

On the sum of sine products

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Abstract

Given $N \geq 1$ and $d \geq 1$, we show the existence of positive integers λ_{kj} for $1 \leq k \leq N$ and $1 \leq j \leq d$ such that

$$\left| \sum_{k=1}^N \prod_{j=1}^d \sin \lambda_{kj} x_j \right| \leq CN^{(d+1)/(2d+1)}$$

for all real x_1, \dots, x_N , where C is a constant that depends only on d . This extends a result of Bourgain for the case $d = 1$.

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1. Introduction

Bourgain [1] has shown the existence of positive integers λ_k such that

$$\left| \sum_{k=1}^N \sin \lambda_k x \right| \leq CN^{2/3}$$

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for all real x , where C is an absolute constant. On the other hand, the trivial lower bound $CN^{1/2}$ cannot be achieved since Konyagin [2] has shown that for N sufficiently large, and any set of distinct positive integers $\lambda_1, \dots, \lambda_N$ there exists an x such that

$$0.15 \sqrt{\frac{N \log N}{\log \log N}} \leq \sum_{k=1}^N \sin \lambda_k x.$$

In this paper we extend Bourgain's result to higher dimensions. We will show that there exist positive integers λ_{kj} for $1 \leq k \leq N$ and $1 \leq j \leq d$ such that

$$\left| \sum_{k=1}^N \prod_{j=1}^d \sin \lambda_{kj} x_j \right| \leq CN^{(d+1)/(2d+1)},$$

where C depends only on d . Questions concerning bounds on sums of sines are delicate. For example it can be shown that if λ_k grows polynomially or exponentially in k then $\|\sum_{k=1}^N \sin \lambda_k x\|_\infty = O(N)$. The (random) construction in this paper produces λ_k which grow like $\exp(k^{1/3})$ to give $\|\sum_{k=1}^N \sin \lambda_k x\|_\infty = O(N^{2/3})$. On the other hand, mere gap conditions on the λ_k are not enough. It can be shown that if $\|\sum_{k=1}^N \sin \lambda_k x\|_\infty = O(N^\delta)$ with $\delta < 1$ then there exists a set of $\epsilon_k = \pm 1$ such that $\|\sum_{k=1}^N \sin(\lambda_k + \epsilon_k)x\|_\infty = O(N)$. These questions regarding bounds on the supremum norm of sine sums are related to and to some extent motivated by questions regarding bounds on the norm of the Hilbert transform on finite-dimensional translation invariant spaces of continuous functions. More specifically, the question is: how large can the Hilbert transform be on a $2N$ -dimensional translation invariant space of continuous functions? If the domain of the functions is the real line, then Bourgain's result shows that it can be $O(N^{1/3})$. If the domain of the functions is d -dimensional Euclidean space, then our result shows that the norm of the Hilbert transform can be $O(N^{d/(2d+1)})$. The authors are grateful to the referee for helpful comments and a careful reading of the manuscript.

2. Proof of the main theorem

In order to make this paper as self-contained as possible, we have chosen to give (the short) proofs of several well-known lemmas. The following three lemmas are well known.

Lemma 2.1. *Let X be a random variable with mean μ satisfying $|x| \leq 1$. For any real λ*

$$E(e^{\lambda(X-\mu)}) \leq e^{\lambda^2/2}.$$

Proof. Let

$$\phi(\lambda) = \log E(e^{\lambda(X-\mu)}) = \log E[e^{\lambda X} e^{-\lambda \mu}] = \log E(e^{\lambda X}) - \lambda \mu.$$

It follows that

$$\phi'(z) = \frac{E(Xe^{zX})}{E(e^{zX})} - \mu$$

and

$$\phi''(z) = \frac{E(e^{zX})E(X^2e^{zX}) - [E(Xe^{zX})]^2}{[E(e^{zX})]^2} \leq \frac{E(X^2e^{zX})}{E(e^{zX})} \leq 1$$

since $|X| \leq 1$. Also $\phi(0) = 0$ and $\phi'(0) = 0$, and we have

$$\begin{aligned} \phi'(z) &= \int_0^z \phi''(y) dy \leq \int_0^z dy = z, \\ \phi(\lambda) &= \int_0^\lambda \phi'(z) dz \leq \int_0^\lambda z dz = \frac{\lambda^2}{2}. \end{aligned}$$

It follows that $E(e^{\lambda(X-\mu)}) = e^{\phi(\lambda)} \leq e^{\lambda^2/2}$ and the proof of the lemma is therefore complete. \square

Lemma 2.2. *Let $a \geq 0$, $\lambda \in \mathbf{R}$, $X: \Omega \rightarrow \mathbf{R}$ be a random variable. Then*

- (a) $\mathcal{P}(X \geq \lambda) \leq e^{-a\lambda} E(e^{aX})$;
- (b) $\mathcal{P}(|X| \geq \lambda) \leq e^{-a\lambda} \{E(e^{aX}) + E(e^{-aX})\}$.

Proof.

$$E(e^{aX}) = \int_{X \geq \lambda} e^{aX} d\mathcal{P} + \int_{X < \lambda} e^{aX} d\mathcal{P} \geq \mathcal{P}(X \geq \lambda)e^{a\lambda}. \tag{1}$$

This gives (a).

$$E(e^{-aX}) = \int_{X \leq -\lambda} e^{-aX} d\mathcal{P} + \int_{X > -\lambda} e^{-aX} d\mathcal{P} \geq \mathcal{P}(X \leq -\lambda)e^{(-a)(-\lambda)}.$$

Hence

$$\mathcal{P}(X \leq -\lambda) \leq e^{-a\lambda} E(e^{-aX}). \tag{2}$$

Combining (1) and (2), we have

$$\mathcal{P}(|X| \geq \lambda) = \mathcal{P}(X \leq -\lambda) + \mathcal{P}(X \geq \lambda) \leq e^{-a\lambda} E(e^{-aX}) + e^{-a\lambda} E(e^{aX}). \quad \square$$

Lemma 2.3. *Let $N > 0$ and let Y_1, \dots, Y_q be random variables such that $E(e^{\lambda b Y_k}) \leq e^{N b^2/2}$ for all real b and $k = 1, \dots, q$. Then*

$$\mathcal{P}(|Y_k| < \sqrt{2N \log 4q}, \text{ for all } k = 1, \dots, q) > \frac{1}{2}.$$

Proof. By Lemma 2.2, given $\lambda \geq 0$,

$$\begin{aligned} \mathcal{P}(|Y_k| \geq \lambda, \text{ for some } k = 1, \dots, q) &\leq \sum_{k=1}^q \mathcal{P}(|Y_k| \geq \lambda) \\ &\leq e^{-a\lambda} \left[\sum_{k=1}^q (E(e^{aY_k}) + E(e^{-aY_k})) \right] \end{aligned}$$

for any $a \geq 0$. By hypothesis, this last expression is less than

$$e^{-a\lambda} 2q e^{Na^2/2}. \quad (3)$$

Setting $a = \lambda/N$ and then setting $\lambda = \sqrt{2N \log 4q}$, (3) becomes $1/2$. Thus,

$$\mathcal{P}(|Y_k| < \sqrt{2N \log 4q}, \text{ for all } k = 1, \dots, q) \geq \frac{1}{2}. \quad \square$$

The next lemma follows from elementary arguments and the mean value theorem.

Lemma 2.4. Let $0 \leq \alpha \leq 1$, $-1 < \gamma \leq 0$, and $u_k = [e^{k^\alpha}]$, where $[\cdot]$ denotes the greatest whole integer function. There exist constants $A, C_1, C_2, C_3, C_4 > 0$ (depending only on α and γ) such that for $k \geq A$

- (a) $u_{k+1} > u_k$;
- (b) $C_1 k^{\alpha-1} e^{k^\alpha} \leq u_k - u_{k-1} \leq C_2 k^{\alpha-1} e^{k^\alpha}$;
- (c) $C_3 k^{\gamma-1} e^{-k^\alpha} \leq \frac{(k-A)^\gamma - (k-A-1)^\gamma}{u_k - \mu_{k-1}} - \frac{(k-A+1)^\gamma - (k-A)^\gamma}{u_{k+1} - u_k} \leq C_4 k^{\gamma-1} e^{-k^\alpha}$.

Lemma 2.5. For $\frac{1}{3} \leq \alpha \leq 1$ and $-1 \leq \beta \leq 0$ there exists a constant C depending only on α and β such that

$$\sum_{k=1}^N k^\beta e^{k^\alpha} \leq C N^{1+\beta-\alpha} e^{N^\alpha}.$$

Proof. Since $x^\beta e^{x^\alpha}$ is monotonic for $x > 27$, we can estimate the sum by an integral of the form $\int_0^N x^\beta e^{x^\alpha} dx$. Change of variables, integration by parts and elementary estimations give the result. \square

Lemma 2.6. Let $m_k, k = 0, \dots, N$, denote integers such that $0 < m_0 < m_1 < \dots < m_N$, and let r_k denote real numbers such that $r_1 > \dots > r_N > 0$. Let $a_n = r_k$ for $m_{k-1} \leq n < m_k$ for $1 \leq n \leq N$, and let $a_n = 0$ otherwise. Define intervals $I_0 = (\pi/m_1, \pi]$, $I_j = (\pi/m_{j+1}, \pi/m_j]$ for $j = 1, \dots, N-1$, and $I_N = [0, \pi/m_{N-1})$. For $x \in I_j$

$$\left| \sum_{n \in \mathbf{Z}} a_n \sin nx \right| \leq \frac{\pi^2}{4} \frac{1}{m_j} \sum_{k=1}^j (r_k - r_{k+1}) m_k^2 + r_{j+1} m_{j+1}$$

with the following conventions:

- (a) If a lower index exceeds an upper index in a summation, the sum is void;
- (b) $r_{N+1} m_{N+1}$ is defined to be 0.

Proof. Let

$$S(x) = \sum_{n \in \mathbf{Z}} a_n \sin nx = \sum_{k=1}^N r_k \sum_{j=m_{k-1}}^{m_k-1} \sin jx.$$

Defining r_{N+1} to be 0, this can be rewritten as

$$\begin{aligned} S(x) &= \sum_{k=1}^N (r_k - r_{k+1}) \sum_{j=m_0}^{m_k-1} \sin jx \\ &= \frac{1}{\sin x/2} \sum_{k=1}^N (r_k - r_{k+1}) \sin \frac{(m_k + m_0 - 1)x}{2} \sin \frac{(m_k - m_0)x}{2}. \end{aligned}$$

Now suppose $(\pi/m_{j+1}) < x \leq (\pi/m_j)$ where $1 \leq j \leq N - 1$. We write $S(x) = S_1(x) + S_2(x)$ where $S_1(x)$ is the sum over $1 \leq k \leq j$ and $S_2(x)$ is the sum over $j + 1 \leq k \leq N$. For such x

$$\begin{aligned} &\left| \frac{1}{\sin x/2} \left| \sin \frac{(m_k + m_0 - 1)x}{2} \sin \frac{(m_k - m_0)x}{2} \right| \right| \\ &\leq (m_k + m_0 - 1)(m_k - m_0)x^2 4 \sin(x/2) \leq (m_k(m_k - 1) - m_0(m_0 - 1)) \frac{\pi^2}{4m_j}. \end{aligned}$$

Hence

$$|S_1(x)| \leq \frac{\pi^2}{4m_j} \sum_{k=1}^j (r_k - r_{k+1})m_k(m_k - 1) - \frac{\pi^2}{4m_j}(r_1 - r_{j+1})m_0(m_0 - 1).$$

Also

$$|S_2(x)| \leq \frac{1}{\sin(x/2)} \sum_{k=j+1}^N (r_k - r_{k+1}) = \frac{1}{\sin(x/2)} r_{j+1} \leq \frac{\pi}{x} r_{j+1} \leq r_{j+1} m_{j+1}.$$

This gives for $(\pi/m_{j+1}) < x \leq (\pi/m_j)$ and $0 \leq j \leq N - 1$

$$\begin{aligned} |S(x)| &\leq \frac{\pi^2}{4} \left(\frac{1}{m_j} \sum_{k=1}^j (r_k - r_{k+1})m_k(m_k - 1) - \frac{m_0(m_0 - 1)}{m_j}(r_1 - r_{j+1}) \right) \\ &\quad + r_{j+1} m_{j+1}. \end{aligned}$$

If $0 \leq x \leq (\pi/m_N)$ then $S(x) = S_1(x)$ and the above estimates for $S_1(x)$ still apply. Therefore, the last estimate for $S(x)$ is valid for $j = N$ where $r_{N+1} = 0$ so that $r_{N+1}m_{N+1} = 0$ even though m_{N+1} has not been defined. If $(\pi/m_1) < x < \pi$ then $S(x) = S_2(x)$ and we get $|S(x)| \leq r_1 m_1$. Hence we complete the proof of the lemma. \square

Lemma 2.7. Let $\frac{1}{3} < \alpha \leq 1$. Let $d \geq 1$ and $N > 1$ be integers. Let $A \geq 0$ and define $m_k = [e^{(A+k)^\alpha}]$ for $k = 0, 1, \dots, N$, where $[\cdot]$ denotes the greatest whole integer function. Suppose A is large enough to give $m_0 < m_1 < \dots < m_N$. Define

$$r_k = \frac{k^{1/d} - (k - 1)^{1/d}}{m_k - m_{k-1}}$$

and

$$a_n = \begin{cases} r_k, & m_{k-1} \leq n < m_k, \quad k = 1, \dots, N, \\ 0, & n < m_0 \text{ or } n \geq m_N. \end{cases}$$

Then there exists constants A and C such that for all x

$$\left| \sum_{n \in \mathbf{Z}} a_n \sin nx \right| \leq CN^{(1/d)-\alpha}.$$

Proof. Let $c_k = r_k - r_{k+1}$. We will apply Lemmas 2.4–2.6. As in Lemma 2.4, let u_k denote $[e^{k^\alpha}]$. First consider

$$\frac{1}{m_j} \sum_{k=1}^j c_k m_k^2 = \frac{1}{u_{A+j}} \sum_{k=A+1}^{A+j} \left[\frac{k^{1/d} - (k-1)^{1/d}}{u_k - u_{k-1}} - \frac{(k+1)^{1/d} - k^{1/d}}{u_{k+1} - u_k} \right] u_k^2.$$

Setting $\gamma = 1/d$ in Lemma 2.4 and $\beta = (1/d) - 1$ in Lemma 2.5 gives

$$\frac{1}{m_j} \sum_{k=1}^j c_k m_k^2 \leq C' e^{-(A+j)\alpha} \sum_{k=A+1}^{A+j} k^{(1/d)-1} e^{k^\alpha} \leq C'' (A+j)^{(1/d)-\alpha} \leq C''' N^{(1/d)-\alpha},$$

where C''' depends only on α and d . Next, if $1 \leq q \leq N$, Lemma 2.4 gives

$$\begin{aligned} r_q m_q &= \left(\frac{q^{1/d} - (q-1)^{1/d}}{m_q - m_{q-1}} \right) m_q \leq D \left(\frac{q^{(1/d)-1}}{(A+q)^{\alpha-1} e^{(A+q)^\alpha}} \right) e^{(A+q)^\alpha} \\ &\leq D' (A+q)^{(1/d)-1} \leq D' N^{(1/d)-1}, \end{aligned}$$

where D' depends only on α and d . Inserting these estimates into Lemma 2.6 gives the result. \square

The following lemma is well known.

Lemma 2.8. Let $f(x) = \sum_k a_k e^{ikx}$ be a real trigonometric polynomial on $[0, 2\pi)^d$. Suppose

$$\sup\{|k_1|, \dots, |k_d|\} \leq M.$$

Then there exist points x_1, \dots, x_q where $q < CM^d$ and C depends only on d , such that

$$\|f\|_\infty \leq 2 \max_{j=1, \dots, q} |f(x_j)|.$$

Proof. Choose z such that $|f(z)| = \|f\|_\infty$. Let K be an integer not less than $4\pi dM$, and let the x_j be the lattice points in $[0, 2\pi)^d$ with coordinate spacing equal to $2\pi/K$. There are $q = K^d$ such points. Consider the x_j that is closest to z . Then by the mean value theorem there is a c_j on the line segment from z to x_j such that

$$f(z) - f(x_j) = \sum_{q=1}^d \frac{\partial f}{\partial t_q}(c_j)(z_q - x_{jq}).$$

Therefore

$$|f(z) - f(x_j)| \leq \sum_{q=1}^d \left| \frac{\partial f}{\partial t_q}(c_j) \right| \max_{r=1, \dots, d} |z_r - x_{jr}|.$$

Define $\epsilon_q = \text{sgn } \partial/\partial t_q(c_j)$, and define $u(\theta) = f(c_{j1} + \epsilon_1\theta, \dots, c_{jd} + \epsilon_d\theta)$. Then u is a single-variable real trigonometric polynomial of degree Md and

$$u'(0) = \sum_{q=1}^d \left| \frac{\partial f}{\partial t_q}(c_j) \right|.$$

By Bernstein's theorem $|u'(0)| \leq Md \|u\|_\infty \leq Md \|f\|_\infty$. Hence

$$|f(z) - f(x_j)| \leq Md \|f\|_\infty (2\pi/K) \leq \|f\|_\infty / 2$$

and the conclusion follows. \square

Before stating the main theorem, we establish some notation. Fix integers $N > 1$ and $d \geq 1$. For $k = 1, \dots, N$ define m_k and r_k as in Lemma 2.7 (we will assign values to α and A later). Let

$$I_k = [m_k, m_{k-1}) \cap \mathbf{Z}, \quad J_k = \bigcup_{j=1}^k I_j, \quad B_k = J_k^d \subset \mathbf{Z}^d,$$

$$L_1 = B_1, \quad L_k = B_k \setminus B_{k-1} \quad \text{for } k = 2, \dots, N.$$

On J_N define the discrete measure ν by $\nu(n) = r_k$ for $n \in I_k$. Then $\nu(I_k) = k^{1/d} - (k-1)^{1/d}$ and $\nu(J_k) = k^{1/d}$. Define a measure ν_k on J_k by the restriction of ν to J_k . Define the measure μ_k on B_k^d by the product $\mu_k = \nu_k^d$. Then $\mu_k(B_k) = k$ and $\mu_k(L_k) = 1$. Define the probability measure P_k on L_k by the restriction of μ_k to L_k .

Theorem 2.9. *Given integers $N > 1$ and $d \geq 1$, setting $\alpha = 1/(1 + 2d)$ and using the notation above, there exists constants $A > 0$ and C depending only on d such that if the multi-integers $\lambda_1, \dots, \lambda_N$ are chosen at random, independently and uniformly from the sets L_1, \dots, L_N , then the probability that*

$$\left| \sum_{k=1}^N \sin \lambda_{k1} x_1 \dots \sin \lambda_{kd} x_d \right| \leq C N^{(d+1)/(2d+1)}$$

is not less than $1/2$.

Proof. Let $\Lambda_1, \dots, \Lambda_N$ be independent random variables where Λ_k is uniformly distributed on L_k . Define for $x = (x_1, \dots, x_d)$ and $\lambda = (\lambda_1, \dots, \lambda_d)$

$$S(x, \lambda) = \prod_{j=1}^d \sin x_j \lambda_j.$$

Let f be the random trigonometric polynomial in $[0, 2\pi)^d$ defined by

$$f(x) = \sum_{k=1}^N S(x, \Lambda_k).$$

Note that the degree M of f is less than dm_N . Let x_j for $j = 1, \dots, q$ be the lattice points in $[0, 2\pi)^d$ as defined in Lemma 2.8. We will estimate the expectation of the random variables $Z_j = f(x_j)$:

$$E(Z_j) = \sum_{k=1}^N E(S(x_j, \Lambda_k)) = \sum_{k=1}^N \sum_{\lambda_k \in L_k} S(x_j, \lambda_k) \mu_k(\lambda_k).$$

Since $\mu_k(\lambda_k) = \prod_{j=1}^d v(\lambda_{kj})$,

$$\begin{aligned} E(Z_j) &= \sum_{k=1}^N \sum_{\lambda_k \in L_k} \left(\prod_{j=1}^d \sin \lambda_{kj} x_j \right) \left(\prod_{j=1}^d v(\lambda_{kj}) \right) \\ &= \sum_{\lambda \in J_N} \prod_{j=1}^d \sin(\lambda_j x_j) v(\lambda_j) = \left(\sum_{\lambda \in J_N} v(\lambda) \sin \lambda x \right)^d. \end{aligned}$$

Recalling that $\alpha = 1/(1+2d)$ and using Lemma 2.7 gives $E(Z_j) \leq CN^{(d+1)/(2d+1)}$. Next, by Lemma 2.1 we have

$$\begin{aligned} E(\exp[\lambda(Z_j - E(Z_j))]) &= E\left(\exp\left(\lambda \sum_{k=1}^N S(x_j, \Lambda_k) - E(S(x_j, \Lambda_k))\right)\right) \\ &= \prod_{j=1}^N E\left(\exp(\lambda[S(x_j, \Lambda_k) - E(S(x_j, \Lambda_k))])\right) \\ &\leq \prod_{k=1}^N \exp(\lambda^2/2) = \exp(N\lambda^2/2). \end{aligned}$$

Therefore, by Lemma 2.3,

$$\mathcal{P}(|Z_j - E(Z_j)| \leq \sqrt{2N \log 4q}, \text{ for } j = 1, \dots, q) \geq \frac{1}{2}$$

and

$$\mathcal{P}\left(\max_{j=1, \dots, q} |f(x_j)| \leq |E(Z_j)| + \sqrt{2N \log 4q}\right) \geq \frac{1}{2}.$$

By Lemma 2.8, $\|f\|_\infty \leq C' \max_{j=1, \dots, q} |f(x_j)|$. Also $q \leq C'' e^{N^\alpha} = e^{N^{1/(2d+1)}}$ and $|E(Z_j)| \leq C''' N^{(d+1)/(2d+1)}$. Therefore there exists a constant C such that

$$\mathcal{P}(\|f\|_\infty \leq CN^{(d+1)/(2d+1)}) \geq \frac{1}{2}.$$

We remark that the probability $1/2$ can be replaced by probability $1 - \epsilon$ for any $\epsilon > 0$, by choosing a larger (ϵ -dependent) constant C . \square

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