeta(\epsilon)=\zeta_0+\sum_{j=0}^{\infty}\zeta_j(\epsilon^{1/p})^j=g(\epsilon^{1/p})$$

with some positive integer p and a holomorphic $g(z)$ near $z = 0$ (see, for instance, [1]). In what follows, we consider the equation

$$w''(y) = \Big(y^3 + \zeta y - \frac{1}{4}\zeta^2 \epsilon^p\Big) w(y), \tag{5.8}$$

so that the equation $C_0(\zeta, -\zeta^2\epsilon^p/4) = 0$ has a solution

$$\zeta(\epsilon^p) = \tilde{\zeta}(\epsilon), \tag{5.9}$$

where $\tilde{\zeta}(\epsilon)$ is holomorphic in a neighborhood of $\epsilon = 0$. Using this zero we see that

$$\mathcal{Y}_0(y; \tilde{\zeta}(\epsilon), -\tilde{\zeta}(\epsilon)^2\epsilon^p/4) = -\omega\mathcal{Y}_2(y; \tilde{\zeta}(\epsilon), -\tilde{\zeta}(\epsilon)^2\epsilon^p/4), \quad \text{for all } y \in \mathbb{C}, \tag{5.10}$$

where $|\epsilon| \ll 1$. Once this choice of root has been made, the function

$$w(y) = \mathcal{Y}_0(y; \tilde{\zeta}(\epsilon), -\tilde{\zeta}(\epsilon)^2\epsilon^p/4)$$

is now subdominant in both S_0 and S_2.

5.2 Estimate of solutions.

Let

$$U(x, \rho) = \exp\big[i\rho^5 x_n + \frac{i}{2}\tilde{\zeta}(\rho^{-2/p})\rho x_0\big]\mathcal{Y}(x_1\rho^2; \tilde{\zeta}(\rho^{-2/p}), -\tilde{\zeta}(\rho^{-2/p})^2\rho^{-2}/4), \tag{5.11}$$

where we have set $\epsilon^p = \rho^{-2}$, $\rho \in \mathbb{C}$, $|\rho|$ large, and the direction of ρ is to be suitably chosen (we have set $\xi_n^{1/5} = \rho$; see Subsection 5.1). More precisely, in what follows, we set $\rho = |\rho|\omega^k$, where $\omega = e^{i\frac{2}{5}\pi}$ and $k = 0, 1$; in both cases, we have $\rho^5 > 0$. Since $C_0(\overline{\omega}\overline{\zeta}_0, 0) = 0$ (see [17]), then considering $\overline{\omega}\overline{\zeta}_0$, instead of ζ_0 if necessary, one can choose $k = 0$ or 1 so that

$$\mathsf{Im}\big[\zeta_0\omega^k\big] < 0$$

because at least one of $\zeta_0, \omega\zeta_0, \overline{\omega}\overline{\zeta}_0, \overline{\zeta}_0$ has negative imaginary part.

Recall that $P(x, D)U(x, \rho) = 0$, for all $\rho \in \mathbb{C}$ and $\mathcal{Y}(z; a_1, a_2)$ is entire analytic in (z, a_1, a_2).

Lemma 5.2. *With* $W(x_1; \rho) = \mathcal{Y}(x_1\rho^2; \tilde{\zeta}(\rho^{-2/p}), -\tilde{\zeta}(\rho^{-2/p})\rho^{-2}/4)$*, we have*

$$\Big|(\frac{\partial}{\partial x_1})^k W(x_1; \rho)\Big| \leq C^{k+1}|\rho|^{2k}k^{3k}, \quad k \in \mathbb{N}. \tag{5.12}$$

To prove this lemma, we show

Lemma 5.3. *Let $\phi(x)$ be a smooth function defined in $I = (R, +\infty)$ (or $I = (-\infty, -R)$) satisfying the estimates*

$$|\phi^{(k)}(x)| \le C^{k+1} k! \langle x\rangle^{\alpha}, \quad k = 1, 2, \dots, \; x \in I.$$

Let $\Phi_\ell(x) = e^{-\phi(x)}\big(\frac{d}{dx}\big)^{\ell} e^{\phi(x)}$. Then

$$|\Phi_\ell(x)| \le C A^{\ell} (\ell + \langle x\rangle^{\alpha})^{\ell}, \quad \ell \in \mathbb{N}, \; x \in I,$$

and hence

$$\Big|\Big(\frac{d}{dx}\Big)^{\ell} e^{\phi(x)}\Big| \le C A^{\ell} (\ell + \langle x\rangle^{\alpha})^{\ell} e^{\mathsf{Re}\phi(x)}, \quad \ell \in \mathbb{N}, \; x \in I.$$

Proof. We show that

$$\Big|\Big(\frac{d}{dx}\Big)^{k} \Phi_\ell(x)\Big| \le C A_1^k A_2^{\ell} \sum_{j=0}^{\ell} \langle x\rangle^{\alpha(\ell-j)} (k+j)!, \quad x \in I, \; k \in \mathbb{N} \tag{5.13}$$

for $\ell \in \mathbb{N}$. For $\ell = 0$, 1, (5.13) is clear. Assume that (5.13) holds for ℓ. Since $\Phi_{\ell+1} = (d/dx)\Phi_\ell + \phi^{(1)}\Phi_\ell$, we have

$$\begin{aligned}
\Big|\Big(\frac{d}{dx}\Big)^{k} \Phi_{\ell+1}(x)\Big| \le & C A_1^{k+1} A_2^{\ell} \sum_{j=0}^{\ell} \langle x\rangle^{\alpha(\ell-j)} (k+1+j)! \\
& + \sum_{j=0}^{k} \binom{k}{j} C^{j+2} (j+1)! \langle x\rangle^{\alpha} C A_1^{k-j} A_2^{\ell} \sum_{i=0}^{\ell} \langle x\rangle^{\alpha(\ell-i)} (k-j+i)! \\
\le & A_1 A_2^{-1} (C A_1^k A_2^{\ell+1}) \sum_{j=1}^{\ell+1} \langle x\rangle^{\alpha(\ell+1-j)} (k+j)! \\
& + C^2 A_2^{\ell} \sum_{i=0}^{\ell} \langle x\rangle^{\alpha(\ell+1-i)} \sum_{j=0}^{k} \binom{k}{j} C^{j+1} (j+1)! A_1^{k-j} (k-j+i)!.
\end{aligned}$$

Here we note that

$$\begin{aligned}
\sum_{j=0}^{k} \binom{k}{j} C^{j+1} (j+1)! A_1^{k-j} (k-j+i)! &\le \sum_{j=0}^{k} \binom{k}{j} (2C)^{j+1} j! (k-j+i)! A_1^{k-j} \\
&\le 2C A_1 (A_1 - 2C)^{-1} A_1^k (k+i)!.
\end{aligned}$$

Inserting this estimate, we obtain

$$\begin{aligned}
\Big|\Big(\frac{d}{dx}\Big)^{k} \Phi_{\ell+1}(x)\Big| \le & A_1 A_2^{-1} (C A_1^k A_2^{\ell+1}) \sum_{j=1}^{\ell+1} \langle x\rangle^{\alpha(\ell+1-j)} (k+j)! \\
& + 2C^2 A_1 A_2^{-1} (A_1 - 2C)^{-1} C A_1^k A_2^{\ell+1} \sum_{i=0}^{\ell} \langle x\rangle^{\alpha(\ell+1-i)} (k+i)!.
\end{aligned}$$

Then, taking A_1, A_2 so that $A_1A_2^{-1}\{1+2C^2(A_1-2C)^{-1}\}\le 1$, we get the desired assertion. $\square$

We turn to the proof of Lemma 5.2. Let us consider $\phi(x)=\sum_{j=0}^k a_jx^{b_j}$ where $b_k>\cdots>b_1>0$. Take m to be the smallest integer such that $m\ge b_k$ and hence $b_k-m\le 0$. Then

$$|\phi^{(k)}(x)|\le C^{k+1}k!|x|^{m-1},\quad \text{for all } x\in I,\ k=1,2,\dots.$$

Indeed, $\phi(x)$ is holomorphic in some sector containing the positive real axis; for large x, one can then write

$$\phi^{(m+j)}(x)=\frac{j!}{2\pi i}\int_{|z-x|=\delta}\frac{\phi^{(m)}(z)}{(z-x)^{j+1}}dz$$

with some $\delta>0$ independent of $x\in I$. Since $|\phi^{(m)}(z)|\le M$ on $|z-x|=\delta$ where M is independent of $x\in I$ and hence

$$|\phi^{(m+j)}(x)|\le M\delta^{-j}j!\le C^{m+j}(m+j)!|x|^{m-1},\quad j=0,1,2,\dots,\ x\in I$$

with some $C>0$. On the other hand, for $\phi^{(i)}(x)$, $i=1,2,\dots,m$, the assertion is clear taking C large.

We apply the above arguments to

$$W(x_1;\rho)=\mathcal{Y}(x_1\rho^2;\tilde\zeta(\rho^{-2/p}),-\tilde\zeta(\rho^{-2/p})^2\rho^{-2}/4).$$

Take $R>0$ enough large. When $|x_1\rho^2|<R$, it is clear that

$$\Big|\big(\frac{d}{dx_1}\big)^kW(x_1;\rho)\Big|\le C^{k+1}|\rho|^{2k}k!$$

because

$$\Big|\big(\frac{d}{dx}\big)^k\mathcal{Y}(x;\zeta,\epsilon)|\le C^{k+1}k!,\quad |x|<R,\ (\zeta,\epsilon)\in L,$$

where L is a some compact set. Recall that $\mathcal{Y}$ has the form

$$\mathcal{Y}(x\omega^{2k};\zeta,\eta)=e^{-3\pi i/4}\omega^{-3k/2}e^{-(\frac{2}{5}i|x|^{5/2}+i\zeta|x|^{1/2}\omega^k)}|x|^{-3/4}[1+R(x)]$$

in $x<-R$, where $\zeta=\tilde\zeta(\rho^{-2/p})$ and $\eta=-\tilde\zeta(\rho^{-2/p})\rho^{-2}/4$, and $R(x)$ is holomorphic in the sector $|\arg x-\pi|<\pi/5$ and satisfies $|R(x)|\le C$ if $|\arg x-\pi|\le\pi/10$, for instance.

We apply the above result with $\phi(x)=-\frac{2}{5}i(-x)^{5/2}-i\zeta(-x)^{1/2}\omega^k$ and $I=(-\infty,-R)$. Note that we may assume that $\mathsf{Im}(\tilde\zeta(\rho^{-2/p})\omega^k)\le -c$ for some $c>0$. Then

$$\Big|\Big(\frac{d}{dx}\Big)^ke^{-(\frac{2}{5}i|x|^{5/2}+i\zeta|x|^{1/2}\omega^k)}\Big|\le C^{k+1}(k+|x|^{3/2})^ke^{-c|x|^{1/2}},\quad x<-R.$$

Noting that $|x|^{3k/2}e^{-c|x|^{1/2}} \le C^{k+1}k^{3k}$ for $x < -R$, we have

$$\Big|\Big(\frac{d}{dx}\Big)^k e^{-(\frac{2}{5}i|x|^{5/2}+i\zeta|x|^{1/2}\omega^k)}\Big| \le C^{k+1}k^{3k}, \quad x < -R.$$

When $x > R$, we have

$$\mathcal{Y}(x\omega^{2k};\zeta,\eta) = x^{-3/4}\omega^{-3k/2}e^{-(\frac{2}{5}x^{5/2}+\tilde{\zeta}(\rho^{-2/p})x^{1/2}\omega^k)}[1+R(x)]$$

in $x > R$ where ζ, η are as before. Here $R(x)$ is holomorphic in the sector $|\arg x| < \pi/5$ and $|R(x)| \le C$ in a closed subsector. The same arguments as above show that

$$\Big|\Big(\frac{d}{dx}\Big)^k e^{-(\frac{2}{5}x^{5/2}+i\tilde{\zeta}(\rho^{-2/p})x^{1/2}\omega^k)}\Big| \le C^{k+1}(k+x^{3/2})^k e^{-cx^{5/2}} \le C_1^{k+1}k^k, \quad x > R.$$

Since $(d/dx_1)^k W(x_1;\rho) = |\rho|^{2k}(d/dx)^k\mathcal{Y}(x_1|\rho|^2\omega^{2k};\tilde{\zeta}(\rho^{-2/p}), -\tilde{\zeta}(\rho^{-2/p})^2\rho^{-2}/4)$, we have the desired assertion.

Since

$$D_0^j U(0,x',\rho) = \big[\rho\tilde{\zeta}(\rho^{-2/p})/2\big]^j e^{i\rho^5 x_n} W(x_1;\rho), \quad j = 0,1,$$

it is easy to verify that for some $A_1 > 0$,

$$|D_{x_n}^p D_{x_1}^q D_0^j U(0,x',\rho)| \le A_1^{k+1}|\rho|^j(\rho^5+|\rho|^2k^3)^k, \quad j = 0,1 \tag{5.14}$$

for $p+q = k$, $k \in \mathbb{N}$ if $|\rho| > B$.

We next consider

$$D_{x_1}^j U(x_0,0,\rho) = \exp\Big[\frac{i}{2}\tilde{\zeta}(\rho^{-2/p})\rho x_0\Big]\Big(\frac{d}{dx_1}\Big)^j W(0;\rho), \quad j = 0,1$$

with $x_0 > 0$. Recall that $\tilde{\zeta}(\epsilon)$ is holomorphic at $\epsilon = 0$ and hence

$$\mathcal{Y}(0;\tilde{\zeta}(\epsilon), -\tilde{\zeta}(\epsilon)^2\epsilon^p/4) = \epsilon^q(c+O(\epsilon))$$

with some $c \ne 0$, $q \in \mathbb{N}$ if $\mathcal{Y}(0;\tilde{\zeta}(\epsilon), -\tilde{\zeta}(\epsilon)^2\epsilon^p/4)$ is not identically zero. If not, we have $(d\mathcal{Y}/dx)(0;\tilde{\zeta}(0),0) \ne 0$ because of the uniqueness of the ordinary differential equation. Since the argument is the same, we may assume that $\mathcal{Y}(0;\tilde{\zeta}(\epsilon) - \zeta(\epsilon)^2\epsilon^p/4)$ is not identically zero. Then we have

$$W(0,\rho) = \rho^{-2q/p}(c+O(\rho^{-2/p})).$$

Recall that $\rho = |\rho|\omega^k$. By our choice of k,

$$\mathsf{Re}\Big[\frac{i}{2}\rho\tilde{\zeta}(\rho^{-2/p})x_0\Big] \ge c|\rho|x_0$$

with some $c > 0$ for $|\rho| > B$, where $x_0 > 0$. Then

$$|U(x_0, 0, \rho)| \geq c'|\rho|^{-2q/p} \exp(c|\rho|x_0)$$

with some $c > 0$, $c' > 0$.

Let $\chi(x') \in \gamma_0^{(s')}(\mathbb{R}^n)$ with $1 < s' < s$ satisfy $\chi(x') = 1$ near $x' = 0$. Note that

$$\begin{aligned}\sum_{j=0}^{1} \sup_{x'} |\partial_{x'}^{\alpha}[\chi(x') D_0^j U(0, x', \rho)]| &\leq A_1^{|\alpha|+1} |\rho| (|\rho|^5 + |\rho|^2 |\alpha|^3 + |\alpha|^{s'})^{|\alpha|} \\ &\leq A_2^{|\alpha|+1} |\rho| (|\rho|^{5|\alpha|} + |\rho|^{2|\alpha|} |\alpha|^{3|\alpha|} + |\alpha|^{s'|\alpha|}).\end{aligned}$$

Since $B^k / k^{sk} \leq e^{scB^{1/s}}$, we have

$$\begin{aligned}&\sum_{j=0}^{1} \sup_{x',\alpha} \frac{|\partial_{x'}^{\alpha}[\chi(x') D_0^j U(0, x', \rho)]|}{h^{|\alpha|} |\alpha|^{s|\alpha|}} \\ &\leq A_2 |\rho| \Big[e^{sc(A_2 h^{-1} |\rho|^5)^{1/s}} + e^{(s-3)c(A_2 h^{-1} |\rho|^2)^{1/(s-3)}} + e^{(s-s')c(A_2 h^{-1})^{1/(s-s')}} \Big].\end{aligned}$$

Then one has

$$\sum_{j=0}^{1} \sup_{x',\alpha} \frac{|\partial_{x'}^{\alpha}[\chi(x') D_0^j U(0, x', \rho)]|}{h^{|\alpha|} |\alpha|^{s|\alpha|}} \leq C |\rho| e^{c|\rho|^{s^*}},$$

where $s^* = \max\{5/s, 2/(s-3)\}$, which is less than 1 if $s > 5$. Summarizing, we have

Proposition 5.4. *We have*

$$\begin{aligned}|U(x_0, 0, \rho)| &\geq c|\rho|^{-2q/p} \exp(c|\rho|x_0), \\ \sum_{j=0}^{1} \sup_{x',\alpha} \frac{\big|\partial_{x'}^{\alpha}[\chi(x') D_0^j U(0, x', \rho)]\big|}{h^{|\alpha|} |\alpha|^{s|\alpha|}} &\leq C|\rho| e^{c|\rho|^{s^*}},\end{aligned}$$

where $s^* = \max\{5/s, 2/(s-3)\}$.

5.3 Proof of Theorem 1.2. We are considering the Cauchy problem for

$$P = -D_0^2 + 2x_1 D_0 D_n + D_1^2 + x_1^3 D_n^2$$

in the Gevrey class of order s. Let $h > 0$ be fixed and denote by $\gamma_0^{(s),h}(K)$ the set of all $\phi(x) \in \gamma^{(s)}(\mathbb{R}^n)$ such that $\text{supp}\phi \subset K$ and (1.4) holds with some $C > 0$ for all $\alpha \in \mathbb{N}^n$. Note that $\gamma_0^{(s),h}(K)$ is a Banach space with the norm

$$\sup_{\alpha,x} \frac{|\partial_x^{\alpha} \phi(x)|}{h^{|\alpha|} |\alpha|!^s}.$$

Let us set

$$D_\delta = \{x \in \mathbb{R}^{n+1} : |x'|^2 + |x_0| < \delta\}$$

and recall the Holmgren theorem (see, for example, [12], Theorem 4.2):

Proposition 5.5. *There exists $\epsilon_0 > 0$ such that if $0 < \epsilon < \epsilon_0$ and $u(x) \in C^2(D_\epsilon)$ satisfies*

$$\begin{cases} Pu = 0 \quad \text{in } D_\epsilon \\ D_0^j u(0, x') = 0, \quad j = 0, 1, \quad x \in D_\epsilon \cap \{x_0 = 0\}, \end{cases}$$

then $u(x)$ vanishes identically in D_ϵ.

Let $(\epsilon_0 \geq)\ \epsilon_1 > 0$ and $h > 0$ be fixed and let $K = \{|x'| \leq \epsilon_1^2\}$. Since $\gamma_0^{(s),h}(K)$ is a Banach space, the next result follows from standard arguments (see [11], also [12]).

Proposition 5.6. *Assume that the Cauchy problem for P is locally solvable in $\gamma^{(s)}$ at the origin. Then there exists $\delta > 0$ such that for any $\Phi = (u_0(x'), u_1(x')) \in \gamma_0^{(s),h}(K)$, there is a unique solution $u(x) \in C^\infty(D_\delta)$*

$$\begin{cases} Pu = 0 \quad \text{in } D_\delta \\ D_0^j u(0, x') = u_j(x'), \quad j = 0, 1, \quad x \in D_\delta \cap \{x_0 = 0\}. \end{cases} \tag{5.15}$$

Let D_δ be determined in Proposition 5.6. Then from the closed graph theorem, we conclude that the mapping from $\Phi = (u_0, u_1) \in \gamma_0^{(s),h}(K)$ to the solution $u(x) \in C^\infty(D_\delta)$ is continuous. This implies

Proposition 5.7. *Suppose that the Cauchy problem for P is locally solvable in $\gamma^{(s)}$ at the origin. Then there exists $\delta > 0$ such that for any compact set $L \subset D_\delta$, there exists $C > 0$ such that*

$$|u(x)|_{C^0(L)} \leq C \sum_{j=0}^{1} \sup_{\alpha, x'} \frac{|\partial^\alpha u_j(x')|}{h^{|\alpha|}|\alpha|!^s} \tag{5.16}$$

holds for any $u_j(x') \in \gamma_0^{(s),h}(K)$, where $u(x)$ is the solution to (5.15).

We have a family of solutions $U(x, \rho)$, $\rho \in \mathbb{C}$, to $PU(x, \rho) = 0$ verifying Proposition 5.4. Let $\chi(x') \in \gamma^{(s')}(\mathbb{R}^n)$ be such that $\chi(x') = 0$ for $|x'| \geq \epsilon_1^2$ and $\chi(x') = 1$ for $|x'| \leq \epsilon_2^2$, where $1 < s' < s$. Since

$$\Phi_\rho = \big(\chi(x')U(0, x', \rho), \chi(x')D_0U(0, x', \rho)\big) \in \gamma_0^{(s),h}(K),$$

there is a unique solution $V_\rho(x) \in C^\infty(D_\delta)$ to the Cauchy problem (5.15). By the Holmgren theorem, we see that

$$V_\rho(x) = U(x, \rho), \quad x \in D_{\epsilon_2}.$$

To conclude, let $L \subset D_{\epsilon_2}$ be a compact set. Then from Proposition 5.4 and Proposition 5.7, there exists $C > 0$ such that

$$|V_\rho(x)|_{C^0(L)} = |U(x, \rho)|_{C^0(L)} \le C|\rho| e^{c|\rho|^{s^*}} \tag{5.17}$$

as $|\rho| \to \infty$. On the other hand, from Proposition 5.4 again, we have

$$|V_\rho(x_0, 0)| = |U(x_0, 0, \rho)| \ge c'|\rho|^{-2p/q} e^{c|\rho| x_0}. \tag{5.18}$$

It is clear that if $s > 5$, and hence $s^* < 1$, the inequalities (5.17) and (5.18) are not compatible when $|\rho| \to \infty$. Hence we have Theorem 1.2.

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