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Polynomial sequences of bounded operators

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Abstract

The basic results of spectral theory are obtained using the sequence of powers of a bounded linear operator $T, T^2, \ldots, T^n, \ldots$. In this paper, we replace the powers T^n by certain polynomials $p_n(T)$, and make use of special properties of the polynomial sequence $\{p_n\}_{n\geqslant 0}$ to derive some new results concerning operators. For example, using an arbitrary polynomial sequence $\{p_n\}_{n\geqslant 0}$, we obtain "binomial" spectral radii and semidistances, which reduce, in the case of the sequence of powers, to the usual spectral radius and semidistance.

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1. Introduction

In this paper, we intend to point out that the analytic functional calculus of bounded operators, and particularly the polynomial calculus, is still able to give new interesting informations on the bounded linear operators on complex Banach spaces. In this connection, we obtain informations about spectra, new nilpotence criteria and some results concerning both operator Lie algebras and algebras of matrix-valued functions (see Section 6 below). Our main tool consists in polynomial sequences of binomial type (see Definition 3.3 and Proposition 3.4 below).

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In this way, the polynomial sequences of binomial type seem to play an important role in both the theory of *Lie algebras of bounded operators* and the theory of *algebras of matrices over function algebras*. The role of these polynomial sequences and of the closely related umbral calculus in connection with diffeomorphism groups and with the corresponding *Lie algebras of vector fields* was also explored, e.g., in [FS00,Gz88]. Thus, there exist some interesting connections between polynomial sequences of binomial type, on the one hand, and, on the other, Lie algebras of bounded operators, Lie algebras of matrices over function algebras and Lie algebras of vector fields. (We note that these are three of the four main classes of infinite-dimensional Lie algebras singled out in the Introduction to [Ka90].)

To describe the contents of the present paper in more detail, we recall that some basic facts in spectral theory are obtained by using the sequence of powers of a bounded linear operator. If T is a bounded linear operator on a complex Banach space, we can replace the polynomial sequence $\{z^n\}_{n\geqslant 0}$ by an arbitrary "binomial polynomial" sequence $\{p_n(z)\}_{n\geqslant 0}$. In other words, the corresponding operator sequence $\{p_n(T)\}_{n\geqslant 0}$ can be employed instead of $\{T^n\}_{n\geqslant 0}$ in order to introduce "binomial" spectral radii and semidistances, which reduce, in the case of the sequence of powers, to the usual spectral radius and semidistance.

To prepare the ground for introducing these new spectral invariants, we first study in Section 2 a special type of points in the spectrum of a bounded operator. These points include the approximate eigenvalues as a special case, and are determined by polynomial sequences, and this is the reason why we call them polynomially approximate eigenvalues (see Definition 2.20).

In Section 3, we introduce the binomial polynomials associated with a formal power series (Definition 3.3) in a manner which is slightly different from the one in [Ro75]. We then prove a Leibniz derivation formula (Theorem 3.7) which will play a key role in Section 4 (see the proof of Lemma 4.14).

In Section 4, the binomial spectral radii (Definitions 4.2 and 4.11) are introduced, and we prove some of their basic properties (Theorems 4.4, 4.6, 4.13 and Proposition 4.17).

We then obtain in Section 5 nilpotence criteria in terms of binomial sequences (see Theorem 5.3 and Example 5.5), which extend some known results from [BS01].

Finally, we prove in Section 6 versions of Engel's classical theorem (see e.g., [BS01]) in terms of binomial sequences. We thus single out certain special cases of solvable Lie algebras (see Theorem 6.4) which are, in some sense, binomially nilpotent (Definition 6.8). It is remarkable that binomial versions of classical facts still hold in this framework (see Proposition 6.13).

To conclude this Introduction, we establish some notation to be used in the sequel. We denote by $\mathbb{C}$ the field of complex numbers, by $\mathscr{X}$ an arbitrary complex Banach space, and by $\mathscr{B}(\mathscr{X})$ the Banach algebra of all bounded linear operators on $\mathscr{X}$. If $T\in\mathscr{B}(\mathscr{X})$, then $\sigma(T)$ stands for the spectrum of T. For any compact subset K of $\mathbb{C}$, we denote by $\mathscr{O}(K)$ the set of all functions which are defined and holomorphic in some neighborhood of K.

If A is an abelian group, then we denote by $A[[X]]$ the abelian group of formal power series in the variable X. If A is a not necessarily associative algebra, then $A[[X]]$ is in turn an algebra (not necessarily associative) whose multiplication will be called *Abel multiplication*. For every $g = \sum_{n=0}^{\infty} g_n X^n \in A[[X]]$ we define as usually

$$\omega(g) = \begin{cases} \min\{k \in \mathbb{N} \mid g_k \neq 0\} & \text{if } g \neq 0, \\ -\infty & \text{if } g = 0. \end{cases}$$

If $\mathscr{A}$ is a complex vector space, then End $\mathscr{A}$ denotes the complex unital associative algebra of all complex-linear maps from $\mathscr{A}$ into itself.

2. Polynomially approximate eigenvalues

To begin with, we prove two basic lemmas.

Lemma 2.1. *If $T \in \mathscr{B}(\mathscr{X})$ and U is a relatively compact open subset of $\mathbb{C}$ with $\sigma(T) \subseteq U$, then there exists a constant $C_U > 0$ such that*

$$(\forall p \in \mathcal{O}(\overline{U}))\ \ (\forall x \in \mathscr{X},\ ||x|| = 1) \quad ||p(T)x|| \geqslant C_U \cdot \inf_U |p|.$$

Proof. The desired inequality is obvious when $\inf_U |p| = 0$.

If $p \in \mathcal{O}(\overline{U})$ and $\inf_U |p| > 0$, then $1/p \in \mathcal{O}(\overline{U}) \subseteq \mathcal{O}(\sigma(T))$, and thus there exists $(p(T))^{-1} = (1/p)(T) \in \mathscr{B}(\mathscr{X})$. Then for every $x \in \mathscr{X}$ with $||x|| = 1$ we can write

$$1 = ||p(T)^{-1} p(T)x|| \leqslant ||p(T)^{-1}|| \cdot ||p(T)x||.$$

On the other hand, by the continuity of the holomorphic functional calculus, we have

$$||p(T)^{-1}|| \leqslant M_U \cdot \sup_U (1/|p|) = M_U \cdot \left(1/\inf_U |p|\right)$$

for some $M_U > 0$ which only depends on U (and T). Hence for every $x \in \mathscr{X}$ with $||x|| = 1$ we have

$$||p(T)x|| \geqslant \frac{1}{||p(T)^{-1}||} \geqslant \frac{1}{M_U} \cdot \inf_U |p|$$

and the desired equality holds with $C_U = 1/M_U$. □

We distinguish the following class of functions.

Definition 2.2. An *i_0^+-function* is an entire function $z \mapsto \sum_{n=0}^{\infty} c_n z^n$ on $\mathbb{C}$ which is not identically zero and has the properties $c_0 = 0$ and $c_n \geqslant 0$ for every $n \geqslant 1$.

Remark 2.3. It is obvious that a non-constant i_0^+-function $q : \mathbb{C} \to \mathbb{C}$ defines by restriction an increasing bijection $q : [0, \infty) \to [0, \infty)$ and

$$(\forall T \in \mathscr{B}(\mathscr{X})) \quad ||q(T)|| \leqslant q(||T||).$$

We can now state the following version of Lemma 2.1.

Lemma 2.4. *Assume that $T \in \mathscr{B}(\mathscr{X})$, U is an open relatively compact subset of $\mathbb{C}$ with $\sigma(T) \subseteq U$, q is an i_0^+-function, δ is a non-negative real number and $p \in \mathscr{O}(\overline{U})$ has the property that*

$$\inf_{z \in U} q^{-1}(|p(z)|) \geqslant \delta.$$

Then there exists a constant $C_U > 0$ such that

$$(\forall x \in \mathscr{X}, ||x|| = 1) \quad ||p(T)x|| \geqslant C_U \cdot q(\delta).$$

Proof. First note that $\inf_U |p| \geqslant q(\delta)$, and then make use of Lemma 2.1. $\square$

Example 2.5. In the setting of Lemma 2.4, let us consider the special case when p is a holomorphic polynomial of degree n, having the canonical decomposition $p(z) = a(z - \alpha_1)(z - \alpha_2) \cdots (z - \alpha_n)$, and q is also a holomorphic polynomial, namely $q(z) = z^n$. Denote $\delta_1 = \min_{1 \leqslant i \leqslant n} \mathrm{d}(\alpha_i, U)$. Then we clearly have

$$(\forall z \in U) \quad |p(z)| \geqslant |a|\delta_1^n,$$

so that

$$(\forall z \in U) \quad q^{-1}(|p(z)|) \geqslant q^{-1}(|a|\delta_1^n) = |a|^{1/n}\delta_1 =: \delta.$$

Thus Lemma 2.4 says that there exists a constant $C_U > 0$ such that for every holomorphic polynomial p in one variable we have

$$(\forall x \in \mathscr{X}, ||x|| = 1) \quad ||p(T)x|| \geqslant C_U |a| \delta_1^{\deg p},$$

which is in turn equivalent to

$$(\forall x \in \mathscr{X}, ||x|| = 1) \quad ||p(T)x||^{1/\deg p} \geqslant C_U^{1/\deg p} |a| \delta_1.$$

We now distinguish some classes of sequences of i_0^+-functions (cf. Definition 2.2) which will play a crucial role in the present paper.

Definition 2.6. Let $\{q_n\}_{n \geqslant 0}$ be a sequence of i_0^+-functions. We say that the sequence $\{q_n\}_{n \geqslant 0}$ has *property* (π_1), (π_2), respectively, if

$$(\forall \alpha, \delta > 0) \quad \liminf_{n \to \infty} q_n^{-1}(\alpha q_n(\delta)) > 0, \tag{π_1}$$

respectively

$$(\forall \alpha, \delta > 0) \quad \limsup_{n\to\infty} q_n^{-1}(\alpha q_n(\delta)) > 0. \tag{π_2}$$

Remark 2.7. We now describe some equivalent forms for the properties (π_1) and (π_2) in Definition 2.6, as well as an easily verifiable sufficient condition for (π_1) to hold.

(1) Property (π_1) is equivalent to

$$(\forall \alpha, \delta > 0)\ (\exists \gamma > 0)\ (\exists n_0 \geqslant 0)(\forall n \geqslant n_0) \quad \gamma < q_n^{-1}(\alpha q_n(\delta)),$$

which is in turn equivalent to

$$(\forall \alpha, \delta > 0)(\exists \gamma > 0)(\exists n_0 \geqslant 0)(\forall n \geqslant n_0) \quad q_n(\gamma) < \alpha q_n(\delta).$$

(2) Property (π_2) is equivalent to

$$(\forall \alpha, \delta > 0)(\exists \gamma > 0)(\exists n_1 < n_2 < \cdots)(\forall k \geqslant 1) \quad \gamma < q_{n_k}^{-1}(\alpha q_{n_k}(\delta)),$$

which is in turn equivalent to

$$(\forall \alpha, \delta > 0)(\exists \gamma > 0)(\exists n_1 < n_2 < \cdots)(\forall k \geqslant 1) \quad q_{n_k}(\gamma) < \alpha q_{n_k}(\delta).$$

(3) The following condition is sufficient in order for the property (π_1) to hold:

$$\text{(c)} \quad \lim_{n\to\infty} \frac{q_n(b)}{q_n(a)} = \infty \quad \text{whenever } 0 < a < b < \infty.$$

If this is the case, then γ in Remark 2.7(1) can be chosen less than δ and arbitrarily close to δ.

The following proposition concerns an assertion which is equivalent to condition (c) in Remark 2.7(3).

Proposition 2.8. *If $\{q_n\}_{n\geqslant 0}$ is a sequence of i_0^+-functions, then condition* (c) *in Remark* 2.7(3) *is equivalent to*

$$\limsup_{n\to\infty} q_n^{-1}(\alpha q_n(\delta)) \leqslant \delta \quad \textit{whenever } \alpha, \delta > 0. \tag{m_1}$$

Proof. Condition (m_1) is clearly equivalent to the following one: for all $\alpha, \delta, \eta > 0$ there exists $n_0 \geqslant 0$ such that for every $n \geqslant n_0$ we have $q_n^{-1}(\alpha q_n(\delta)) < \delta + \eta$, that is,

$\alpha q_n(\delta) < q_n(\delta + \eta)$. The latter condition is further equivalent to

$$(\forall \delta, \eta > 0) \quad \lim_{n \to \infty} \frac{q_n(\delta + \eta)}{q_n(\eta)} = \infty,$$

which is just condition (c) in Remark 2.7(3). □

The following statement concerns some conditions which are stronger than condition (m_1) in Proposition 2.8, and condition (π_2) in Definition 2.6, respectively.

Proposition 2.9. *Let $\{q_n\}_{n \geqslant 0}$ be a sequence of i_0^+-functions. Then condition*

$$(\mathrm{m}_2) \ (\forall \alpha > 0)(\forall \beta_0, \beta_1, \dots > 0) \qquad \limsup_{n \to \infty} q_n^{-1}(\beta_n) \leqslant \limsup_{n \to \infty} q_n^{-1}(\alpha \beta_n)$$

is equivalent to

$$(\forall \alpha \in (0,1))(\forall \beta_0, \beta_1, \dots > 0) \quad \limsup_{n \to \infty} q_n^{-1}(\beta_n) = \limsup_{n \to \infty} q_n^{-1}(\alpha \beta_n). \qquad (\mathrm{m}_2')$$

Also, condition

$$(\mathrm{m}_3) \ (\forall \alpha > 0)(\forall \beta_0, \beta_1, \dots > 0) \quad \limsup_{n \to \infty} q_n^{-1}(\alpha \beta_n) \leqslant \limsup_{n \to \infty} q_n^{-1}(\beta_n).$$

is equivalent to

$$(\forall \alpha > 1)(\forall \beta_0, \beta_1, \dots > 0) \quad \limsup_{n \to \infty} q_n^{-1}(\alpha \beta_n) = \limsup_{n \to \infty} q_n^{-1}(\beta_n). \qquad (\mathrm{m}_3')$$

Finally, the implications

$$(\mathrm{m}_3) \Rightarrow (\mathrm{m}_1) \quad \text{and} \quad (\mathrm{m}_2) \Rightarrow (\pi_2)$$

hold.

Proof. Let us notice firstly that, if for some $\alpha_0 > 0$ we have

$$(\forall \beta_0, \beta_1, \dots > 0) \quad \limsup_{n \to \infty} q_n^{-1}(\beta_n) \leqslant \limsup_{n \to \infty} q_n^{-1}(\alpha_0 \beta_n),$$

then

$$(\forall \alpha \geqslant \alpha_0)(\forall \beta_0, \beta_1, \dots > 0) \quad \limsup_{n \to \infty} q_n^{-1}(\beta_n) \leqslant \limsup_{n \to \infty} q_n^{-1}(\alpha \beta_n).$$

Indeed, since

$$(\forall \alpha \geqslant \alpha_0)(\forall \beta_0, \beta_1, \dots > 0) \quad q_n^{-1}(\alpha_0 \beta_n) \geqslant q_n^{-1}(\alpha \beta_n),$$

it follows that for all $\alpha \geqslant \alpha_0$ and $\beta_0, \beta_1, \dots > 0$ we have

$$\limsup_{n\to\infty} q_n^{-1}(\beta_n) \leqslant \limsup_{n\to\infty} q_n^{-1}(\alpha_0\beta_n) \leqslant \limsup_{n\to\infty} q_n^{-1}(\alpha\beta_n).$$

It obviously follows by this remark that we have $(\mathrm{m}_2{}') \Rightarrow (\mathrm{m}_2)$.

To prove the converse implication, assume that (m_2) holds but there exist $\alpha_0 \in (0,1)$ and $\beta_0, \beta_1, \ldots \in (0,1)$ such that $\limsup_{n\to\infty} q_n^{-1}(\beta_n) < \limsup_{n\to\infty} q_n^{-1}(\alpha_0\beta_n)$. Then there exists a positive number θ such that

$$\limsup_{n\to\infty} q_n^{-1}(\beta_n) < \theta < \limsup_{n\to\infty} q_n^{-1}(\alpha_0\beta_n).$$

In particular, we have $q_n^{-1}(\beta_n) < \theta$ for n large enough, and $\theta < q_n^{-1}(\alpha_0\beta_n)$ for infinitely many values of n. Thus there exists n with $q_n^{-1}(\beta_n) < \theta < q_n^{-1}(\alpha_0\beta_n)$. Therefore $\beta_n < \alpha_0\beta_n$, and then $1 < \alpha_0$. Since this contradicts the assumption $\alpha_0 \in (0,1)$, it follows that we have $(\mathrm{m}_2) \Rightarrow (\mathrm{m}_2{}')$.

The equivalence $(\mathrm{m}_3) \Leftrightarrow (\mathrm{m}_3{}')$ can be proved in the same manner.

To see that $(\mathrm{m}_3) \Rightarrow (\mathrm{m}_1)$, we just have to note that, putting $\beta_n = q_n(\delta)$ in (m_3), we get

$$\limsup_{n\to\infty} q_n^{-1}(\alpha q_n(\delta)) \leqslant \delta,$$

which is just condition (m_1) in Proposition 2.8.

Analogously, for implication $(\mathrm{m}_2) \Rightarrow (\pi_1)$, remark that, putting $\beta_n = q_n(\delta)$ in (m_2), we obtain

$$0 < \delta \leqslant \limsup_{n\to\infty} q_n^{-1}(\alpha q_n(\delta)),$$

which is just condition (π_2) in Definition 2.6. □

Next we draw some consequences of Lemma 2.4 on properties (π_1) and (π_2) in Definition 2.6.

Proposition 2.10. *Let $T \in \mathscr{B}(\mathscr{X})$, U an open relatively compact subset of $\mathbb{C}$ with $\sigma(T) \subseteq U$, and $\{p_n\}_{n\geqslant 0}$ a sequence in $\mathcal{O}(\overline{U})$. Also let $\{q_n\}_{n\geqslant 0}$ be a sequence of i_0^+-functions.*

If the sequence $\{q_n\}_{n\geqslant 0}$ has property (π_1), then the following assertions hold.

(i) *If*

$$\liminf_{n\to\infty}\left(\inf_{z\in U} q_n^{-1}(|p_n(z)|)\right) > 0,$$

then there exists a real number $\gamma > 0$ such that for every sequence $\{x_n\}_{n\geqslant 0}$ of unit vectors in $\mathscr{X}$ we have

$$\liminf_{n\to\infty} q_n^{-1}(\|p_n(T)x_n\|) \geqslant \gamma > 0.$$

(ii) *If*

$$\limsup_{n\to\infty}\left(\inf_{z\in U} q_n^{-1}(|p_n(z)|)\right)>0,$$

then there exists a real number $\gamma'>0$ *such that for every sequence* $\{x_n\}_{n\geqslant 0}$ *of unit vectors in* $\mathscr{X}$ *we have*

$$\limsup_{n\to\infty}\ q_n^{-1}(||p_n(T)x_n||)\geqslant\gamma>0.$$

On the other hand, if the sequence $\{q_n\}_{n\geqslant 0}$ *has property* (π_2), *then the following assertion holds.*

(iii) *If*

$$\liminf_{n\to\infty}\left(\inf_{z\in U} q_n^{-1}(|p_n(z)|)\right)>0,$$

then there exists a real number $\gamma''>0$ *such that for every sequence* $\{x_n\}_{n\geqslant 0}$ *of unit vectors in* $\mathscr{X}$ *we have*

$$\limsup_{n\to\infty}\ q_n^{-1}(||p_n(T)x_n||)\geqslant\gamma''>0.$$

Proof. We first prove assertions (i) and (iii). Assume that

$$\eta:=\liminf_{n\to\infty}\left(\inf_{z\in U} q_n^{-1}(|p_n(z)|)\right)>\delta>0.$$

We then deduce by Lemma 2.4 that for every $x\in\mathscr{X}$ with $||x||=1$ we have

$$||p_n(T)x||\geqslant C_U q_n(\delta)\quad\text{for sufficiently large } n.$$

If the sequence $\{q_n\}_{n\geqslant 0}$ verifies (π_1) (respectively (π_2)), then there exists $\gamma>0$ such that for sufficiently large n (respectively, for infinitely many values of n) we have

$$C_U q_n(\delta)>q_n(\gamma).$$

Thus we have for every unit vector $x\in\mathscr{X}$ and sufficiently large n (respectively, infinitely many values of n) that

$$||p_n(T)x||>q_n(\gamma).$$

Since every $q_n : [0,\infty)\to[0,\infty)$ is an increasing function, it further follows that for sufficiently large n (respectively, infinitely many values of n) we have

$$q_n^{-1}(||p_n(T)x||)>\gamma.$$

Thus, for every sequence $\{x_n\}_{n\geqslant 0}$ of unit vectors in $\mathscr{X}$, we get

$$0<\gamma\leqslant \liminf_{n\to\infty} q_n^{-1}(\|p_n(T)x_n\|) \quad \left(\text{resp.}, 0<\gamma\leqslant \limsup_{n\to\infty} q_n^{-1}(\|p_n(T)x_n\|)\right),$$

and both assertions (i) and (iii) are proved.

To prove (ii), assume that $\limsup_{n\to\infty} q_n^{-1}(|p_n(z)|)>\delta>0$. Then, by Lemma 2.4, the inequality

$$\|p_n(T)x\|\geqslant C_U q_n(\delta)$$

holds for infinitely many values of n and all unit vectors $x\in\mathscr{X}$. On the other hand, the sequence $\{q_n\}_{n\geqslant 0}$ verifies (π_1), hence there exists $\gamma>0$ such that $q_n^{-1}(C_U q_n(\delta))>\gamma$ for n large enough, and thus the inequality

$$q_n^{-1}(\|p_n(T)x\|)\geqslant\gamma$$

holds for infinitely many values of n and all unit vectors $x\in\mathscr{X}$. It then follows that for every sequence of unit vectors $\{x_n\}_{n\geqslant 0}$ in $\mathscr{X}$ we have

$$\limsup_{n\to\infty} q_n^{-1}(\|p_n(T)x_n\|)\geqslant\gamma>0,$$

and the proof ends. □

Remark 2.11. We note that the situation when the sequence $\{q_n\}_{n\geqslant 0}$ verifies (m_1) (see Proposition 2.8) is a special case where Proposition 2.10 can be applied. Indeed, as we have already proved in Proposition 2.10, condition (m_1) is equivalent to condition (c) in Remark 2.7(3), hence the sequence $\{q_n\}_{n\geqslant 0}$ verifies (π_1) and the number γ in (π_1) can be chosen arbitrarily close to δ (see Remark 2.7(3)).

Consequently, in the above proof of assertions (i) and (ii), the number γ can be chosen arbitrarily close to δ because δ can be arbitrarily close to η. Thus, in this case (that is, when $\{q_n\}_{n\geqslant 0}$ verifies (m_1)), for every sequence $\{x_n\}_{n\geqslant 0}$ of unit vectors in $\mathscr{X}$, every $T\in\mathscr{B}(\mathscr{X})$, every open relatively compact subset U of $\mathbb{C}$ with $\sigma(T)\subseteq U$ and every sequence $\{p_n\}_{n\geqslant 0}$ in $\mathscr{O}(\overline{U})$ we have

$$\liminf_{n\to\infty}\ \inf_{z\in U} q_n^{-1}(|p_n(z)|)\leqslant \liminf_{n\to\infty} q_n^{-1}(\|p_n(T)x_n\|)$$

and

$$\limsup_{n\to\infty}\ \inf_{z\in U} q_n^{-1}(|p_n(z)|)\leqslant \limsup_{n\to\infty} q_n^{-1}(\|p_n(T)x_n\|).$$

Proposition 2.10 can be stated in a dual manner as follows.

Proposition 2.12. *Let $T \in \mathscr{B}(\mathscr{X})$, U an open relatively compact subset of $\mathbb{C}$ with $\sigma(T) \subseteq U$, and $\{p_n\}_{n\geqslant 0}$ a sequence in $\mathscr{O}(\overline{U})$. Also let $\{q_n\}_{n\geqslant 0}$ be a sequence of i_0^+-functions.*

If the sequence $\{q_n\}_{n\geqslant 0}$ has property (π_1), then the following assertions hold.

(i) *If for some sequence $\{x_n\}_{n\geqslant 0}$ of unit vectors in $\mathscr{X}$ we have*

$$\liminf_{n\to\infty} q_n^{-1}(\|p_n(T)x_n\|) = 0,$$

then

$$\liminf_{n\to\infty}\left(\inf_{z\in U} q_n^{-1}(|p_n(z)|)\right) = 0.$$

(ii) *If for some sequence $\{x_n\}_{n\geqslant 0}$ of unit vectors in $\mathscr{X}$ we have*

$$\lim_{n\to\infty} q_n^{-1}(\|p_n(T)x_n\|) = 0,$$

then

$$\lim_{n\to\infty}\left(\inf_{z\in U} q_n^{-1}(|p_n(z)|)\right) = 0.$$

If the sequence $\{q_n\}_{n\geqslant 0}$ has property (π_2), then the following assertion holds.

(iii) *If for some sequence $\{x_n\}_{n\geqslant 0}$ of unit vectors in $\mathscr{X}$ we have*

$$\limsup_{n\to\infty} q_n^{-1}(\|p_n(T)x_n\|) = 0,$$

then

$$\liminf_{n\to\infty}\left(\inf_{z\in U} q_n^{-1}(|p_n(z)|)\right) = 0.$$

Proposition 2.12 allows us to describe some special points in the spectrum of T thus:

Corollary 2.13. *Let $T \in \mathscr{B}(\mathscr{X})$ and $\{p_n\}_{n\geqslant 0}$ a sequence of holomorphic polynomials in one variable. For each $n \geqslant 0$, let $\deg p_n = k_n$, a_{k_n} the leading coefficient in p_n, and denote by $Z_n = p_n^{-1}(0)$ the null-set of p_n. If there exists a sequence $\{q_n\}_{n\geqslant 0}$ of i_0^+-functions verifying (π_1) such that $\liminf_{n\to\infty} q_n^{-1}(|a_{k_n}|\delta^{k_n}) > 0$ for every $\delta > 0$, and $\liminf_{n\to\infty} q_n^{-1}(\|p_n(T)x_n\|) = 0$ for some sequence $\{x_n\}_{n\geqslant 0}$ of unit vectors in $\mathscr{X}$, then* *we have*

$$\sigma(T) \cap \overline{\bigcup_{n \geqslant 0} Z_n} \neq \emptyset.$$

Proof. Proposition 2.12(i) shows that for every open relatively compact subset U of $\mathbb{C}$ containing $\sigma(T)$ we have

$$\liminf_{n \to \infty} \inf_{z \in U} q_n^{-1}(|p_n(z)|) = 0,$$

because the hypothesis ensures that $\liminf_{n \to \infty} q_n^{-1}(\|p_n(T)x_n\|) = 0$. Now the desired assertion follows by Lemma 2.14 below, applied for $K = \sigma(T)$. $\square$

Lemma 2.14. *Let* $K \subseteq \mathbb{C}$ *be a compact set and* $\{p_n\}_{n \geqslant 0}$ *a sequence of holomorphic polynomials in one variable. For each* $n \geqslant 0$, *let* $\deg p_n = k_n$, a_{k_n} *the leading coefficient in* p_n, *and denote by* $Z_n = p_n^{-1}(0)$ *the null-set of* p_n. *If there exists a sequence* $\{q_n\}_{n \geqslant 0}$ *of* i_0^+*-functions verifying* (π_1) *such that* $\liminf_{n \to \infty} q_n^{-1}(|a_{k_n}|\delta^{k_n}) > 0$ *for every* $\delta > 0$, *and* $\liminf_{n \to \infty} \inf_{z \in U} q_n^{-1}(|p_n(z)|) = 0$ *for every relatively compact open neighborhood* U *of* K, *then*

$$K \cap \overline{\bigcup_{n \geqslant 0} Z_n} \neq \emptyset.$$

Proof. If $K \cap \overline{\bigcup_{n \geqslant 0} Z_n} = \emptyset$, then there exist $\delta > 0$ and an open relatively compact set $U \subseteq \mathbb{C}$ with $K \subseteq U$ such that $\mathrm{dist}(z, \bigcup_{n \geqslant 0} Z_n) \geqslant \delta$ whenever $z \in U$. On the other hand, for each $n \geqslant 0$ we have according to the hypothesis

$$(\forall z \in \mathbb{C}) \quad p_n(z) = a_{k_n}(z - \alpha_1) \cdots (z - \alpha_{k_n}),$$

hence

$$(\forall z \in U) \quad |p_n(z)| \geqslant |a_{k_n}|\delta^{k_n}.$$

So

$$0 = \liminf_{n \to \infty} \inf_{z \in U} q_n^{-1}(|p_n(z)|) \geqslant \liminf_{n \to \infty} q_n^{-1}(|a_{k_n}|\delta^{k_n}) > 0,$$

which is impossible. Consequently, the desired assertion holds. $\square$

Corollary 2.15. *In the setting of Corollary* 2.13 *we have*

$$\sigma(T) \cap \bigcap_{n \geqslant 0} \overline{\bigcup_{k \geqslant n} Z_k} \neq \emptyset.$$

Proof. By Corollary 2.13 we deduce that for each $k \geqslant 0$ we have $\sigma(T) \cap \overline{\bigcup_{k \geqslant n} Z_k} \neq \emptyset$. Since the latter sets constitute a non-increasing sequence of non-empty closed subsets of the compact set $\sigma(T)$, it follows that their intersection is non-empty, and this is just the desired conclusion. $\square$

Remark. We can obtain a result similar to the one of Corollary 2.15, involving a sequence $\{q_n\}_{n \geqslant 0}$ of i_0^+-functions satisfying (π_2), provided $\lim_{n \to \infty} q_n^{-1}(\| p_n(T)x_n \|) = 0$ for some sequence $\{x_n\}_{n \geqslant 0}$ of unit vectors in $\mathscr{X}$.

2.1. Special cases

1. Let $\{q_n\}_{n \geqslant 0}$ be a sequence of i_0^+-functions verifying the following *property* (p) with respect to a sequence of non-negative integers $\{k_n\}_{n \geqslant 0}$:

$$\liminf_{n \to \infty} q_n^{-1}(\gamma^{k_n}) \neq 0 \quad \text{whenever } \gamma > 0. \tag{p}$$

Under this assumption, if $\{a_n\}_{n \geqslant 0}$ is any sequence of complex numbers such that $\liminf_{n \to \infty} |a_{k_n}|^{1/k_n} > 0$, then

$$(\forall \delta > 0) \quad \liminf_{n \to \infty} q_n^{-1}(|a_{k_n}| \delta^{k_n}) > 0.$$

Indeed, there exists $\alpha > 0$ such that $|a_{k_n}|^{1/k_n} \geqslant \alpha$ for n large enough. If $\delta > 0$, for n as before we get $|a_{k_n} \mid \delta^{k_n} \geqslant (\alpha\delta)^{k_n}$. Since each $q_n : [0, \infty) \to [0, \infty)$ is an increasing function, it then follows that for n large enough we have $q_n^{-1}(|a_{k_n} \mid \delta^{k_n}) \geqslant q_n^{-1}((\alpha\delta)^{k_n})$, and then the claimed assertion follows by the assumption that the sequence $\{q_n\}_{n \geqslant 0}$ has property (p).

The preceding remark leads us to the following version of Corollary 2.15.

Corollary 2.16. *Let $T \in \mathscr{B}(\mathscr{X})$ and $\{p_n\}_{n \geqslant 0}$ a sequence of holomorphic polynomials in one variable. For each $n \geqslant 0$, let $\deg p_n = k_n$, a_{k_n} the leading coefficient in p_n, and denote by $Z_n = p_n^{-1}(0)$ the null-set of p_n. Assume that $\liminf_{n \to \infty} |a_{k_n}|^{1/k_n} > 0$ and there exists a sequence $\{q_n\}_{n \geqslant 0}$ of i_0^+-functions verifying both (π_1) and the above condition* (p) *with respect to $\{k_n\}_{n \geqslant 0}$, and $\liminf_{n \to \infty} q_n^{-1}(\| p_n(T)x_n \|) = 0$ for some sequence $\{x_n\}_{n \geqslant 0}$ of unit vectors in $\mathscr{X}$. Then*

$$\sigma(T) \cap \bigcap_{n \geqslant 0} \overline{\bigcup_{k \geqslant n} Z_k} \neq \emptyset.$$

2. For every sequence $\{m_n\}_{m \geqslant 0}$ of positive integers, the sequence $\{q_n\}_{n \geqslant 0}$ of i_0^+-functions defined by

$$q_n(w) = w^{m_n}$$

has property (π_1). Indeed, for all $\alpha, \delta > 0$ we have

$$\liminf_{n\to\infty} q_n^{-1}(\alpha q_n(\delta)) = \liminf_{n\to\infty} (\alpha\delta^{m_n})^{1/m_n} = \liminf_{n\to\infty} \alpha^{1/m_n}\delta > 0.$$

Using this remark, we get the following easy consequence of Corollary 2.16.

Corollary 2.17. *Let $T \in \mathscr{B}(\mathscr{X})$ and $\{p_n\}_{n\geqslant 0}$ a sequence of holomorphic polynomials in one variable. For each $n \geqslant 0$, let $\deg p_n = k_n$, a_{k_n} the leading coefficient in p_n, and denote by $Z_n = p_n^{-1}(0)$ the null-set of p_n. Assume that $\liminf_{n\to\infty} |a_{k_n}|^{1/k_n} > 0$ and $\{m_n\}_{n\geqslant 0}$ is a sequence of positive integers such that the sequence $\{k_n/m_n\}_{n\geqslant 0}$ is bounded. If $\liminf_{n\to\infty} ||p_n(T)x_n||^{1/m_n} = 0$ for some sequence $\{x_n\}_{n\geqslant 0}$ of unit vectors in $\mathscr{X}$, then*

$$\sigma(T) \cap \bigcap_{n\geqslant 0} \overline{\bigcup_{k\geqslant n} Z_k} \neq \emptyset.$$

Proof. As we have already noticed, $q_n(w) = w^{m_n}$ defines a sequence of i_0^+-functions which verifies both (π_1) and (p) (with respect to $\{k_n\}_{n\geqslant 0}$), hence we can apply Corollary 2.16. □

We also note the following special case of Corollary 2.17.

Corollary 2.18. *Let $T \in \mathscr{B}(\mathscr{X})$ and $\{p_n\}_{n\geqslant 0}$ a sequence of holomorphic polynomials in one variable. For each $n \geqslant 0$, let $\deg p_n = k_n$, a_{k_n} the leading coefficient in p_n, and denote by $Z_n = p_n^{-1}(0)$ the null-set of p_n. If $\liminf_{n\to\infty} |a_{k_n}|^{1/k_n} > 0$ and $\liminf_{n\to\infty} ||p_n(T)x_n||^{1/k_n} = 0$ for some sequence $\{x_n\}_{n\geqslant 0}$ of unit vectors in $\mathscr{X}$, then*

$$\sigma(T) \cap \bigcap_{n\geqslant 0} \overline{\bigcup_{k\geqslant n} Z_k} \neq \emptyset.$$

In Corollary 2.18 we can have, for instance, $k_n = n$ for all $n \geqslant 1$, or $k_n = 1$ for all $n \geqslant 1$. In other words, $q_n(w) = w^n$ if $\deg p_n = n$, or $q_n(w) = w$ if $\deg p_n = 1$.

Example 2.19. It is obvious that there are many examples illustrating the above corollaries. Some simplest special cases are the following assertions concerning an operator $T \in \mathscr{B}(\mathscr{X})$.

(1) If $0 \neq x \in \mathscr{X}$ and $\lim_{n\to\infty} (\lambda_n - T)x = 0$, then there exists $\lambda \in \sigma(T)$ such that $\lim_{n\to\infty} \lambda_n = \lambda$ and λ is a eigenvalue of T, more precisely $Tx = \lambda x$.
(2) If $\{y_n\}_{n\geqslant 0}$ is a sequence in $\mathscr{X}$ such that $\limsup_{n\to\infty} ||y_n|| > 0$ (that is, we do *not* have $\lim_{n\to\infty} ||y_n|| = 0$) and $\lim_{n\to\infty} (\lambda - T)y_n = 0$, then $\lambda \in \sigma(T)$, in fact λ is an approximate eigenvalue of T.

For the assertion (2) note that the hypothesis $\limsup_{n\to\infty} ||y_n|| > 0$ means that there exist $\varepsilon > 0$ and positive integers $n_1 < n_2 < \cdots < n_k < \cdots$ such that

$||y_{n_k}|| \geqslant \varepsilon$ for all $k \geqslant 1$. Denoting $x_k = y_{n_k}/||y_{n_k}||$, we again have $\lim_{k\to\infty} (\lambda - T)x_k = 0$, and moreover $||x_k|| = 1$ for all $k \geqslant 1$, and then $\lambda \in \sigma(T)$ by Corollary 2.18.

Another special case of Corollary 2.18 is the following assertion.

(3) If $\{y_n\}_{n\geqslant 0}$ is a sequence in $\mathscr{X}$ such that $\limsup_{n\to\infty} ||y_n|| > 0$, and $\{\lambda_n\}_{n\geqslant 0}$ is a sequence of complex numbers such that $\lim_{n\to\infty} (\lambda_n - T)y_n = 0$, then there exists $\lambda \in \sigma(T)$ such that $\lambda \in \bigcap_{n\geqslant 0} \overline{\{\lambda_n, \lambda_{n+1}, \dots\}}$.

Indeed, we can deduce as above that $\lim_{n\to\infty} (\lambda_n - T)x_n = 0$ for some sequence $\{x_n\}_{n\geqslant 0}$ of unit vectors in $\mathscr{X}$, and then Corollary 2.18 can be applied.

Assertion (2) in Example 2.19 suggests the following definition.

Definition 2.20. In the setting of Corollary 2.16, the values $\lambda \in \sigma(T) \cap \bigcap_{n\geqslant 0} \bigcap_{k\geqslant n} Z_k$ are called *polynomially approximate eigenvalues*.

3. Binomial sequences, functional properties of binomial polynomials

Throughout the present section, $g = \sum_{n=1}^{\infty} g_n X^n$ stands for a fixed formal series with complex coefficients, that is, $g \in \mathbb{C}[[X]]$ and $\omega(g) \geqslant 1$. Every formal power series can be composed with g in the well-known way, which we describe in Lemma 3.1 below for later reference.

For each $s \in \mathbb{N}$, denote by $\{g_{n,s}\}_{n\geqslant 0}$ the sequence of coefficients of the sth power of g. Then $\omega(g^s) \geqslant s$, that is, $g_{n,s} = 0$ whenever $s < n$ and then

$$g^s = \sum_{n=s}^{\infty} g_{n,s} X^n.$$

Lemma 3.1. *There exists a family of complex numbers $a_{i,k}$, where i, k are positive integers with $1 \leqslant i \leqslant k$, describing the composition with g mappings in the following way. For every complex vector space $\mathscr{V}$ and every $B = \sum_{n=0}^{\infty} \beta_s X^s \in \mathscr{V}[[X]]$ we have*

$$B \circ g = \sum_{s=0}^{\infty} \beta_s g^s(X) = \beta_0 + \sum_{k=1}^{\infty} \gamma_k(\beta_1, \dots, \beta_k) X^k \in \mathscr{V}[[X]],$$

where for each $k \geqslant 1$ we define

$$\gamma_k : \underbrace{\mathscr{V} \times \cdots \times \mathscr{V}}_{k \text{ times}} \to \mathscr{V}, \quad (\beta_1, \dots, \beta_k) \mapsto a_{k,1}\beta_1 + \cdots + a_{k,k}\beta_k.$$

Remark 3.2. In the statement of Lemma 3.1, the meaning of the equality $B \circ g = \sum_{s=0}^{\infty} \beta_s g^s(X)$ is that $B \circ g$ is the sum of the summable family $\{\beta_s g^s\}_{s \geqslant 0}$, where for each $s \geqslant 0$ we have $\beta_s g^s = \sum_{n=s}^{\infty} g_{n,s} \beta_s X^n$. (See the notation introduced before Lemma 3.1.)

Now, we consider the special case when $\mathscr{V} = \mathbb{C}[[W]]$ and

$$B := e^{W\cdot} := \sum_{n=0}^{\infty} \frac{W^n}{n!} X^n \in (\mathbb{C}[[W]])[[X]].$$

It then follows by Lemma 3.1 that

$$e^{W\cdot} \circ g = \sum_{n=0}^{\infty} \frac{1}{n!} W^n g^n(X) = 1 + \sum_{k=1}^{\infty} \gamma_k \left(\frac{1}{1!} W, \dots, \frac{1}{k!} W^k \right) X^k$$

and, for every $k \geqslant 1$,

$$\gamma_k \left(\frac{1}{1!} W, \dots, \frac{1}{k!} W^k \right) = a_{k,1} \cdot \frac{1}{1!} W + \cdots + a_{k,k} \cdot \frac{1}{k!} W^k \in \mathbb{C}[[X]].$$

Defining $p_0 = 1$ and for $k \geqslant 1$

$$p_k = k! \gamma_k \left(\frac{1}{1!} W, \dots, \frac{1}{k!} W^k \right) \in \mathbb{C}[W],$$

we have

$$e^{W\cdot} \circ g = \sum_{k=0}^{\infty} p_k(W) X^k.$$

Definition 3.3. The above defined polynomial sequence $\{p_k\}_{k \in \mathbb{N}}$ is called the *binomial sequence* associated with g, and its terms are called *binomial polynomials* (associated with g).

The name introduced in Definition 3.3 is justified by the well-known properties described in the following proposition.

Proposition 3.4. *The polynomial sequence* $\{p_k\}_{k \in \mathbb{N}}$ *introduced in Definition* 3.3 *has the following properties.*

(i) *For every* $k \geqslant 0$ *we have* $\deg p_k = k$ *and the leading coefficient of* p_k *equals* 1.
(ii) *We have* $p_k(0) = 0$ *whenever* $k \geqslant 1$.
(iii) *For every* $n \geqslant 0$ *we have*

$$(\forall w_1, w_2 \in \mathbb{C}) \quad p_n(w_1 + w_2) = \sum_{k=0}^{n} \binom{n}{k} p_k(w_1) p_{n-k}(w_2).$$

Remark 3.5. In Lemma 3.1, we can put $\mathscr{V} = \mathscr{B}$, where $\mathscr{B}$ is a unital associative algebra over $\mathbb{C}$. Then we have for every $\mathfrak{w} \in \mathscr{B}$

$$e^{\mathfrak{w}\cdot} = \sum_{n=0}^{\infty} \left(\frac{1}{n!}\mathfrak{w}^n\right) X^n \in \mathscr{B}[[X]],$$

whence

$$e^{\mathfrak{w}\cdot} \circ g = \sum_{n=0}^{\infty} \frac{1}{n!}\mathfrak{w}^n g^n(X) = \sum_{k=0}^{\infty} \frac{1}{k!} p_k(\mathfrak{w}) X^k \in \mathscr{B}[[X]].$$

In the latter series, $p_k(\cdot)$ actually denote the polynomial functions on $\mathscr{B}$ corresponding to the polynomials $p_k \in \mathbb{C}[X]$.

Remark 3.6. Obviously, an equality similar to the one in Proposition 3.4 (iii) also holds when w_1 and w_2 are commuting elements of some complex unital associative algebra $\mathscr{B}$.

Proof of Proposition 3.4. Properties (i) and (ii) are obvious. Next, it is easy to check the equality

$$(\forall w_1, w_2 \in \mathbb{C}) \quad e^{(w_1+w_2)\cdot} = e^{w_1\cdot} e^{w_2\cdot},$$

involving the Abel multiplication in $\mathbb{C}[[X]]$. Identifying the coefficients in this equality we obtain the assertion (iii). $\square$

We now return to the situation of Remark 3.5, where $\mathscr{V} = \mathscr{B}$ is a unital complex algebra. Then for every $\mathfrak{w} \in \mathscr{B}$ we have

$$B := e^{\mathfrak{w}\cdot} = \sum_{n=0}^{\infty} \frac{1}{n!}\mathfrak{w}^n X^n \in \mathscr{B}[[X]], \tag{1}$$

and

$$e^{\mathfrak{w}\cdot} \circ g = \sum_{n=0}^{\infty} \mathfrak{w}^n g^n(X) = 1 + \sum_{k=1}^{\infty} \gamma_k\left(\frac{1}{1!}\mathfrak{w}, \ldots, \frac{1}{k!}\mathfrak{w}^k\right) X^k = \sum_{k=0}^{\infty} \frac{1}{k!} p_k(\mathfrak{w}) X^k.$$

Here $\mathfrak{w} \mapsto p_k(\mathfrak{w}) = k!\gamma_k(\frac{1}{1!}\mathfrak{w}, \ldots, \frac{1}{k!}\mathfrak{w}^k)$ are polynomial functions from $\mathscr{B}$ into itself, with $p_0 \equiv 1$.

We can consider the special case $\mathscr{B} = \operatorname{End} \mathscr{A}$, where $\mathscr{A}$ is a complex vector space. Then for every $x \in \mathscr{A}$ and $\mathfrak{W} \in \operatorname{End} \mathscr{A}$, denoting

$$e^{\mathfrak{W}x\cdot} := \sum_{n=0}^{\infty} \frac{1}{n!}\mathfrak{W}^n x X^n \in \mathscr{A}[[X]],$$

we get

$$e^{\mathfrak{W}x\cdot}\circ g=\sum_{n=0}^{\infty}\frac{1}{n!}\mathfrak{W}^n x\, g^n(X)=x+\sum_{k=1}^{\infty}\gamma_k\left(\frac{1}{1!}\mathfrak{W}x,\ldots,\frac{1}{k!}\mathfrak{W}^k x\right)$$

and for each $k\geqslant 1$,

$$\gamma_k\left(\frac{1}{1!}\mathfrak{W}x,\ldots,\frac{1}{k!}\mathfrak{W}^k x\right)=\gamma_k\left(\frac{1}{1!}\mathfrak{W},\ldots,\frac{1}{k!}\mathfrak{W}^k\right)x.$$

Consequently, for every complex vector space $\mathscr{A}$, $x\in\mathscr{A}$ and $\mathfrak{W}\in\operatorname{End}\mathscr{A}$ we have

$$\begin{cases} e^{\mathfrak{W}\cdot}\circ g=\sum_{n=0}^{\infty}\mathfrak{W}^n g^n(X)=\sum_{k=0}^{\infty}\frac{1}{k!}p_k(\mathfrak{W})X^k \\ e^{\mathfrak{W}x\cdot}\circ g=\sum_{n=0}^{\infty}\mathfrak{W}^n x g^n(X)=\sum_{k=0}^{\infty}\frac{1}{k!}p_k(\mathfrak{W})xX^k. \end{cases}\tag{2}$$

3.1. Leibniz derivation formula

The Leibniz derivation formula, concerning the nth order derivative of a product, has the following version in terms of the binomial polynomials associated with the formal series g (see Definition 3.3).

Theorem 3.7. *Let $\mathscr{A}$ be a complex (not necessarily associative) algebra, and $D\in\operatorname{End}\mathscr{A}$ a derivation of $\mathscr{A}$. Then the following formula holds:*

$$p_n(D)(xy)=\sum_{k=0}^{n}\binom{n}{k}(p_k(D)x)(p_{n-k}(D)y)$$

for all $n\geqslant 0$ and $x,y\in\mathscr{A}$.

Proof. We first recall that, denoting by $\cdot$ the Abel multiplication in $\mathscr{A}[[X]]$, we have

$$(S\cdot T)\circ g=(S\circ g)\cdot(T\circ g),\tag{3}$$

whenever $S,T\in\mathscr{A}[[X]]$, $S=\sum_{n=0}^{\infty}s_nX^n$, $T=\sum_{n=0}^{\infty}t_nX^n$.

We now consider the special case given by $s_n=\frac{1}{n!}D^nx$, $t_n=\frac{1}{n!}D^ny$, for $n\in\mathbb{N}$, and $x,y\in\mathscr{A}$. Obviously,

$$S\cdot T=\sum_{n=0}^{\infty}\frac{1}{n!}D^n(xy)X^n$$

by the classical Leibniz derivation formula. Thus $S \cdot T = e^{D(xy)\cdot}$ and we then have by (2) that

$$(S \cdot T) \circ g = \sum_{k=0}^{\infty} \frac{1}{k!} p_k(D)(xy)X^k.$$

Similarly, since $S = e^{Dx\cdot}$ and $T = e^{Dy\cdot}$, we get

$$S \circ g = \sum_{k=0}^{\infty} \frac{1}{k!} p_k(D)xX^k \quad \text{and} \quad T \circ g = \sum_{k=0}^{\infty} \frac{1}{k!} p_k(D)yX^k.$$

Now (3) shows that

$$\sum_{k=0}^{\infty} \frac{1}{k!} p_k(D)(xy)X^k = \left(\sum_{k=0}^{\infty} \frac{1}{k!} p_k(D)yX^k\right) \cdot \left(\sum_{k=0}^{\infty} \frac{1}{k!} p_k(D)yX^k\right),$$

and the desired formula follows by identifying the coefficients in the later equality. $\square$

Corollary 3.8. *In the setting of Theorem* 3.7 *we also have*

$$p_n(D - (\lambda + \mu))(xy) = \sum_{k=0}^{n} \binom{n}{k} (p_k(D - \lambda)x)(p_{n-k}(D - \mu)y)$$

for all $\lambda, \mu \in \mathbb{C}$.

Proof. Remark 3.6 shows that

$$p_n(D - (\lambda + \mu))(xy) = \sum_{k=0}^{n} \binom{n}{k} p_k(D)(xy) \cdot p_{n-k}(-(\lambda + \mu))(xy).$$

Now a simple computation finishes the proof, applying the Leibniz derivation formula of Theorem 3.7 for $p_k(D)(xy)$, and Proposition 3.4(iii) for $p_{n-k}(-(\lambda + \mu))(xy)$. $\square$

4. Binomial spectral radii and semidistances

The ever-present power sequence $\{X^n\}_{n \geqslant 0}$ can be replaced by binomial sequences in order to define analogues of the spectral radius of bounded linear operators on a complex Banach space. Throughout this section, we denote by $\mathscr{X}$ a complex Banach space.

In order to simplify the next statements, we introduce the following terminology.

Definition 4.1. A binomial sequence $(p_n)_{n\geqslant 0}$ associated with some $g\in\mathbb{C}[[X]]$ with $\omega(g)\geqslant 1$ is said to be a *positive binomial sequence* if for each $n\geqslant 0$ the polynomial p_n has only non-negative coefficients and at least one of these coefficients is non-zero.

It is clear that every positive binomial sequence defines for $n\geqslant 1$ a sequence of i_0^+-functions. For the next definition we recall that, if q is a polynomial in one variable such that $\deg q\geqslant 1$, $q(0)=0$ and all of the coefficients of q are non-negative real numbers, then it defines an increasing bijection $q:[0,\infty)\to[0,\infty)$ and $\|q(T)\|\leqslant q(\|T\|)$ for all $T\in\mathscr{B}(\mathscr{X})$ (compare Remark 2.3).

Definition 4.2. Let $\mathscr{P}=\{p_n\}_{n\geqslant 0}$ be a binomial sequence, and $\mathscr{Q}=\{q_n\}_{n\geqslant 0}$ a positive binomial sequence. Then for every $T\in\mathscr{B}(\mathscr{X})$ we define

$$\rho_{\mathscr{P},\mathscr{Q}}(T)\coloneqq\limsup_{n\to\infty}\, q_n^{-1}(\|p_n(T)\|)$$

and call this quantity the *binomial radius* corresponding to the sequences $\mathscr{P}$ and $\mathscr{Q}$. Accordingly, the function

$$\rho_{\mathscr{P},\mathscr{Q}}\,:\,\mathscr{B}(\mathscr{X})\to[0,\infty]$$

is called the *binomial radius function* corresponding to the sequences $\mathscr{P}$ and $\mathscr{Q}$. In the case when $\mathscr{P}=\mathscr{Q}$ (a positive binomial sequence), we denote simply $\rho_{\mathscr{P},\mathscr{Q}}\coloneqq\rho_{\mathscr{P}}$ and call it the binomial radius function corresponding to the positive binomial sequence $\mathscr{P}$.

It is clear that one can define binomial radii for elements in any unital associative Banach algebra.

Proposition 4.3. *Let $T\in\mathscr{B}(\mathscr{X})$. Then the following assertions hold.*

(i) *If $\mathscr{P}=\{X^n\}_{n\geqslant 0}$, then $\rho_{\mathscr{P}}(T)$ is just the usual spectral radius $\rho(T)$.*

(ii) *If both $\mathscr{P}=\{p_n\}_{n\geqslant 0}$ and $\mathscr{Q}=\{q_n\}_{n\geqslant 0}$ are positive binomial sequences and $p_n(\lambda)\leqslant q_n(\lambda)$ whenever $\lambda\geqslant 0$ and $n\in\mathbb{N}$, then $\rho_{\mathscr{P},\mathscr{Q}}(T)\leqslant\|T\|$.*

(iii) *If $\mathscr{P}=\{p_n\}_{n\geqslant 0}$ is a binomial sequence and $\mathscr{Q}=\{q_n\}_{n\geqslant 0}$ is a positive binomial sequence which verifies condition* (m_1) *in Proposition* 2.8, *then*

$$\limsup_{n\to\infty}\,\inf_{z\in U}\,q_n^{-1}(|p_n(z)|)\leqslant\rho_{\mathscr{P},\mathscr{Q}}(T)$$

for every open relatively compact set $U\subseteq\mathbb{C}$ with $\sigma(T)\subseteq U$.

Proof. Assertion (i) is obvious.

Assertion (ii) is a consequence of the inequalities: $q_n^{-1}(\mu)\leqslant p_n^{-1}(\mu)$ for every $\mu\geqslant 0$ and $n\in\mathbb{N}$; $\|p_n(T)\|\leqslant p_n(\|T\|)$ and $p_n^{-1}(\|p_n(T)\|)\leqslant\|T\|$ for every $n\in\mathbb{N}$.

Assertion (iii) is a consequence of Remark 2.11. □

Some spectral radius functions $\rho_{\mathscr{P},\mathscr{Q}}$ have a subadditive behavior in the sense described in the following theorem.

Theorem 4.4. *Let $T_1, T_2 \in \mathscr{B}(\mathscr{X})$ be a couple of commuting bounded linear operators on $\mathscr{X}$, $\mathscr{P}$ a binomial sequence and $\mathscr{Q}$ a positive binomial sequence which verifies condition* (m_1) *in Proposition* 2.8. *Then the following inequality holds*:

$$\rho_{\mathscr{P},\mathscr{Q}}(T_1 + T_2) \leqslant \rho_{\mathscr{P},\mathscr{Q}}(T_1) + \rho_{\mathscr{P},\mathscr{Q}}(T_2).$$

Proof. Denote $\rho_j := \rho_{\mathscr{P},\mathscr{Q}}(T_j)$ and let $\varepsilon > 0$ arbitrary. Also let $\mathscr{P} = \{p_n\}_{n \geqslant 0}$ and $\mathscr{Q} = \{q_n\}_{n \geqslant 0}$. There exists $n_\varepsilon \geqslant 1$ such that

$$(\forall j \in \{1,2\})(\forall n \geqslant n_\varepsilon) \quad q_n^{-1}(\|p_n(T_j)\|) < \rho_j + \varepsilon.$$

So we deduce

$$(\forall j \in \{1,2\})(\forall n \geqslant n_\varepsilon) \quad \|p_n(T_j)\| < q_n(\rho_j + \varepsilon).$$

Then there exists $M_\varepsilon \geqslant 1$ such that

$$(\forall j \in \{1,2\})(\forall n \geqslant 1) \quad \|p_n(T_j)\| < M_\varepsilon q_n(\rho_j + \varepsilon).$$

But $T_1 T_2 = T_2 T_1$ and p_n are binomial polynomials, hence we may compute by Remark 3.6

$$p_n(T_1 + T_2) = \sum_{k=0}^{n} \binom{n}{k} p_{n-k}(T_1) p_k(T_2).$$

Hence for every $n \geqslant 0$ we have

$$\|p_n(T_1 + T_2)\| \leqslant \sum_{k=0}^{n} \binom{n}{k} M_\varepsilon q_{n-k}(\rho_1 + \varepsilon) \cdot M_\varepsilon q_k(\rho_2 + \varepsilon) = M_\varepsilon^2 q_n(\rho_1 + \rho_2 + 2\varepsilon),$$

for q_n are also binomial polynomials.

On the other hand, condition (m_1) for $\{q_n\}_{n \geqslant 0}$ means that

$$(\forall \alpha, \delta > 0) \quad \limsup_{n \to \infty} q_n^{-1}(\alpha q_n(\delta)) \leqslant \delta.$$

So for arbitrary $\varepsilon > 0$ we have

$$\limsup_{n \to \infty} q_n^{-1}(\|p_n(T_1 + T_2)\|) \leqslant \rho_1 + \rho_2 + 2\varepsilon, \tag{4}$$

and the theorem is proved. $\square$

Remark 4.5. The properties described in Proposition 4.3 and Theorem 4.4 can be proved in an analogue manner for binomial radii $\rho_{\mathscr{P},\mathscr{Q}}$ in complex unital associative Banach algebras.

The following theorem shows that some binomial radii are invariant under inner automorphisms of Banach algebras.

Theorem 4.6. *Let $\mathscr{P}$ be a binomial sequence, $\mathscr{Q}$ a positive binomial sequence which verifies condition* (m_1) *in Proposition* 2.8, *and $\mathscr{A}$ a complex unital associative Banach algebra. If $S, T \in \mathscr{A}$ and S is invertible, then we have*

$$\rho_{\mathscr{P},\mathscr{Q}}(STS^{-1}) = \rho_{\mathscr{P},\mathscr{Q}}(T).$$

Proof. Let $\mathscr{P} = \{p_n\}_{n\geqslant 0}$ and $\mathscr{Q} = \{q_n\}_{n\geqslant 0}$. Firstly, we shall prove the inequality

$$\rho_{\mathscr{P},\mathscr{Q}}(STS^{-1}) \leqslant \rho_{\mathscr{P},\mathscr{Q}}(T). \tag{5}$$

We denote $\rho \coloneqq \rho_{\mathscr{P},\mathscr{Q}}(T) = \limsup_{n\to\infty} q_n^{-1}(||p_n(T)||)$. Then for arbitrary $\varepsilon > 0$ there exists $M_\varepsilon \geqslant 1$ such that

$$(\forall n \geqslant 0) \quad ||p_n(T)|| \leqslant M_\varepsilon q_n(\rho + \varepsilon).$$

Hence with a obvious notation we can write for every $n \in \mathbb{N}$

$$||p_n(STS^{-1})|| = ||Sp_n(T)S^{-1}|| \leqslant ||S|| \cdot ||S^{-1}|| \cdot M_\varepsilon \cdot q_n(\rho + \varepsilon) = C_\varepsilon \cdot q_n(\rho + \varepsilon),$$

whence

$$\rho_{\mathscr{P},\mathscr{Q}}(STS^{-1}) = \limsup_{n\to\infty} q_n^{-1}(||p_n(STS^{-1}||) \leqslant \limsup_{n\to\infty} q_n(C_\varepsilon q_n(\rho + \varepsilon)) \leqslant \rho + \varepsilon,$$

for $\{q_n\}_{n\geqslant 0}$ verifies condition (m_1) in Proposition 2.8. The last inequality holds for every $\varepsilon > 0$. Therefore inequality (5) is proved.

If we apply (5) with S^{-1} instead of S we obtain that for every invertible $S \in \mathscr{A}$

$$(\forall T \in \mathscr{A}) \quad \rho_{\mathscr{P},\mathscr{Q}}(S^{-1}TS) \leqslant \rho_{\mathscr{P},\mathscr{Q}}(T).$$

Using the last inequality with STS^{-1} instead of T we get

$$\rho_{\mathscr{P},\mathscr{Q}}(T) \leqslant \rho_{\mathscr{P},\mathscr{Q}}(STS^{-1}),$$

which is just the inequality converse to (5). Therefore, the equality in the conclusion of the theorem is proved. □

Now we shall prove some consequences of the above properties of binomial radii. We begin with some applications of the subadditivity property of the binomial radius (see Theorem 4.4).

Proposition 4.7. *Let $\mathscr{A}$ be a unital associative complex Banach algebra, $\mathscr{P}$ a binomial sequence, and $\mathscr{Q}$ a positive binomial sequence with property* (m_1) *in Proposition* 2.8. *If $T, S \in \mathscr{A}$ and $C(T,S) : \mathscr{A} \to \mathscr{A}$, $U \mapsto C(T,S)U := TU - US$, then the following inequality holds*:

$$\rho_{\mathscr{P},\mathscr{Q}}(C(T,S)) \leqslant \rho_{\mathscr{P},\mathscr{Q}}(T) + \rho_{\mathscr{P},\mathscr{Q}}(-S).$$

Proof. For each $X \in \mathscr{A}$ define $L_X, R_X : \mathscr{A} \to \mathscr{A}$ by $L_X(U) = XU$ and $R_X U = UX$ for all $U \in \mathscr{A}$. Then

$$C(T,S) = L_T + R_{-S}, \quad L_T R_{-S} = R_{-S} L_T,$$

and for every holomorphic polynomial p in one variable we have

$$||p(L_T)|| = ||L_{p(T)}|| = ||p(T)||, \quad ||p(R_{-S})|| = ||R_{p(-S)}|| = ||p(-S)||.$$

Consequently, $\rho_{\mathscr{P},\mathscr{Q}}(L_T) = \rho_{\mathscr{P},\mathscr{Q}}(T)$, $\rho_{\mathscr{P},\mathscr{Q}}(R_{-S}) = \rho_{\mathscr{P},\mathscr{Q}}(-S)$ and the desired conclusion follows since we have

$$\rho_{\mathscr{P},\mathscr{Q}}(C(T,S)) \leqslant \rho_{\mathscr{P},\mathscr{Q}}(L_T)) + \rho_{\mathscr{P},\mathscr{Q}}(R_{-S})$$

as a consequence of Theorem 4.4. □

Corollary 4.8. *Consider the setting of Proposition* 4.7. *If $\mathscr{G}$ is a complex closed Lie subalgebra of $\mathscr{A}$ and $T \in \mathscr{G}$ such that $\rho_{\mathscr{P},\mathscr{Q}}(\pm T) = 0$, then $\rho_{\mathscr{P},\mathscr{Q}}((\mathrm{ad}\, T)|_{\mathscr{G}}) = 0$.*

Proof. Since $\mathrm{ad}\, T = C(T,T)$, we can write as a consequence of Proposition 4.7,

$$\rho_{\mathscr{P},\mathscr{Q}}(\mathrm{ad}\, T) \leqslant \rho_{\mathscr{P},\mathscr{Q}}(T) + \rho_{\mathscr{P},\mathscr{Q}}(-T) = 0.$$

It is easy to see that $\rho_{\mathscr{P},\mathscr{Q}}((\mathrm{ad}\, T)|_{\mathscr{G}}) \leqslant \rho_{\mathscr{P},\mathscr{Q}}(\mathrm{ad}\, T)$, which finishes the proof. □

Remark 4.9. The assertion of Corollary 4.8 is well known when $\mathscr{P} = \mathscr{Q} = \{X^n\}_{n \geqslant 0}$. In this case, it simply says that $\sigma(\mathrm{ad}\, T) = \{0\}$ provided $\sigma(T) = \{0\}$.

As in the case of spectral radius (see e.g., [BD71, Section 3, Theorem 10]), we now obtain a commutativity criterion for a closed Lie subalgebra of a unital associative complex Banach algebra, as a consequence of the invariance property for binomial radii (see Theorem 4.6).

Corollary 4.10. *Let $\mathscr{A}$ be a unital associative complex Banach algebra $\mathscr{P}$ a binomial sequence, and $\mathscr{Q}$ a positive binomial sequence with property* (m_1) *in Proposition* 2.8. *If a complex closed Lie subalgebra $\mathscr{G}$ of $\mathscr{A}$ verifies*

$$(\exists k > 0)(\forall T \in \mathscr{G}) \qquad k||T|| \leqslant \rho_{\mathscr{P},\mathscr{Q}}(T) < \infty$$

then $\mathscr{G}$ is commutative.

Proof. We recall that the formula for the commutators of the values of holomorphic functional calculus, in the special case of Rosenblum's formula (see [BS01, Section 15, Remark 1]) says that

$$(\forall T, S \in \mathscr{A})(\forall \lambda \in \mathbb{C}) \quad e^{\mathrm{ad}\, \lambda S} T = e^{-\lambda S} T e^{\lambda S}.$$

It then follows that

$$(\forall T, S \in \mathscr{G})(\forall \lambda \in \mathbb{C}) \quad \Phi(\lambda) := e^{-\lambda S} T e^{\lambda S} \in \mathscr{G},$$

for $\mathscr{G}$ is a closed complex Lie subalgebra of $\mathscr{A}$.

Next fix $S, T \in \mathscr{G}$ arbitrary and consider $\Phi : \mathbb{C} \to \mathscr{G}$, $\Phi(\lambda) = e^{-\lambda S} T e^{\lambda S}$. By the hypothesis we have

$$(\forall \lambda \in \mathbb{C}) \quad k||\Phi(\lambda)|| \leqslant \rho_{\mathscr{P},\mathscr{Q}}(\Phi(\lambda)) = \rho_{\mathscr{P},\mathscr{Q}}(T) < \infty,$$

where the equality follows by Theorem 4.6. The Liouville Theorem then shows that the entire function Φ is constant. Therefore, $\Phi'(0) = 0$, which means that $TS = ST$, and $\mathscr{G}$ is commutative since $S, T \in \mathscr{G}$ are arbitrary. □

We can also define the local binomial radii of $T \in \mathscr{B}(\mathscr{X})$ at a point $x \in \mathscr{X}$.

Definition 4.11. The *local binomial radius* of $T \in \mathscr{B}(\mathscr{X})$ at $x \in \mathscr{X}$ is defined by

$$\rho_{\mathscr{P},\mathscr{Q}}(T; x) = \limsup_{n \to \infty} q_n^{-1}(||p_n(T)x||),$$

provided $\mathscr{P} = \{p_n\}_{n \geqslant 0}$ and $\mathscr{Q} = \{q_n\}_{n \geqslant 0}$ are binomial sequences, $\mathscr{Q}$ being in addition a positive binomial sequence. □

The following statement is a local version of Proposition 4.3.

Proposition 4.12. *Let $T \in \mathscr{B}(\mathscr{X})$, $x \in \mathscr{X}$, $\mathscr{P} = \{p_n\}_{n \geqslant 0}$ is a binomial sequence and $\mathscr{Q} = \{q_n\}_{n \geqslant 0}$ is a positive binomial sequence. Then the following assertions hold.*

(i) *If $\mathscr{P} = \{X^n\}_{n \geqslant 0}$, then $\rho_{\mathscr{P}}(T; x)$ is just the usual local spectral radius $\rho(T; x)$.*
(ii) *If both $\mathscr{P} = \{p_n\}_{n \geqslant 0}$ and $\mathscr{Q} = \{q_n\}_{n \geqslant 0}$ are positive binomial sequences and $p_n(\lambda) \leqslant q_n(\lambda)$ whenever $\lambda \geqslant 0$ and $n \in \mathbb{N}$, then $\rho_{\mathscr{P},\mathscr{Q}}(T; x) \leqslant ||T||$.*
(iii) *The inequality*

$$\limsup_{n \to \infty} \inf_{z \in U} q_n^{-1}(|p_n(z)|) \leqslant \rho_{\mathscr{P},\mathscr{Q}}(T)$$

holds for every open relatively compact set $U \subseteq \mathbb{C}$ with $\sigma(T) \subseteq U$,
  (a) *whenever $||x|| \leqslant 1$, provided $\mathscr{Q}$ verifies condition* (m_1) *in Proposition* 2.8, *or*
  (b) *for arbitrary $x \in \mathscr{X}$, provided $\mathscr{Q}$ verifies condition* (m_3) *in Proposition* 2.9.
(iv) *If $\mathscr{Q}$ verifies condition* (m_1) *in Proposition* 2.8 *and $\rho_{\mathscr{P},\mathscr{Q}}(T; x) = 0$, then we have $\rho_{\mathscr{P},\mathscr{Q}}(T; \alpha x) = 0$ for all $\alpha \in \mathbb{C}$.*

Proof. Assertion (i) is obvious. For proving (ii), we notice that

$$\|p_n(T)x\| \leqslant p_n(\|T\|)\|x\|$$

and

$$q_n^{-1}(\|p_n(T)x\|) \leqslant q_n^{-1}(\|x\|p_n(\|T\|)) \leqslant q_n^{-1}(\|x\|q_n(\|T\|)).$$

Therefore

$$\rho_{\mathscr{P},\mathscr{Q}}(T;x) \leqslant \limsup_{n\to\infty} q_n^{-1}(\|x\|q_n(\|T\|)) \leqslant \|T\|,$$

because Q verifies condition (m_1) in Proposition 2.8.

By Remark 2.11, we have for every $y \in \mathscr{X}$ with $\|y\| = 1$

$$\limsup_{n\to\infty} \inf_{z\in U} q_n^{-1}(|p_n(z)|) \leqslant \limsup_{n\to\infty} q_n^{-1}(\|p_n(T)y\|) = \rho_{\mathscr{P},\mathscr{Q}}(T;y).$$

For an arbitrary $x \neq 0$ we have in the case (b):

$$\begin{aligned}\limsup_{n\to\infty} \inf_{z\in U} q_n^{-1}(|p_n(z)|) &\leqslant \limsup_{n\to\infty} q_n^{-1}(\|p_n(T)(x/\|x\|)\|)\\ &= \limsup_{n\to\infty} q_n^{-1}((1/\|x\|)\cdot\|p_n(T)x\|)\\ &\leqslant \limsup_{n\to\infty} q_n^{-1}(\|p_n(T)x\|)\end{aligned}$$

because $\mathscr{Q}$ verifies condition (m_3) in Proposition 2.9.

For proving (iv), fix $\alpha \in \mathbb{C}$ and $\eta > 0$, and then pick $\varepsilon \in (0,\eta)$. Since the sequence $\mathscr{Q}$ has property (m_1) in Proposition 2.8, it follows that there exists $n_0 \in \mathbb{N}$ such that $|\alpha| < q_n(\eta)/q_n(\varepsilon)$ whenever $n \geqslant n_0$. Since $\rho_{\mathscr{P},\mathscr{Q}}(T;x) = 0$, there exists $n_1 \in \mathbb{N}$ such that $\|p_n(T)x\| < q_n(\varepsilon)$ for every $n \geqslant n_1$. So, for $n \geqslant \max(n_0, n_1)$, we have

$$|\alpha|\cdot\|p_n(T)x\| < |\alpha|q_n(\varepsilon) < q_n(\eta),$$

whence

$$q_n^{-1}(\|p_n(T)(\alpha x)\|) < \eta.$$

Since η is an arbitrary positive number, it follows that $\rho_{\mathscr{P},\mathscr{Q}}(T;\alpha x) = 0$. □

We now prove a local version of Theorem 4.4.

Theorem 4.13. *Let $T_1, T_2 \in \mathscr{B}(\mathscr{X})$ be a couple of commuting bounded linear operators on $\mathscr{X}$, $\mathscr{P}$ a binomial sequence and $\mathscr{Q}$ a positive binomial sequence which verifies condition* (m_1) *in Proposition* 2.8. *Then the following inequality holds*:

$$\rho_{\mathscr{P},\mathscr{Q}}(T_1 + T_2;x) \leqslant \rho_{\mathscr{P},\mathscr{Q}}(T_1) + \rho_{\mathscr{P},\mathscr{Q}}(T_2;x).$$

Proof. Denote $\rho_1 := \rho_{\mathscr{P},\mathscr{Q}}(T_1)$ and $\rho_2 := \rho_{\mathscr{P},\mathscr{Q}}(T_2;x)$, and let $\varepsilon>0$ arbitrary. Also let $\mathscr{P} = \{p_n\}_{n\geqslant 0}$ and $\mathscr{Q} = \{q_n\}_{n\geqslant 0}$. Then there exists $M_\varepsilon>0$ such that

$$(\forall n\in\mathbb{N}) \qquad ||p_n(T_1)|| \leqslant M_\varepsilon q_n(\rho_1+\varepsilon)$$

and

$$(\forall n\in\mathbb{N}) \qquad ||p_n(T_2)x|| \leqslant M_\varepsilon q_n(\rho_2+\varepsilon).$$

We have

$$(\forall n\in\mathbb{N}) \quad p_n(T_1+T_2) = \sum_{k=0}^{n}\binom{n}{k} p_k(T_1)p_{n-k}(T_2),$$

for $T_1T_2 = T_2T_1$ and $\{p_n\}_{n\in\mathbb{N}}$ is a binomial sequence (see Remark 3.6). Hence

$$(\forall n\in\mathbb{N}) \quad ||p_n(T_1+T_2)x|| \leqslant M_\varepsilon^2 q_n(\rho_1+\rho_2+2\varepsilon),$$

using the fact that $\{q_n\}_{n\in\mathbb{N}}$ is also a binomial sequence. But the fact that $\mathscr{Q}$ verifies condition (m_1) in Proposition 2.8 means that $\limsup_{n\to\infty} q_n^{-1}(\alpha q_n(\delta)) \leqslant \delta$ for all $\alpha,\delta>0$. So we deduce

$$(\forall\varepsilon>0) \quad \limsup_{n\to\infty} q_n^{-1}(||p_n(T_1+T_2)x||) \leqslant \rho_1+\rho_2+\varepsilon,$$

and the theorem is proved. $\square$

4.1. Binomial semidistances

In a similar manner as in the case of binomial radii, we can attach a *binomial semidistance* to some pairs $(\mathscr{P},\mathscr{Q})$ of binomial sequences. In the case $\mathscr{P} = \mathscr{Q} = \{X^n\}_{n\geqslant 0}$, these reduce to the well-known spectral semidistance (see [Ap68,Va67]).

Until the end of this section, $\mathscr{A}$ stands for a unital associative complex Banach algebra. We denote by $\mathbf{1}$ the unit of $\mathscr{A}$ and assume that $||\mathbf{1}|| = 1$. For $A,B\in\mathscr{A}$, we define $\mathbf{C}(A,B) : \mathscr{A}\to\mathscr{A}$ by $\mathbf{C}(A,B)X = AX - XB$. The following lemma is an easy consequence of Theorem 3.7.

Lemma 4.14. *For every binomial sequence $\{p_n\}_{n\geqslant 0}$ and $A,B,C,X,Y\in\mathscr{A}$ we have*

$$(\forall n\in\mathbb{N}) \quad p_n(\mathbf{C}(A,C))(XY) = \sum_{k=0}^{n}\binom{n}{k} p_k(\mathbf{C}(A,B))X\cdot p_{n-k}(\mathbf{C}(B,C))Y.$$

Proof. Consider the complex unital algebra $\mathscr{O}(\mathbb{C},\mathscr{A})$ of holomorphic $\mathscr{A}$-valued functions on $\mathbb{C}$, and its derivation

$$D : \mathscr{O}(\mathbb{C},\mathscr{A})\to\mathscr{O}(\mathbb{C},\mathscr{A}), \quad f\mapsto\frac{\partial f}{\partial z}.$$

Let $f, g \in \mathcal{O}(\mathbb{C}, \mathscr{A})$ defined for all $z \in \mathbb{C}$ by $f(z) = e^{zA}Xe^{-zB}$ and $g(z) = e^{zB}Ye^{-zC}$, respectively. Then Theorem 3.7 implies that for all $n \geqslant 0$ we have

$$p_n(D)(fg) = \sum_{k=0}^{n} \binom{n}{k} p_k(D)f \cdot p_{n-k}(D)g.$$

But for every $j \geqslant 0$ we have $p_j(D)f|_{z=0} = p_j(\mathbf{C}(A,B))$, $p_j(D)g|_{z=0} = p_j(\mathbf{C}(B,C))$ and $p_j(D)(fg)|_{z=0} = p_j(\mathbf{C}(A,C))$ (see [Va82, Chapter III, Proof of Lemma 4.1]), hence the desired formula follows. $\square$

In all which follows, $\mathscr{P} = \{p_n\}_{n \geqslant 0}$ is a binomial sequence, and $\mathscr{Q} = \{q_n\}_{n \geqslant 0}$ is a positive binomial sequence which verifies condition (m_1) in Proposition 2.8.

Definition 4.15. For $A, B \in \mathscr{A}$ we define

$$(\forall X \in \mathscr{A}) \quad \rho_{\mathscr{P},\mathscr{Q};X}(A,B) = \limsup_{n \to \infty} q_n^{-1}(\| p_n(\mathbf{C}(A,B))X \|) = \rho_{\mathscr{P},\mathscr{Q}}(\mathbf{C}(A,B);X).$$

The *binomial semidistance* between A and B is defined by

$$\rho_{\mathscr{P},\mathscr{Q}}(A,B) = \max\{\rho_{\mathscr{P},\mathscr{Q};\mathbf{1}}(A,B), \rho_{\mathscr{P},\mathscr{Q};\mathbf{1}}(B,A)\} = \max\{\rho_{\mathscr{P},\mathscr{Q}}(\mathbf{C}(A,B);\mathbf{1}), \rho_{\mathscr{P},\mathscr{Q}}(\mathbf{C}(B,A);\mathbf{1}).$$

The following result is an extension of Lemma 4.1 in Chapter III in [Va82].

Lemma 4.16. *For every $A, B, C, X, Y \in \mathscr{A}$ we have*

$$\rho_{\mathscr{P},\mathscr{Q};XY}(A,C) \leqslant \rho_{\mathscr{P},\mathscr{Q};X}(A,B) + \rho_{\mathscr{P},\mathscr{Q};Y}(B,C).$$

Proof. We use Lemma 4.14 and proceed as in the proof of Theorem 4.4. $\square$

Proposition 4.17. *The function $\rho_{\mathscr{P}\mathscr{Q}}(\cdot,\cdot) : \mathscr{A} \times \mathscr{A} \to [0,\infty]$ of Definition 4.15 is a semidistance. If in addition $\mathscr{P}$ is a positive binomial sequence and $p_n(x) \leqslant q_n(x)$ for every $x \geqslant 0$ and $n \in \mathbb{N}$, then ρ has only finite values.*

Proof. Using Lemma 4.16 it is easy to prove that $\rho_{\mathscr{P},\mathscr{Q}}(\cdot,\cdot)$ satisfies the triangle inequality.

If $p_n(x) \leqslant q_n(x)$ for every $x \geqslant 0$ and $n \in \mathbb{N}$, then

$$\| p_n(\mathbf{C}(A,B))\mathbf{1} \| \leqslant p_n(\|\mathbf{C}(A,B)\|)\|\mathbf{1}\| \leqslant q_n(\|\mathbf{C}(A,B)\|).$$

Since each $q_n^{-1} : [0, \infty) \to [0, \infty)$ is an increasing function, it then follows that

$$(\forall n \in \mathbb{N}) \quad q_n^{-1}(||p_n(\mathbf{C}(A,B))\mathbf{1}||) \leqslant ||\mathbf{C}(A,B)||.$$

Finally, take $\limsup_{n\to\infty}$ and the proof ends. □

5. Nilpotence criteria

In this section we shall apply the previous results to bounded derivations of Banach algebras, in particular to inner derivations $\operatorname{ad} A$ of $\mathscr{B}(\mathscr{X})$ defined by $A \in \mathscr{B}(\mathscr{X})$. In this way, we shall obtain some nilpotence properties.

For the following statement, we recall condition (m_1) in Proposition 2.8 and the fact that, according to Proposition 2.9, condition (m_3) is sufficient for (m_1) to hold.

Proposition 5.1. *Let $\mathscr{A}$ be a complex Banach algebra, $D : \mathscr{A} \to \mathscr{A}$ a bounded linear derivation of $\mathscr{A}$, $\mathscr{P} = \{p_n\}_{n\geqslant 0}$ and $\mathscr{Q} = \{q_n\}n\geqslant 0$ two positive binomial sequences such that $\mathscr{Q}$ verifies condition* (m_1) *in Proposition* 2.8. *Then the inequality*

$$\limsup_{n\to\infty} q_n^{-1}(||p_n(D-(\lambda+\mu))(x_ny_n)||) \leqslant \limsup_{n\to\infty} q_n^{-1}(||p_n(D-\lambda)x_n||) + \limsup_{n\to\infty} q_n^{-1}(||p_n(D-\mu)y_n||)$$

holds for every $\lambda, \mu \in \mathbb{C}$ and $x_n, y_n \in \mathscr{A}$, $n \in \mathbb{N}$.

Proof. The inequality is obvious if one of the terms in the right-hand side equals ∞. It then remains to prove the inequality when the right-hand side is finite. We denote

$$\rho = \limsup_{n\to\infty} q_n^{-1}(||p_n(D-\lambda)x_n||) \quad \text{and} \quad \theta = \limsup_{n\to\infty} q_n^{-1}(||p_n(D-\mu)y_n||).$$

Then for every $\varepsilon > 0$ there exists $M_\varepsilon > 0$ such that for every $n \in \mathbb{N}$ we have

$$||p_n(D-\lambda)x_n|| \leqslant M_\varepsilon q_n(\rho+\varepsilon), \quad ||p_n(D-\mu)y_n|| \leqslant M_\varepsilon q_n(\theta+\varepsilon).$$

If we proceed as in the proof of Theorem 4.4 we obtain

$$(\forall n \in \mathbb{N}) \quad ||p_n(D-(\lambda+\mu))(x_ny_n)|| \leqslant M_\varepsilon^2 q_n(\rho+\theta+2\varepsilon).$$

But $\mathscr{Q}$ verifies condition (m_1) in Proposition 2.8. Hence for every $\varepsilon > 0$ we have

$$\limsup_{n\to\infty} q_n^{-1}(||p_n(D-(\lambda+\mu))(x_ny_n)||) \leqslant \rho+\theta+2\varepsilon,$$

and the proof is finished. □

If we set, in Proposition 5.1, $x_n = x$ and $y_n = y$ for every $n \in \mathbb{N}$, then the conclusion can be expressed as follows.

Corollary 5.2. *If* $\mathscr{A}$ *is a complex Banach algebra,* $D : \mathscr{A} \to \mathscr{A}$ *is a bounded linear derivation,* $\mathscr{P}$ *and* $\mathscr{Q}$ *are two positive binomial sequences such that* $\mathscr{Q}$ *verifies condition* (m_1) *in Proposition* 2.8, *then we have*

$$\rho_{\mathscr{P},\mathscr{Q}}(D - (\lambda + \mu); xy) \leqslant \rho_{\mathscr{P},\mathscr{Q}}(D - \lambda; x) + \rho_{\mathscr{P},\mathscr{Q}}(D - \mu; y)$$

for every $x, y \in \mathscr{A}$ *and* $\lambda, \mu \in \mathbb{C}$.

Now we will prove that a bounded linear derivation D of a Banach algebra $\mathscr{A}$ produces nilpotent elements. More precisely, in certain conditions about $\mathscr{P}$, $\mathscr{A}$ and $\lambda \in \mathbb{C}$, the elements λ with $\rho_{\mathscr{P},\mathscr{Q}}(D - \lambda; x) = 0$ are nilpotent (see Theorem 5.3 below). A well-known example is provided by the following implication (see [FS74]):

$$A, X \in \mathscr{B}(\mathscr{X}); \quad 0 \neq \lambda \in \mathbb{C}; \quad [A, X] = \lambda X \Rightarrow X \text{ nilpotent.}$$

More generally, if $\lambda \neq 0$ and $\lim_{n\to\infty} ||(\mathrm{ad}\, A - \lambda)^n X||^{1/n} = 0$, then X is nilpotent (see [BS01, Section 17, Theorem 4]).

A much more general nilpotence criterion is provided by the following theorem, whose condition $\rho_{\mathscr{P},\mathscr{Q}}(D - \lambda; x) = 0$ means $\lim_{n\to\infty} q_n^{-1}(||p_n(D - \lambda)x||) = 0$.

Theorem 5.3. *Let* $\mathscr{A}$ *be a unital associative complex Banach algebra,* $D : \mathscr{A} \to \mathscr{A}$ *a bounded linear derivation,* $\mathscr{P} = \{p_n\}_{n\geqslant 0}$ *a binomial sequence and* $\mathscr{Q} = \{q_n\}_{n\geqslant 0}$ *a positive binomial sequence which verifies condition* (m_1) *in Proposition* 2.8 *and such that* $\liminf_{n\to\infty} q_n^{-1}(\delta^n) > 0$ *for every* $\delta > 0$. *Denote* $Z_m = p_m^{-1}(\{0\})$ *for each* $m \geqslant 0$, *and* $Z = \bigcap_{n\geqslant 0} \overline{\bigcup_{m\geqslant n} Z_m}$. *If* $x \in \mathscr{A}$ *and there exists* $\lambda \in \mathbb{C}$ *such that* $\rho_{\mathscr{P},\mathscr{Q}}(D - \lambda; x) = 0$ *and* $\mathrm{dist}(0, Z + k_0\lambda) > ||D||$ *for some* $k_0 \in \mathbb{N}$, *then* x *is nilpotent.*

Proof. By Corollary 5.2 we have

$$(\forall k \geqslant 1) \quad \rho_{\mathscr{P},\mathscr{Q}}(D - k\lambda; x^k) = 0,$$

because $\mathscr{Q}$ satisfies (m_1).

If x is not nilpotent, then $x^k \neq 0$ for every $k \geqslant 1$. Then we notice that, for each $k \geqslant 1$, we can apply Proposition 4.12 (iv) since $\mathscr{Q}$ verifies condition (m_1) in Proposition 2.8. We thus obtain

$$\rho_{\mathscr{P},\mathscr{Q}}(D - k\lambda; x^k/||x^k||) = 0.$$

Now, using Proposition 2.12(ii) with the constant sequence $y_n = x^k/||x^k||$ for all $n \in \mathbb{N}$, we deduce

$$\lim_{n\to\infty} \inf_{z\in U} q_n^{-1}(|p_n(z)|) = 0$$

for any relatively compact subset U of $\mathbb{C}$ with $U \supseteq \sigma(D-k\lambda)$. We can apply Lemma 2.14 for $\deg p_n = k_n = n$, $a_{k_n} = a_n = 1$, $\mathcal{Q}$ verifies condition (m_1) in Proposition 2.8, hence also condition (π_1) (see Remark 2.7(3) and Proposition 2.8). We deduce that

$$\sigma(D-k\lambda)\cap\overline{\bigcup_{n\geqslant 1} Z_n}\neq\emptyset.$$

As in the proof of Corollary 2.15 it then follows that

$$\sigma(D-k\lambda)\cap\bigcap_{n\geqslant 1}\overline{\bigcup_{m\geqslant n} Z_m}\neq\emptyset.$$

So, we have, using the notation in the statement of Theorem 5.3,

$$(\forall k\geqslant 1)\quad \sigma(D)\cap(Z+k\lambda)\neq\emptyset.$$

But this is a nonsense, since the hypothesis says that there exists $k_0\in\mathbb{N}$ such that $\mathrm{dist}(0, Z+k_0\lambda) > ||D||$, which implies that we actually have

$$\sigma(D)\cap(Z+k_0\lambda)=\emptyset.$$

(It is well known that $\sigma(D)\subseteq\{z\in\mathbb{C} \mid z| < ||D||\}$.) The proof is finished. □

Remark 5.4. In the hypothesis of Theorem 5.3, one can replace the condition on λ by the following one:

$$\limsup_{k\to\infty}\ \mathrm{dist}(0, Z+k\lambda) = \infty.$$

Example 5.5. As in Theorem 5.3, let $\mathscr{A}$ be a unital associative complex Banach algebra and $D:\mathscr{A}\to\mathscr{A}$ a bounded linear derivation. Then $\mathcal{Q}=\{X^n\}_{n\geqslant 0}$ is a positive binomial sequence with property (m_1) in Proposition 2.8, and $\liminf_{n\to\infty} q_n^{-1}(\delta^n) > 0$ for every $\delta > 0$. We specialize the statement of Theorem 5.3 according to various choices of the sequence $\mathscr{P}=\{p_n\}_{n\geqslant 0}$. Let $Z=\bigcap_{n\geqslant 1}\overline{\bigcup_{m\geqslant n} p_m^{-1}(0)}$ as in that statement.

(1) If we consider the sequence of powers

$$(\forall n\in\mathbb{N})\qquad p_n(X)=X^n,$$

then

$$Z=\{0\},$$

hence $\lim_{k\to\infty} \mathrm{dist}(0, Z + k\lambda) = \infty$ whenever $\lambda \neq 0$. Thus we get Corollary 1 in Section 17 in [BS01]: If $x \in \mathscr{A}$, $0 \neq \lambda \in \mathbb{C}$ and

$$\lim_{n\to\infty} ||(D-\lambda)^n x||^{1/n} = 0,$$

then x is nilpotent.

(2) For the sequence of difference polynomials

$$(\forall n \in \mathbb{N}) \qquad p_n(X) = X(X-1)\cdots(X-n+1),$$

which is the binomial sequence associated as in Definition 3.3 with the formal series $g(X) = \log(1+X) = \sum_{n\geqslant 0} \frac{(-1)^n}{n+1} X^{n+1}$ (see [Ro75, Section 13] or [BB64], and also [BL96]), then

$$Z = \mathbb{N},$$

hence $\lim_{k\to\infty} \mathrm{dist}(0, Z + k\lambda) = \infty$ if and only if $-\lambda \notin \mathbb{N}$. Thus, Theorem 5.3 says that, if $x \in \mathscr{A}$, $-\lambda \notin \mathbb{N}$ and

$$\lim_{n\to\infty} ||(D-\lambda)(D-\lambda-1)\cdots(D-\lambda-n+1)x||^{1/n} = 0,$$

then x is nilpotent.

(3) If we consider the Abel polynomials

$$(\forall n \in \mathbb{N}) \qquad p_n(X) = X(X-n)^{n-1},$$

which is the binomial sequence associated as in Definition 3.3 with the formal series $g = h^{-1}$ (inverse with respect to formal series composition), where $h(X) = Xe^X = \sum_{n\geqslant 0} \frac{1}{n!} X^{n+1}$ (see [Ro75, Section 13] or [BB64]), then

$$Z = \{0\},$$

hence $\lim_{k\to\infty} \mathrm{dist}(0, Z + k\lambda) = \infty$ if and only if $\lambda \neq 0$. Thus, according to Theorem 5.3, if $x \in \mathscr{A}$, $\lambda \neq 0$ and

$$\lim_{n\to\infty} ||(D-\lambda)(D-\lambda-n)^{n-1} x||^{1/n} = 0,$$

then x is nilpotent.

6. Applications to Lie algebras

As in the previous sections, we denote by $\mathscr{X}$ a complex Banach space and by $\mathscr{B}(\mathscr{X})$ the algebra of all bounded linear operators on $\mathscr{X}$. We recall that, as actually it is the case with every associative algebra, $\mathscr{B}(\mathscr{X})$ has a natural structure of Lie algebra.

Lemma 6.1. *If $\mathscr{S}$ is a finite-dimensional complex semisimple Lie subalgebra of $\mathscr{B}(\mathscr{X})$, then there exists $H \in \mathscr{S}$ such that*

$$\sigma(H) \cap \sigma(-H) \nsubseteq \{0\}.$$

Proof. By Corollary 5 in Section 30 in [BS01], there exists a non-zero finite-dimensional subspace $\mathscr{X}_0$ of $\mathscr{X}$ such that $\mathscr{X}_0$ is invariant to each operator in $\mathscr{S}$. On the other hand, it is well known that $\mathscr{S}$ has a Lie subalgebra $\mathscr{S}_0$ isomorphic to $\mathfrak{sl}(2, \mathbb{C})$. Then the representation $\mathscr{S}_0 \to \mathscr{B}(\mathscr{X}_0)$, $S \mapsto S|_{\mathscr{X}_0}$, is a direct sum of irreducible representations. Now, the structure theory of irreducible representations of $\mathfrak{sl}(2, \mathbb{C})$ (see e.g., subsections 1.8.4 and 1.8.1 in [Di74]) shows that there exist $H \in \mathscr{S}_0$ and $n \in \mathbb{N}$, $n \neq 0$, such that $\pm n$ are eigenvalues of $H|_{\mathscr{X}_0}$. So $0 \neq n \in \sigma(H) \cap \sigma(-H)$. □

Lemma 6.2. *Every finite-dimensional complex Lie subalgebra of $\mathscr{B}(\mathscr{X})$, all of whose elements T have the property*

$$\sigma(T) \cap \sigma(-T) \subseteq \{0\},$$

is a solvable Lie algebra.

Proof. If such a Lie algebra is not solvable, then it contains a semisimple Lie subalgebra. But this contradicts the hypothesis, according to Lemma 6.1. □

Lemma 6.3. *If $\{p_n\}_{n \geqslant 0}$ is a binomial sequence and we denote*

$$S = \bigcup_{n \geqslant 0} \bigcap_{k \geqslant n} p_k^{-1}(0),$$

then the following assertions hold:

(i) *The set S is a subsemigroup of $(\mathbb{C}, +)$.*
(ii) *We have $S \cap (-S) = \{0\}$.*

Proof. Assertion (i) is an easy consequence of Proposition 3.4(iii).

To prove (ii), assume that we have $0 \neq \alpha \in S \cap (-S)$. Then there exists $n_0 \in \mathbb{N}$ such that $p_k(\alpha) = p_k(-\alpha) = 0$ whenever $k > n_0$. If the sequence $\{p_n\}_{n \geqslant 0}$ is associated as in Definition 3.3 with $g \in \mathbb{C}[[X]]$, where $\omega(g) \geqslant 1$, then

$$e^{\pm\alpha\cdot} \circ g = \sum_{k=0}^{\infty} \frac{1}{k!} p_k(\pm\alpha) X^k = \sum_{k=0}^{n_0} \frac{1}{k!} p_k(\pm\alpha) X^k.$$

So we have

$$1 = (e^{\alpha\cdot} \circ g)(e^{-\alpha\cdot} \circ g) = \left(\sum_{k=0}^{n_0} \frac{1}{k!} p_k(\alpha) X^k\right)\left(\sum_{k=0}^{n_0} \frac{1}{k!} p_k(-\alpha) X^k\right),$$

which implies that $n_0 = 0$, whence $p_n(\alpha) = p_n(-\alpha) = 0$ whenever $n \geqslant 1$. In particular $p_1(\alpha) = p_1(-\alpha) = 0$, which is impossible since $\deg p_1 = 1$. $\square$

Theorem 6.4. *Let $\mathscr{G}$ be a finite-dimensional complex Lie subalgebra of $\mathscr{B}(\mathscr{X})$. If for each $T \in \mathscr{G}$ there exist a binomial sequence $\{p_n\}_{n\geqslant 0}$ and $n_0 \in \mathbb{N}$ such that the operator $p_n(T)$ is quasinilpotent whenever $n \geqslant n_0$, then $\mathscr{G}$ is a solvable Lie algebra.*

Proof. Let $T \in \mathscr{G}$ arbitrary, and $\{p_n\}_{n\geqslant 0}$ the corresponding binomial sequence as in the hypothesis. By the spectral mapping theorem it then follows that

$$\sigma(T) \subseteq \bigcup_{n\geqslant 0} \bigcap_{k\geqslant n} p_k^{-1}(0).$$

So by Lemma 6.3 we have

$$\sigma(T) \cap \sigma(-T) \subseteq \{0\}.$$

Then $\mathscr{G}$ is solvable by Lemma 6.2. $\square$

In the next statement, we use the notions of ideally finite, respectively quasisolvable, Lie algebra in the sense of Definition 5(b) in Section 2 in [BS01].

Corollary 6.5. *Let $\mathscr{G}$ be an ideally finite complex Lie algebra. If for every $G \in \mathscr{G}$ there exist a binomial sequence $\{p_n\}_{n\geqslant 0}$ and $n_0 \in \mathbb{N}$ such that $p_n(\mathrm{ad}\, G) = 0$ whenever $n \geqslant n_0$, then $\mathscr{G}$ is a quasisolvable Lie algebra.*

Proof. Obviously, it suffices to prove the assertion in the case when $\dim \mathscr{G} < \infty$. Then $\mathrm{ad}\, \mathscr{G} := \{\mathrm{ad}\, G \mid G \in \mathscr{G}\}$ is a Lie subalgebra of $\mathrm{End}\, \mathscr{G} = \mathscr{B}(\mathscr{G})$. If follows by Theorem 6.4 that $\mathrm{ad}\, \mathscr{G}$ is a solvable Lie algebra. Then the solvability of $\mathscr{G}$ can be deduced by the following exact short sequence:

$$0 \to Z_{\mathscr{G}} \hookrightarrow \mathscr{G} \to \mathrm{ad}\, \mathscr{G} \to 0,$$

where both $Z_{\mathscr{G}}$ (the center of $\mathscr{G}$) and $\mathrm{ad}\, \mathscr{G}$ are solvable Lie algebras. $\square$

The point of the next corollary is that we make no finite-dimensionality assumption on the Lie algebra $\mathscr{G}$.

Corollary 6.6. *Let K be a compact topological space, $m \in \mathbb{N}$, and the unital C^*-algebra*

$$\mathscr{A} = \mathscr{C}(K) \otimes M_m(\mathbb{C}) = \{a : K \to M_m(\mathbb{C}) \mid a \text{ continuous}\}.$$

If $\mathcal{G}$ is an arbitrary complex Lie subalgebra of $\mathcal{A}$ and for every $a \in \mathcal{G}$ there exist a binomial sequence $\{p_n\}_{n \geqslant 0}$ and $n_0 \in \mathbb{N}$ such that $p_n(a)$ is quasinilpotent whenever $n \geqslant n_0$, then $\mathcal{G}$ is a solvable Lie algebra.

Proof. To begin with, we recall that for every $a \in \mathcal{A}$

$$\sigma(a) = \bigcup_{t \in K} \sigma(a(t))$$

(see e.g., [Be02, Theorem 2.5]), and denote for every $t \in K$,

$$\mathcal{G}_t := \{a(t) \mid a \in \mathcal{G}\} \subseteq M_m(\mathbb{C}).$$

Next let $a \in \mathcal{A}$ and let $\{p_n\}_{n \geqslant 0}$ be the binomial sequence and $n_0 \in \mathbb{N}$ corresponding to a by the hypothesis. By the beginning remark, we have for every $t \in K$ and $n \geqslant n_0$ that $\sigma((p_n(a(t))) \subseteq \sigma(p_n(a)) = \{0\}$ (where the latter equality follows by the assumption on p_n). So we can apply Theorem 6.4 to the *finite-dimensional* Lie algebra $\mathcal{G}_t$, and we obtain that $\mathcal{G}_t$ is solvable for each $t \in K$.

But $\dim \mathcal{G}_t \leqslant \dim M_m(\mathbb{C}) = m^2$, hence $(\mathcal{G}_t)^{(m^2)} = \{0\}$ for all $t \in K$. (We denote by $\mathcal{L}^{(k)}$ the kth term in the descending derived series of a Lie algebra $\mathcal{L}$; see e.g., the very beginning of Section 2 in [BS01]).

On the other hand, for every $a \in \mathcal{G}^{(m^2)}$ and $t \in K$ we have $a(t) \in (\mathcal{G}_t)^{(m^2)}$. Therefore, $a(t) = 0$ for every $t \in K$ and $a \in \mathcal{G}^{(m^2)}$. Hence $\mathcal{G}^{(m^2)} = \{0\}$, and then $\mathcal{G}$ is solvable. □

Example 6.7. Let $m \geqslant 1$ and $\mathcal{T}$ a Lie algebra of $m \times m$ upper-triangular matrices with complex entries. For each $A \in \mathcal{T}$, we denote its diagonal entries in the following way:

$$A = \begin{pmatrix} \lambda_1(A) & * & * \\ 0 & \ddots & * \\ 0 & 0 & \lambda_m(A) \end{pmatrix}.$$

It is clear that $\lambda_1, \ldots, \lambda_m : \mathcal{T} \to \mathbb{C}$ are characters of $\mathcal{T}$. That is, all of them are linear functionals vanishing on $[\mathcal{T}, \mathcal{T}]$.

Next denote

$$\mathcal{G} = \{A \in \mathcal{T} \mid (\exists \alpha \in \mathbb{C})(\forall k \in \{1, \ldots, m\}) \quad \lambda_k(A) = \lambda_1(A) + (k-1)\alpha\}.$$

Then $\operatorname{ad} \mathcal{G} = \{\operatorname{ad} A|_{\mathcal{G}} \mid A \in \mathcal{G}\}$ is a (solvable) Lie algebra which falls under the hypothesis of Theorem 6.4.

Indeed, $\mathcal{G} = \mathcal{T}$ for $m = 2$, and for $m \geqslant 3$ we can write

$$\mathcal{G} = \bigcap_{i=1}^{n-2} \operatorname{Ker}(\lambda_i - 2\lambda_{i+1} + \lambda_{i+2}).$$

Thus $\mathscr{G}$ is the intersection of the kernels of some characters of $\mathscr{T}$, which shows that $\mathscr{G}$ is an ideal of the solvable Lie algebra $\mathscr{T}$. In particular, $\mathscr{G}$ is a solvable Lie algebra.

Now we notice that for each $A \in \mathscr{T}$ we have

$$\sigma(\text{ad } A|_{\mathscr{T}}) \subseteq \{\lambda_j(A) - \lambda_i(A) \mid 1 \leqslant i \leqslant j \leqslant m\}.$$

(This is obvious if A is a diagonal matrix. The general case then follows by Corollary 5 in Section 27, p. 170 in [BS01]).

If $A \in \mathscr{G}$, let $\alpha \in \mathbb{C}$ such that $\lambda_k(A) = \lambda_1(A) + (k-1)\alpha$ whenever $1 \leqslant k \leqslant m$. If we consider the binomial sequence defined by $p_0 \equiv 1$ and $p_n(X) = X(X-\alpha)\cdots(X-(n-1)\alpha)$ for $n \geqslant 1$ (which, in the terminology of Definition 3.3, is associated with $g(X) = \log(1+\alpha X)^{1/\alpha}$ for $\alpha \neq 0$, or $g(X) = X$ for $\alpha = 0$; see Section 13 in [Ro75] or [BB64], and also the previous Example 5.5(2)), then

$$\sigma(\text{ad } A|_{\mathscr{G}}) \subseteq \sigma(\text{ad } A|_{\mathscr{T}}) \subseteq p_m^{-1}(0).$$

By the spectral mapping theorem we deduce

$$\sigma(p_m(\text{ad } A|_{\mathscr{G}})) = \{0\}.$$

Therefore $p_m(\text{ad } A|_{\mathscr{G}})$ is a nilpotent operator on $\mathscr{G}$. For every $n \geqslant m$ we have $p_m \mid p_n$, hence $p_n(\text{ad } A|_{\mathscr{G}})$ is in turn nilpotent. Thus ad $\mathscr{G}$ falls under the hypothesis of Theorem 6.4. □

We now give a name to the Lie algebras occurring in Corollary 6.5.

Definition 6.8. A complex Lie algebra $\mathscr{G}$ is *binomially nilpotent* if for every $b \in \mathscr{G}$ there exist a binomial sequence $\{p_n\}_{n \geqslant 0}$ and $n_0 \in \mathbb{N}$ such that $p_n(\text{ad } b) = 0$ whenever $n \geqslant n_0$. In other words, for every $b \in \mathscr{G}$ there exists $g \in \mathbb{C}[[X]]$ with $\omega(g) \geqslant 1$ such that $e^{(\text{ad } b)\cdot} \circ g$ is a polynomial.

Remark 6.9. The nilpotent Lie algebras are binomially nilpotent, corresponding to the binomial sequence $\{X^n\}_{n \geqslant 0}$.

Some properties of nilpotent Lie algebras can be extended to binomially nilpotent Lie algebras as follows.

Proposition 6.10. *If $\mathscr{G}$ is a Lie algebra of endomorphisms of some complex vector space $\mathscr{V}$, then*

$$p_n(\text{ad } Q)T = \sum_{k=0}^{n} \frac{1}{k!(n-k)!} p_k(Q) T p_{n-k}(-Q)$$

for all $Q, T \in \mathscr{G}$, $n \in \mathbb{N}$, and every binomial sequence $\{p_n\})n \geqslant 0$.

Proof. Indeed, if $\{p_n\}_{n\geqslant 0}$ is associated with $g\in\mathbb{C}[[X]]$, where $\omega(g)\geqslant 1$, the we can obtain in the same manner as in the proof of Theorem 3.7 the following formal version of Rosenblum's formula:

$$(e^{Q\cdot}\circ g)T(e^{Q\cdot}\circ g) = (e^{(\mathrm{ad}\ Q)\cdot}\circ g)T,$$

that is,

$$\left(\sum_{n=0}^{\infty}\frac{p_n(Q)}{n!}X^nT\right)\left(\sum_{n=0}^{\infty}\frac{p_n(-Q)}{n!}X^n\right) = \sum_{n=0}^{\infty}\frac{p_n(\mathrm{ad}\ Q)}{n!}X^nT,$$

where $\sum_{n=0}^{\infty}\frac{p_n(Q)}{n!}X^nT$ actually means $\sum_{n=0}^{\infty}\frac{p_n(Q)}{n!}TX^n$, the left-hand side of the above equality means the Abel product of the series

$$\sum_{n=0}^{\infty}\frac{p_n(Q)}{n!}TX^n, \sum_{n=0}^{\infty}\frac{p_n(-Q)}{n!}X^n\in(\mathrm{End}\ \mathscr{V})[[X]],$$

and the right-hand side of the equality means $\sum_{n=0}^{\infty}\frac{p_n(\mathrm{ad}\ Q)T}{n!}X^n$. If we identify the coefficients in the above equality we obtain the desired conclusion. □

Definition 6.11. An endomorphism Q of a complex vector space $\mathscr{V}$ is *binomially nilpotent* if there exist a binomial sequence $\{p_n\}_{n\geqslant 0}$ and $n_0\in\mathbb{N}$ such that $p_n(Q) = p_n(-Q) = 0$ whenever $n\geqslant n_0$.

Remark 6.12.

(1) The nilpotent endomorphisms are binomially nilpotent, relatively to the binomial sequence $\{X^n\}_{n\geqslant 0}$.
(2) If $\mathscr{V}$ is a complex vector space, then $Q\in\mathrm{End}\ \mathscr{V}$ is binomially nilpotent if and only if there exists $h\in\mathbb{C}[[X]]$, $\omega(h)\geqslant 1$, such that both $e^{Q\cdot}\circ h$ and $e^{(-Q)\cdot}\circ h$ are polynomials (with coefficients in End $\mathscr{V}$).
(3) In the setting of (2), if h is odd (in the sense that all coefficients of even powers of the variable are zero), then the fact that $e^{Q\cdot}\circ h$ is a polynomial, automatically implies the similar property for $e^{(-Q)\cdot}\circ h$.

Proposition 6.13. *Every Lie algebra $\mathscr{G}$ of binomially nilpotent endomorphisms of some complex vector space is binomially nilpotent.*

Proof. Indeed, Proposition 6.10 implies that, if $\{p_n\}_{n\geqslant 0}$ is a binomial sequence and $p_n(Q) = p_n(-Q) = 0$ for every $n\geqslant n_0$, then $p_n(\mathrm{ad}\ Q) = 0$ whenever $n > 2n_0$. □

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