



Weak compactness and fixed point property for affine mappings

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Abstract

It is shown that a closed convex bounded subset of a Banach space is weakly compact if and only if it has the generic fixed point property for continuous affine mappings. The class of continuous affine mappings can be replaced by the class of affine mappings which are uniformly Lipschitzian with some constant $M > 1$ in the case of c_0 , the class of affine mappings which are uniformly Lipschitzian with some constant $M > \sqrt{6}$ in the case of quasi-reflexive James' space J and the class of nonexpansive affine mappings in the case of L -embedded spaces.

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1. Introduction

Lin and Sternfeld [10] gave the complete characterization of norm compactness for convex subsets of a Banach space in terms of a fixed point property. They proved that if a convex set K is not compact, then there exists a Lipschitzian mapping $f : K \rightarrow K$ with $\inf\{\|x - f(x)\| : x \in K\} > 0$. It follows that a convex set in a Banach space has the fixed point property for Lipschitzian mappings if and only if it is compact. In this paper we study similar problems for weak compactness.

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Let X be a Banach space. We say that a closed convex bounded subset C of X has the generic fixed point property for a class of mappings, if every mapping from a convex closed subset of C into itself belonging to this class has a fixed point. We will show that weak compactness of convex sets can be characterized in terms of the generic fixed point property for some classes of affine mappings. A continuous affine self-mapping of a closed convex set is weakly continuous. The well-known Schauder–Tychonoff theorem (see [4], p. 74) shows therefore that a continuous affine self-mapping of a convex weakly compact subset C of a Banach space X has a fixed point (see also [15]). To complete the characterization, in a closed convex bounded but not weakly compact set C we construct a closed convex subset K which admits a continuous affine self-mapping without a fixed point. Thus, in general a closed convex bounded subset of a Banach space is weakly compact if and only if it has the generic fixed point property for continuous affine mappings.

Moreover, in some spaces it is possible to replace the class of all continuous affine mappings by a smaller one. For instance, in [3] convex weakly compact subsets of the space $L_1[0, 1]$ are characterized as the only ones which have the generic fixed point property for nonexpansive (i.e. Lipschitzian with constant 1) affine mappings. A similar result was proved for the preduals of semi-finite von Neumann algebra equipped with a faithful normal semi-finite trace.

In this paper, we prove that a closed convex bounded subset C of c_0 is weakly compact if and only if C has the generic fixed point property for affine mappings which are uniformly Lipschitzian. In fact, we prove that these mappings can be chosen with Lipschitz constant arbitrarily close to 1. As far as we know, it is an open problem if the constant can be chosen equal to 1. Since Maurey [13] proved that convex weakly compact subsets of c_0 have the generic fixed point property for nonexpansive mappings, a positive answer to the above problem would give the inverse of Maurey’s result (see [12] for related results).

The main tools for proving our results are basic sequences equivalent to the summing basis of c_0 . We will show that such a sequence can be extracted from any sequence (x_n) in c_0 which converges weak[★] in ℓ_∞ to an element $x \in \ell_\infty \setminus c_0$. The summing basis can also be considered in generalized James’ spaces J_p , $1 < p < \infty$. Their definition extends the notion of quasi-reflexive James’ space J , which is J_2 in this notation. We will show that a convex closed bounded subset C of J_p is weakly compact if and only if there is $M > 3^{1/p}2^{1/q}$, where $1/p + 1/q = 1$, such that C has the generic fixed point property for uniformly Lipschitzian affine mappings with constant M . In the last section we will extend the results given in [3] to a larger class of spaces, the so-called L -embedded Banach spaces.

2. Preliminaries

The notation and terminology used in this paper are standard. They can be found for instance in [2, 11]. For convenience of the reader we recall the basic definitions. Let C be a nonempty subset of a Banach space. The convex hull of C will be denoted

by co C . Let us recall that a self-mapping T of a convex set C is said to be affine if

$$T(\lambda x + (1 - \lambda)y) = \lambda Tx + (1 - \lambda)Ty$$

whenever $x, y \in C$ and $\lambda \in [0, 1]$. A mapping $T : C \rightarrow C$ is nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|$$

for all $x, y \in C$. We say that T is uniformly Lipschitzian with a constant M if

$$\|T^n x - T^n y\| \leq M \|x - y\|$$

for every $n \in \mathbb{N}$ and all $x, y \in C$.

Basic sequences will be our main tool in this paper. Let (x_n) be a sequence in a Banach space X . Its closed linear span will be denoted by $[x_n]$. Let us recall that (x_n) is a basic sequence if each $x \in [x_n]$ has a unique expansion of the form $x = \sum_{n=1}^{\infty} t_n x_n$ for some scalars t_1, t_2, \dots . Then the projections P_n defined on $[x_n]$ by the formula

$$P_n \left(\sum_{i=1}^{\infty} t_i x_i \right) = \sum_{i=1}^n t_i x_i$$

are uniformly bounded and $S = \sup\{\|P_n\| : n \in \mathbb{N}\}$ is called the basis constant of (x_n) (see [11]). It is clear that

$$\inf\{\|x - y\| : x \in [x_i]_{i=1}^n, \|x\| \geq a, y \in [x_i]_{i=n+1}^{\infty}, n \in \mathbb{N}\} \geq \frac{a}{S} \tag{1}$$

for every $a > 0$. Additionally, we put $R_n = Id_{[x_n]} - P_n$ and

$$S^+(\{x_n\}) = \left\{ x = \sum_{n=1}^{\infty} t_n x_n : t_n \geq 0 \text{ for every } n \in \mathbb{N}, \sum_{n=1}^{\infty} t_n = 1 \right\}.$$

In the sequel we will use the following fact.

Fact 2.1. *Let (x_n) be a bounded sequence in a Banach space X without weak convergent subsequences. Then*

(i) *there exist a subsequence (x_{n_k}) and a functional $f \in X^*$ such that (x_{n_k}) is a basic sequence and $a = \inf\{f(x_{n_k}) : k \in \mathbb{N}\} > 0$. Consequently, setting $g = (1/a)f$ and $y_k = (a/f(x_{n_k}))x_{n_k}$, we have $g(y_k) = 1$ for every $k \in \mathbb{N}$.*

(ii) $\overline{\text{co}}(y_k) = S^+(\{y_k\})$.

Proof. (i) By Kadec and Pełczyński [7], (x_n) has a basic subsequence (x_{n_k}) . Our assumption guarantees that (x_{n_k}) does not weakly converge to zero. Passing to a subsequence, we can therefore find $f \in X^*$ so that $\inf\{f(x_{n_k}) : k \in \mathbb{N}\} > 0$. (ii) is trivial. \square

Remark 2.1. The reasoning in the proof of (i) works only for real spaces. In the case of a complex space it is necessary to replace the functional f by its real or imaginary part.

A sequence (y_k) of nonzero vectors of X is said to be a block basic sequence of a basic sequence (x_n) if there exist a sequence (α_n) of scalars and an increasing sequence of integers $0 \leq p_1 < p_2 < \dots$ such that

$$y_k = \sum_{i=p_k+1}^{p_{k+1}} \alpha_i x_i$$

for every k . Clearly, (y_k) is also a basic sequence and the basis constant of (y_k) does not exceed that of (x_n) .

Let (x_n) and (y_n) be basic sequences. We say that (x_n) is equivalent to (y_n) provided that a series $\sum_{n=1}^{\infty} t_n x_n$ converges if and only if $\sum_{n=1}^{\infty} t_n y_n$ converges. This is the case if and only if there exist constants $M_1, M_2 \in (0, \infty)$ such that

$$M_1 \left\| \sum_{n=1}^{\infty} t_n x_n \right\| \leq \left\| \sum_{n=1}^{\infty} t_n y_n \right\| \leq M_2 \left\| \sum_{n=1}^{\infty} t_n x_n \right\| \tag{2}$$

for every sequence (t_n) of scalars such that the above series converge. We say that (x_n) is λ -equivalent to (y_n) if $M_2/M_1 \leq \lambda$. Clearly, the relation of λ -equivalence is symmetric. We will apply the following result (see [11, Proposition 1.a.9]).

Theorem 2.3. Let (x_n) be a basic sequence with the basis constant K in a Banach space X and let $M = \inf\{\|x_n\| : n \in \mathbb{N}\} > 0$. If (y_n) is a sequence in X such that

$$s = \sum_{n=1}^{\infty} \|x_n - y_n\| < \frac{M}{2K},$$

then (2) holds with $M_1 = 1 - 2Ks/M$ and $M_2 = 1 + 2Ks/M$. Consequently, (y_n) is a basic sequence $(1 + 2Ks/M)(1 - 2Ks/M)^{-1}$ -equivalent to (x_n) .

3. Characterization of weakly compact convex sets

Let (e_n) be the standard basis of the space c_0 . The sequence of vectors $\sigma_n = \sum_{k=1}^n e_k = (1, \dots, 1, 0, 0, \dots)$ is called the summing basis. It is easy to see that

$$\left\| \sum_{n=1}^{\infty} t_n \sigma_n \right\| = \sup_{n \in \mathbb{N}} \left| \sum_{k=n}^{\infty} t_k \right|$$

for every sequence (t_n) of scalars such that the series $\sum_{n=1}^{\infty} t_n$ converges.

Let $1 < p < \infty$. By J_p we denote the space of all sequences $x = (x(n))$ of real numbers such that $\lim_{n \rightarrow \infty} x(n) = 0$ and

$$\|x\| = \sup \left(\sum_{k=1}^{m-1} |x(q_k) - x(q_{k+1})|^p \right)^{1/p} < \infty,$$

where the supremum is taken over all finite sequences $q_1 < \dots < q_m$ of positive integers. In case $p = 2$ this gives us the well-known definition of James' space (see [6]). Let P_n be the projection associated to the standard basis (e_n) of J_p . Then

$$P_n x = (x(1), \dots, x(n), 0, 0, \dots)$$

for every $x \in J_p$. Using this formula, we extend P_n to the linear space of all sequences. For each p , the space J_p is not reflexive and $J_p^{\star\star}$ is the space of all convergent sequences x such that

$$\|x\|_{J_p^{\star\star}} = \sup_{n \in \mathbb{N}} \|P_n x\|_{J_p}$$

is finite (see [11], Proposition 1.b.2)). Moreover, J_p does not contain c_0 and ℓ_1 isomorphically.

As in the case of c_0 , the vectors $\sigma_n = (1, \dots, 1, 0, 0, \dots)$ form a basis of J_p . If (t_k) is a sequence of scalars such that the series $\sum_{k=1}^{\infty} t_k \sigma_k$ converges in J_p , then

$$\left\| \sum_{k=1}^{\infty} t_k \sigma_k \right\| = \sup \left(\sum_{k=1}^{m-1} \left| \sum_{i=q_k}^{q_{k+1}-1} t_i \right|^p \right)^{1/p},$$

where the supremum is taken over all finite sequences $q_1 < \dots < q_m$ of positive integers.

Proposition 3.1. (a) Let C be a closed convex bounded set in a Banach space X and (e_n) be the natural basis of ℓ_1 . If C is not weakly compact, then C contains a basic sequence (y_n) such that there is an affine homeomorphism $\phi : S^+(\{y_n\}) \rightarrow S^+(\{e_n\})$ with $\phi(y_n) = e_n$ for every $n \in \mathbb{N}$.

(b) Let C be a closed convex bounded set in J_p and (σ_n) be the summing basis of J_p . If C is not weakly compact, then C contains a basic sequence (y_n) equivalent to (σ_n) . In particular, C contains a closed convex subset $K = S^+(\{y_n\})$ which is bi-Lipschitz homeomorphic to $S^+(\{\sigma_n\})$.

(c) Let C be a closed convex bounded set in c_0 and (σ_n) be the summing basis of c_0 . If C is not weakly compact, then C contains a basic sequence (y_n) equivalent to (σ_n) . In particular, C contains a closed convex subset $K = S^+(\{y_n\})$ which is bi-Lipschitz homeomorphic to $S^+(\{\sigma_n\})$.

Proof. (a) Translating the set C , we can assume that $0 \in C$. Then Fact 2.1 gives a basic sequence (y_n) in C and a functional $g \in X^\star$ with $g(y_n) = 1$ for every $n \in \mathbb{N}$. Let $K = S^+(\{y_n\})$ and $\phi : K \rightarrow S^+(\{e_n\})$ be the affine mapping such that $\phi(y_n) = e_n$ for every $n \in \mathbb{N}$. We will check that ϕ is a homeomorphism.

Take $x = \sum_{n=1}^\infty t_n y_n \in K$ and $\varepsilon > 0$. Fix an index m such that $\sum_{n=m+1}^\infty t_n < \varepsilon/4$ and put $\delta = \varepsilon/(8m\|g\|(S+1))$ where S is the basis constant of (y_n) . If $u = \sum_{n=1}^\infty b_n y_n \in K$ is such that $\|x - u\| < \delta$, then $|t_n - b_n| < 2S\|g\|\delta$ for every $n \in \mathbb{N}$ and hence, $\sum_{n=1}^m |t_n - b_n| < \varepsilon/4$. Next,

$$\left| \sum_{n=m+1}^\infty (b_n - t_n) \right| = |g(R_m(u - x))| \leq \|g\| \|R_m\| \|u - x\| < \frac{\varepsilon}{4},$$

and therefore

$$\sum_{n=m+1}^\infty b_n \leq \left| \sum_{n=m+1}^\infty (b_n - t_n) \right| + \sum_{n=m+1}^\infty t_n < \frac{\varepsilon}{2}.$$

Finally, we have

$$\|\phi(u) - \phi(x)\|_{\ell_1} = \sum_{n=1}^\infty |b_n - t_n| \leq \sum_{n=1}^m |b_n - t_n| + \sum_{n=m+1}^\infty b_n + \sum_{n=m+1}^\infty t_n < \varepsilon,$$

which shows that ϕ is continuous.

Clearly, $\|\phi^{-1}(u) - \phi^{-1}(v)\| \leq \max_{n \in \mathbb{N}} \|y_n\| \|u - v\|_{\ell_1}$ for all $u, v \in S^+(\{e_n\})$. Thus ϕ^{-1} is continuous.

(b) Since C is not weakly compact, there exists a sequence (x_n) in C such that (x_n) converges weak * in J_p^\star to some $x \in J_p^\star \setminus J_p$. Passing to a subsequence, we can assume that (x_n) is a basic sequence (see [7]). Let S be its basis constant.

We put $M_1 = \inf\{\|x_n\| : n \in \mathbb{N}\}$, $M_2 = \sup\{\|x_n\| : n \in \mathbb{N}\}$. Since x is a convergent sequence and $x \notin J_p$, $L = \lim_{k \rightarrow \infty} \|x(k)\| > 0$. Given $\varepsilon \in (0, 1)$, we set $M = (1 - \varepsilon/16)L$ and $\gamma_k = M_1 \varepsilon (S(\varepsilon + 16))^{-1} 2^{-k-2}$ for $k \in \mathbb{N}$. It is easy to see that there exists $m_0 \in \mathbb{N}$ such that if $m_0 \leq q_1 < \dots < q_m$, then

$$\sum_{k=1}^{m-1} |x(q_k) - x(q_{k+1})|^p \leq \left(\frac{L\varepsilon}{16}\right)^p. \tag{3}$$

Next, we choose two increasing sequences (m_k) and (n_k) so that $m_1 \geq m_0$,

$$\|P_{m_k}(x_{n_k} - x)\| < \gamma_k, \quad \|R_{m_{k+1}}(x_{n_k})\| < \gamma_{k+1},$$

and $|x(j)| > M$ for every $j \geq m_1$.

We put $u_k = P_{m_k}x$, $v_k = (P_{m_{k+1}} - P_{m_k})(x_{n_k})$ and $w_k = u_k + v_k$. Then

$$\begin{aligned} \|w_k - x_{n_k}\| &\leq \|P_{m_k}(u_k - x_{n_k})\| + \|R_{m_k}(v_k - x_{n_k})\| \\ &= \|P_{m_k}(x - x_{n_k})\| + \|R_{m_{k+1}}(x_{n_k})\| < 2\gamma_k \end{aligned}$$

for every k . Applying Theorem 2.3, we see that (w_k) is a basic sequence $(1 + \varepsilon/8)$ -equivalent to (x_{n_k}) . Let K_1 denote the basis constant of (w_k) . We choose a sequence (p_n) of nonnegative integers such that $\Delta_k > 2(M_2 K_1 2^{k+1} (\varepsilon + 16) / (M\varepsilon))^q$ for every k where $\Delta_k = p_{k+1} - p_k$ and $1/p + 1/q = 1$. Let

$$z_k = \frac{1}{\Delta_k} \sum_{i=p_{k+1}}^{p_{k+1}} u_i, \quad z'_k = \frac{1}{\Delta_k} \sum_{i=p_{k+1}}^{p_{k+1}} w_i.$$

Then $\|z'_k\| \geq |z'_k(m_{p_{k+1}})| > M$ and it is easy to see that

$$\|z'_k - z_k\| = \frac{1}{\Delta_k} \left\| \sum_{i=p_{k+1}}^{p_{k+1}} v_i \right\| \leq \frac{2^q}{\Delta_k} \left(\sum_{i=p_{k+1}}^{p_{k+1}} \|v_i\|^p \right)^{1/p} < \frac{M\varepsilon}{K_1(\varepsilon + 16)2^{k+1}}$$

for every k . Theorem 2.3 shows that (z_k) is a basic sequence $(1 + \varepsilon/8)$ -equivalent to (z'_k) . Consequently, (z_k) is $(1 + \varepsilon/8)^2$ -equivalent to a block basic sequence (y_n) of (x_{n_k}) whose terms belong to $\text{co}\{x_{n_k}\}$.

We will show that (z_k) is equivalent to (σ_k) . To this end let us fix a sequence (t_k) such that the series $\sum_{k=1}^\infty t_k \sigma_k$ converges and put $N = \|\sum_{k=1}^\infty t_k \sigma_k\|$, $y = \sum_{k=1}^\infty t_k z_k$. We take a finite sequence $q_1 < \dots < q_m$ of positive integers. By A_1 we denote the set of all $1 \leq j < m$ such that there exists $k \geq 2$ with $q_j \leq m_{p_k} < q_{j+1}$ and let A_2 be the set of the remaining indices. Given $j \in A_1$, we find $k \geq 1$, $0 \leq i_1 \leq \Delta_k - 1$, $l \geq 2$ and $0 \leq i_2 \leq \Delta_l - 1$ such that $m_{p_k+i_1} < q_j \leq m_{p_k+i_1+1} < m_{p_l+i_2} < q_{j+1} \leq m_{p_l+i_2+1}$. Then

$$\begin{aligned} & |y(q_j) - y(q_{j+1})| \\ &= \left| x(q_j) \left(\lambda t_k + \sum_{i=k+1}^{l-1} t_i + (1 - \mu) t_l \right) + (x(q_j) - x(q_{j+1})) \left(\mu t_l + \sum_{i=l+1}^\infty t_i \right) \right| \\ &\leq |x(q_j)| \max \left\{ \left| \sum_{i=k+v}^{l-v} t_i \right| : v, v = 0, 1 \right\} + |x(q_j) - x(q_{j+1})| \sup_{n \in \mathbb{N}} \left| \sum_{i=n}^\infty t_i \right|, \end{aligned}$$

where $\lambda = 1 - i_1/\Delta_k$, $\mu = 1 - i_2/\Delta_l$. This gives us the estimate

$$|y(q_j) - y(q_{j+1})| \leq M_2 \left| \sum_{i=k_j}^{l_j} t_i \right| + |x(q_j) - x(q_{j+1})| N \tag{4}$$

for some $k \leq k_j \leq l_j \leq l$. Observe that

$$\sum_{j \in A_1} \left| \sum_{i=k_j}^{l_j} t_i \right|^p \leq 2N^p. \tag{5}$$

Let us now consider the set A_2 . We decompose it into disjoint intervals A_2^k where $A_2^1 = \{j \in A_2 : 1 \leq q_j < q_{j+1} \leq m_{p_2}\}$ and $A_2^k = \{j \in A_2 : m_{p_k} < q_j < q_{j+1} \leq m_{p_{k+1}}\}$ for $k \geq 2$.

Assume that A_2^k is not empty. It is easy to see that if $j \in A_2^k$, then

$$|y(q_j) - y(q_{j+1})| \leq \lambda_j^k |x(q_j)t_k| + |x(q_j) - x(q_{j+1})|N \tag{6}$$

for some nonnegative λ_j^k such that $\sum_{j \in A_2^k} \lambda_j^k \leq 1$. Clearly,

$$\sum_{k=1}^{\infty} \sum_{j \in A_2^k} (\lambda_j^k |t_k|)^p \leq \sum_{k=1}^{\infty} |t_k|^p \leq N^p. \tag{7}$$

Here we regard sums over the empty set as zero. Using (4)–(7), we obtain

$$\begin{aligned} \left(\sum_{j=1}^{m-1} |y(q_j) - y(q_{j+1})|^p \right)^{1/p} &\leq M_2 \left(2N^p + \sum_{k=1}^{\infty} |t_k|^p \right)^{1/p} + N|x| \\ &\leq M_2(3^{1/p} + 1)N. \end{aligned} \tag{8}$$

We now set $r_k = m_{p_{k+1}}$ for $k \in \mathbb{N}$. If $i < j$, then

$$\begin{aligned} |y(r_i) - y(r_j)| &= \left| x(r_i) \left(\sum_{k=i}^{j-1} t_k \right) + (x(r_i) - x(r_j)) \left(\sum_{k=j}^{\infty} t_k \right) \right| \\ &\geq |x(r_i)| \left| \sum_{k=i}^{j-1} t_k \right| - |x(r_i) - x(r_j)| \left| \sum_{k=j}^{\infty} t_k \right| \\ &\geq M \left| \sum_{k=i}^{j-1} t_k \right| - |x(r_i) - x(r_j)|N. \end{aligned}$$

This and (3) show that

$$\|y\| \geq \left(\sum_{k=1}^{m-1} |y(r_{q_k}) - y(r_{q_{k+1}})|^p \right)^{1/p} \geq M \left(\sum_{k=1}^{m-1} \left| \sum_{i=q_k}^{q_{k+1}-1} t_i \right|^p \right)^{1/p} - \frac{L\varepsilon}{16} N$$

for every sequence $q_1 < \dots < q_m$ of positive integers. Hence

$$\|y\| \geq N \left(M - \frac{L\varepsilon}{16} \right) = NL \left(1 - \frac{\varepsilon}{8} \right). \tag{9}$$

This completes the proof of (b).

(c) To prove (c) we can apply an argument similar to that in the proof of (b) (now $L = \limsup_{k \rightarrow \infty} |x(k)| > 0$). In this way it is not difficult to construct a basic sequence (z_k) which is M_2/M equivalent to the summing basis (σ_n) of c_0 and such that (z_k) is $(1 + \varepsilon/8)^2$ equivalent to a block basic (y_n) of (x_{n_k}) whose terms belong to $\text{co}\{x_{n_k}\}$. \square

In the special case when C is the unit ball of a nonreflexive space part (a) of Proposition 3.1 was obtained in [15] (see also [14]). We will generalize another result from [15].

Let $C \neq \emptyset$ be a convex subset of a Banach space and $T : C \rightarrow C$ be an affine mapping. We put

$$\theta(T) = \inf \left\{ \liminf_{n \rightarrow \infty} \|x - T^n y\| : x, y \in C \right\}.$$

From the first part of the proof of [15, Theorem 3] we see that $\inf \{ \|x - Tx\| : x \in C \} = 0$. In spite of this fact, if C is not weakly compact, it is possible to construct a set $K \subset C$ and a continuous affine mapping $T : K \rightarrow K$ such that $\theta(T) > 0$. In particular, T fails to have fixed points.

Let (x_n) be a bounded basic sequence. The right shift T_0 with respect to (x_n) is the mapping defined by the formula

$$T_0 \left(\sum_{n=1}^{\infty} t_n x_n \right) = \sum_{n=1}^{\infty} t_n x_{n+1}.$$

By the bilateral shift T_1 with respect to (x_n) we in turn mean the mapping

$$T_1 \left(\sum_{n=1}^{\infty} t_n x_n \right) = t_2 x_1 + \sum_{k=1}^{\infty} t_{2k-1} x_{2k+1} + \sum_{k=2}^{\infty} t_{2k} x_{2k-2}.$$

Clearly, T_0 and T_1 are affine self-mappings of $S^+(\{x_n\})$ and T_1 is onto.

Theorem 3.2. (a) *Let C be a closed convex bounded set in a Banach space X . If C is not weakly compact, then there are a closed convex subset $K \subset C$ and an affine continuous mapping $T : K \rightarrow K$ such that $T(K) = K$ and $\theta(T) > 0$.*

(b) *Let $C \subset J_p$ be a closed convex bounded set. If C is not weakly compact, then there are a closed convex subset $K \subset C$ and an affine uniformly Lipschitzian mapping $T : K \rightarrow K$ such that $\theta(T) > 0$.*

(c) *Let $C \subset c_0$ be a closed convex bounded set. If C is not weakly compact, then there are a closed convex subset $K \subset C$ and an affine uniformly Lipschitzian mapping $T : K \rightarrow K$ such that $\theta(T) > 0$.*

Proof. (a) Let (e_n) be the standard basis of ℓ_1 and $T_1 : S^+(\{e_n\}) \rightarrow S^+(\{e_n\})$ be the bilateral shift. Fact 2.1 gives us a sequence (y_n) in C and a functional $g \in X^*$. Let S be the basis constant of (y_n) and $K = S^+(\{y_n\})$. The proof of Proposition 3.1(a) shows that the affine mapping $\phi : K \rightarrow S^+(\{e_n\})$ such that $\phi(y_n) = e_n$ is a homeomorphism. Define $T : K \rightarrow K$ by the formula $T = \phi^{-1} T_1 \phi$. Let $x = \sum_{n=1}^{\infty} t_n y_n$ and $y = \sum_{n=1}^{\infty} b_n y_n$ belong to K . We fix $\varepsilon > 0$ and find m such that $\|R_m x\| < \varepsilon$. It is not

difficult to see that $\|P_m T^n y\| < \varepsilon$ for n large enough. By (1) we obtain

$$\begin{aligned} \|x - T^n y\| &\geq \|P_m x - R_m T^n y\| - \|R_m x\| - \|P_m T^n y\| \geq \frac{\|P_m x\|}{S} - 2\varepsilon \\ &\geq \frac{1}{S} (\|x\| - \varepsilon) - 2\varepsilon \geq \frac{1}{S} (\|g\|^{-1} - \varepsilon) - 2\varepsilon. \end{aligned}$$

Thus $\theta(T) \geq 1/(S\|g\|)$.

(b) Let (y_n) in C be a sequence given in Proposition 3.1(b). Then there is a bi-Lipschitz homeomorphism ϕ between $K = S^+(\{y_n\})$ and $S^+(\{\sigma_n\})$. We can therefore define $T : K \rightarrow K$ by $T = \phi^{-1} T_0 \phi$ where T_0 is the right shift with respect to (σ_n) . Since T_0 is nonexpansive, the mapping T is uniformly Lipschitzian.

Let $x = \sum_{n=1}^\infty t_n y_n$, $y = \sum_{n=1}^\infty b_n y_n$ belong to K . Since (y_n) is equivalent to the summing basis (σ_n) of J_p , there is a positive constant M such that $\|x\| \geq M \|\sum_{n=1}^\infty t_n \sigma_n\| \geq M$. Let $\varepsilon \in (0, M)$. An argument as above yields to

$$\|x - T^n y\| \geq \frac{M - \varepsilon}{S} - 2\varepsilon.$$

Thus $\theta(T) \geq M/S$. The proof of (c) is analogous to (b). \square

Remark 3.3. Parts (b) and (c) of Theorem 3.2 may be strengthened. Namely, for any $\varepsilon > 0$ in case (b) we can choose T so that it is uniformly Lipschitzian with the constant $2^{1/q} 3^{1/p} + \varepsilon$, where $1/q + 1/p = 1$, and in case (c) we can choose T so that it is uniformly Lipschitzian with the constant $1 + \varepsilon$.

Indeed, let (y_n) be a sequence in J_p given in Proposition 3.1(b). We can find $v \in \mathbb{N}$ such that if $n \geq v$, then

$$\left\| \sum_{k=1}^\infty t_k y_{n+k} \right\| \leq 2^{1/q} 3^{1/p} (1 + \varepsilon) \left\| \sum_{k=1}^\infty t_k y_{v+k} \right\|$$

for every sequence (t_k) of scalars such that the series $\sum_{k=1}^\infty t_k \sigma_k$ converges. For this purpose, in the proof of Proposition 3.1(b) we choose $v \in \mathbb{N}$ so that $|x(j)| < L(1 + \varepsilon/16)$ for every $j > m_{p_n}$. Let (t_k) be a sequence of scalars such that the series $\sum_{k=1}^\infty t_k \sigma_k$ converges. Given $n \geq v$, we write $z = \sum_{k=1}^\infty t_k z_{v+k}$ and $z' = \sum_{k=1}^\infty t_k z_{n+k}$. We put

$$N_1 = \sup \left(\sum_{k=1}^{s-1} \left| \sum_{i=j_k}^{j_{k+1}-1} t_i \right|^p \right)^{1/p}$$

and

$$N_2 = \sup \left(\sum_{k=1}^{s-1} |z(j_k) - z(j_{k+1})|^p \right)^{1/p},$$

where both the suprema are taken over all finite sequences $m_{p_v} < j_1 < \dots < j_s$. The same reasoning as in the case of (9) shows that

$$N_1 L \left(1 - \frac{\varepsilon}{8} \right) \leq N_2.$$

Let us consider a sequence $q_1 < \dots < q_m$ of positive integers. If $q_l = m_{p_v} + 1$ for some $1 < l < m$, then $z(q_k) = z'(q_k)$ for every $k \leq l$ and modifying the proof of (8), we obtain

$$\left(\sum_{k=l+1}^{m-1} |z'(q_k) - z'(q_{k+1})|^p \right)^{1/p} \leq 3^{1/p} \left(1 + \frac{\varepsilon}{8} \right) N_1 L.$$

Hence

$$\begin{aligned} \sum_{k=1}^{m-1} |z'(q_k) - z'(q_{k+1})|^p &\leq \sum_{k=1}^{l-1} |z(q_k) - z(q_{k+1})|^p + 3 \left(\left(1 + \frac{\varepsilon}{8} \right) N_1 L \right)^p \\ &\leq \sum_{k=1}^{l-1} |z(q_k) - z(q_{k+1})|^p + 3 \left(\frac{1 + \frac{\varepsilon}{8}}{1 - \frac{\varepsilon}{8}} \right)^p N_2^p \\ &\leq 3 \left(\frac{1 + \frac{\varepsilon}{8}}{1 - \frac{\varepsilon}{8}} \right)^p \left(\sum_{k=1}^{l-1} |z(q_k) - z(q_{k+1})|^p + N_2^p \right) \\ &\leq 3 \left(\frac{1 + \frac{\varepsilon}{8}}{1 - \frac{\varepsilon}{8}} \right)^p \|z\|^p. \end{aligned}$$

In the case when there exists $1 < l < m$ such that $q_l < m_{p_v} + 1 < q_{l+1}$ we replace the sequence q_1, \dots, q_m by q'_1, \dots, q'_{m+1} where $q'_i = q_i$ if $1 \leq i \leq l$, $q'_{l+1} = m_{p_v} + 1$ and $q'_i = q_{i-1}$ if $l + 2 \leq i \leq m + 1$. Clearly,

$$\begin{aligned} \left(\sum_{k=1}^{m-1} |z'(q_k) - z'(q_{k+1})|^p \right)^{1/p} &\leq 2^{\frac{1}{q}} \left(\sum_{k=1}^m |z'(q'_k) - z'(q'_{k+1})|^p \right)^{1/p} \\ &\leq 2^{\frac{1}{q}} 3^{\frac{1}{p}} \frac{1 + \frac{\varepsilon}{8}}{1 - \frac{\varepsilon}{8}} \|z\|. \end{aligned}$$

It follows that $\|z'\| \leq 2^{1/q} 3^{1/p} (1 + \varepsilon/8)(1 - \varepsilon/8)^{-1} \|z\|$. But (y_n) is $(1 + \varepsilon/8)^2$ -equivalent to (z_n) . This implies that

$$\begin{aligned} \left\| \sum_{k=1}^{\infty} t_k y_{n+k} \right\| &\leq 2^{\frac{1}{q}} 3^{\frac{1}{p}} \left(1 + \frac{\varepsilon}{8}\right)^3 \left(1 - \frac{\varepsilon}{8}\right)^{-1} \left\| \sum_{k=1}^{\infty} t_k y_{v+k} \right\| \\ &\leq 2^{\frac{1}{q}} 3^{\frac{1}{p}} (1 + \varepsilon) \left\| \sum_{k=1}^{\infty} t_k y_{v+k} \right\|. \end{aligned}$$

Let now $K = S^+(\{y_{n+v}\}_n)$ and $T : K \rightarrow K$ be the right shift. The above inequality shows that the Lipschitz constant of T^{n-v} does not exceed $2^{1/q} 3^{1/p} + \varepsilon$.

In the case of c_0 we can, analogously, find $v \in \mathbb{N}$ such that if $n \geq v$, then

$$\left\| \sum_{k=1}^{\infty} t_k y_{n+k} \right\| \leq (1 + \varepsilon) \left\| \sum_{k=1}^{\infty} t_k y_{v+k} \right\|$$

for every sequence (t_k) of scalars such that the series $\sum_{k=1}^{\infty} t_k$ converges. Thus the right shift T on $K = S^+(\{y_{n+v}\}_n)$ is uniformly Lipschitzian with the constant $1 + \varepsilon$.

Let Γ be an infinite set. Every sequence (x_n) in $c_0(\Gamma)$ is contained in a subspace Y of $c_0(\Gamma)$ such that Y is isometrically isomorphic to c_0 . It follows that Proposition 3.1(c), and Theorem 3.2(c) hold also for the space $c_0(\Gamma)$.

From Theorem 3.2 and Remark 3.3 we deduce the following characterization of convex bounded weakly compact sets.

Corollary 3.4. (a) *Let $C \neq \emptyset$ be a closed convex bounded subset of a Banach space X . The set C is weakly compact if and only if C has the generic fixed point property for continuous affine mappings.*

(b) *Let $C \neq \emptyset$ be a convex closed bounded subset of J_p . The set C is weakly compact if and only if there exists $M > 2^{1/q} 3^{1/p}$ such that C has the generic fixed point property for affine mappings which are uniformly Lipschitzian with the constant M .*

(c) *Let Γ be an infinite set and $C \neq \emptyset$ be a convex closed bounded subset of $c_0(\Gamma)$. The set C is weakly compact if and only if there exists $M > 1$ such that C has the generic fixed point property for affine mappings which are uniformly Lipschitzian with the constant M .*

4. Characterization of weakly compact convex subsets of L -embedded Banach spaces

Let Y be a Banach space and P be a projection in Y . P is called an L -projection if $\|x\| = \|Px\| + \|(Id - P)x\|$ for all $x \in Y$. A closed subspace $X \subset Y$ is called an L -summand in Y if X is the range of an L -projection on Y . A Banach space X is said to be L -embedded if X is an L -summand in X^{***} . Then there exists a closed subspace $X_s \subset X^{***}$ such that $X^{***} = X \oplus_1 X_s$. Examples of L -embedded Banach spaces are the $L_1(\mu)$ -spaces, preduals of von Neumann algebras, the dual of the disk algebra A^*

and the quotient space L_1/H_0^1 . Another class of L -embedded Banach spaces are the duals of M -embedded Banach spaces. A Banach space E is called an M -embedded space (also called an M -ideal in its bidual) if its annihilator $E^\perp = \{w \in E^{\star\star\star} : w(e) = 0 \text{ for all } e \in E\}$ is an L -summand in $E^{\star\star\star}$. In this case, the L -projection is just the adjoint of the canonical embedding of E in $E^{\star\star}$. It is clear that $X = E^\star$ is an L -embedded Banach space and $X^{\star\star} = X \oplus_1 E^\perp$. Particular cases of duals of M -embedded Banach spaces are $\ell_1(\Gamma)$, the Hardy space H_1 , the space $\mathcal{C}_1(H)$ (dual of the space of compact operators on a Hilbert space), the duals of certain Orlicz spaces, some Lorentz spaces, etc. A wide study and more examples of these classes of Banach spaces can be found in the monograph [5].

In order to establish the main theorem in this section we recall the following result which can be found in [3, Theorem 1].

Theorem 4.1. *Let X be a Banach space and let C be a closed convex bounded subset of X . Let (ε_n) be a null sequence in $(0, 1)$. If C contains a sequence (x_n) such that*

$$\sum_{n=1}^{\infty} (1 - \varepsilon_n)|t_n| \leq \left\| \sum_{n=1}^{\infty} t_n x_n \right\| \leq \sum_{n=1}^{\infty} (1 + \varepsilon_n)|t_n|$$

for all $(t_n) \in \ell_1$, then C contains a nonempty closed convex subset K such that there is a nonexpansive affine mapping $T : K \rightarrow K$ which fails to have a fixed point in K .

Theorem 4.2. *Let X be an L -embedded Banach space and $C \neq \emptyset$ be a closed convex bounded subset of X . Then the following conditions are equivalent:*

- (1) C is weakly compact.
- (2) C has the generic fixed point property for nonexpansive affine mappings.

Proof. In view of Theorem 3.2 we only need to prove that (2) implies (1). Assume that C is not weakly compact. Then there exists a net $(u_\alpha) \subset C$ such that (u_α) converges weak * in $X^{\star\star}$ to some $w \in X^{\star\star} \setminus X$. Thus $w = x_0 + x_s$ with $x_0 \in X$, $x_s \in X_s$ and $x_s \neq 0$. From the proof of Lemma 8 in [16] there exist a null sequence (ε_n) in $(0, 1)$ and a sequence (y_n) in $\text{co}\{u_\alpha\} \subset C$ such that

$$\sum_{n=1}^{\infty} (1 - \varepsilon_n)|t_n| \leq \left\| \sum_{n=1}^{\infty} t_n \left(\frac{y_n - x_0}{\|x_s\|} \right) \right\| \leq \sum_{n=1}^{\infty} (1 + \varepsilon_n)|t_n|$$

for all $(t_n) \in \ell_1$.

We put $z_n = (y_n - x_0)/\|x_s\|$ for $n \in \mathbb{N}$. This gives us a sequence (z_n) in the closed convex and bounded set $C_0 = (C - x_0)/\|x_s\|$. By Theorem 3.1, there exist a convex closed subset K_0 of C_0 and a nonexpansive affine mapping $T_0 : K_0 \rightarrow K_0$ which is fixed point free. Consider now $K = x_0 + \|x_s\|K_0$ which is a closed convex subset of C and

define $T : K \rightarrow K$ by

$$T(x_0 + \|x_s\|x) = x_0 + \|x_s\|T_0(x)$$

for all $x \in K_0$. Then T is fixed point free nonexpansive affine mapping from a closed convex subset of C into itself, which contradicts (2). \square

Remark 4.3. Analysis of the proof of Theorem 4.1 shows that the mapping T given in its conclusion satisfies the condition $\theta(T) > 0$. Consequently, the mapping constructed in the proof of Theorem 4.2 also has this property.

Note that the word *affine* cannot be dropped from the statement of the above theorem. This is due to Alspach's example [1], which shows that there is a convex weakly compact subset C of $L_1[0, 1]$ and a nonexpansive mapping $T : C \rightarrow C$ without fixed points.

In the case when an L -embedded Banach space X has the w -FPP, i.e. every nonexpansive mapping from a convex weakly compact subset of X into itself has a fixed point, we can drop the word *affine*. This gives us the following modification of Theorem 4.2.

Corollary 4.4. *Let X be an L -embedded Banach space with the w -FPP and let $C \neq \emptyset$ be a closed convex bounded subset of X . Then the following conditions are equivalent:*

- (1) C is weakly compact.
- (2) C has the generic fixed point property for nonexpansive mappings.

Corollary 4.4 may be applied for instance to the sequence space ℓ_1 , the space of the trace operators $\mathcal{C}_1(H)$ (see [9]), the Hardy space H_1 (see [13]) and space of the nuclear operators $\mathcal{C}_1(\ell_q, \ell_p)$, dual of the compact operators $K(\ell_p, \ell_q)$ with $1/p + 1/q = 1$ (see [8]).

Since the uniform Lipschitz condition is preserved under renorming, the following result is a direct consequence of Corollary 3.4 and Theorem 4.2.

Corollary 4.5. *Let X denote a renorming of $c_0(\Gamma)$ or J_p or an L -embedded Banach space and $C \neq \emptyset$ be a closed convex bounded subset of X . The following conditions are equivalent:*

- (1) C is weakly compact.
- (2) C has the generic fixed point property for uniformly Lipschitzian affine mappings.

Remark 4.6. The bilateral shift with respect to the standard basis of ℓ_1 is an isometry. It follows that if X is a renorming of an L -embedded space and C is a closed convex bounded but not weakly compact subset of X , then there exist a closed convex subset K of C and a uniformly Lipschitzian mapping T from K onto K without a fixed point. A similar result holds for renormings of c_0 and J_p . This time,

however, we obtain a mapping which is only Lipschitzian. Indeed, the bilateral shift with respect to the summing basis is Lipschitzian in c_0 and J_p .

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