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On averaging operators for Atanassov's intuitionistic fuzzy sets

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Atanassov's intuitionistic fuzzy set (AIFS) is a generalization of a fuzzy set. There are various averaging operators defined for AIFSs. These operators are not consistent with the limiting case of ordinary fuzzy sets, which is undesirable. We show how such averaging operators can be represented by using additive generators of the product triangular norm, which simplifies and extends the existing constructions. We provide two generalizations of the existing methods for other averaging operators. We relate operations on AIFS with operations on interval-valued fuzzy sets. Finally, we propose a new construction method based on the Łukasiewicz triangular norm, which is consistent with operations on ordinary fuzzy sets, and therefore is a true generalization of such operations.

1. Introduction

Since the introduction of fuzzy sets by Zadeh [39] many attempts have been made to generalize the notion of fuzzy sets. Among others, Zadeh introduced the idea of type-2 fuzzy sets and interval valued fuzzy sets [40], see also [18,41,42]. Later in [2], Atanassov introduced the idea of intuitionistic fuzzy sets (AIFS). Recently, several authors [23,25–30] have used AIFS in different applications. In [3,8,16] authors advanced the theory of operators and relations for intuitionistic fuzzy sets and interval-valued intuitionistic fuzzy sets. Bustince and Burillo [8] and Deschrijver and Kerre [11] made theoretical development relating to composition of intuitionistic fuzzy relations.

In many decision making applications, it becomes necessary to aggregate several fuzzy sets, particularly when preference is expressed by fuzzy sets [6,7,22,37]. Weighted means and the Ordered Weighted Averaging functions (OWA) [37] have been applied to this problem, along with triangular norms, conorms and uninorms. Various aggregation functions are discussed in detail in [7,19,20,32].

In a recent series of papers [31,33–36,38,43] the authors defined some averaging aggregation functions for Atanassov's intuitionistic fuzzy sets, including weighted means, OWA and Choquet integrals. Their definitions are based on the operations of addition and multiplication for AIFS [3,4], which involve the product t-norm and its dual t-conorm. A problem with their definitions is that they do not lead to the standard aggregation operations on fuzzy sets in the limiting case (see Example 2).

On the other hand, as noticed in [5] and later developed in [8,10,14], the AIFS are mathematically equivalent to the interval-valued fuzzy sets (IVFS). There exist several papers that relate other extensions of fuzzy set theory to AIFS [9,17,24]. It is

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straightforward to define aggregation functions for IVFS, see, e.g. [38], by applying a fixed aggregation function to the ends of the membership interval, the membership and (transformed) non-membership degrees. Such functions are called representable in [15]. This approach is fully consistent with the limiting case of the ordinary fuzzy sets, but there are various non-representable AIFS aggregation functions, such as those presented in [12,13,15], and those discussed in this paper.

In this paper we develop an approach to extending aggregation operators to AIFS and IVFS, by using additive generators of the t-norm and t-conorm in the arithmetical operations for AIFS. We provide simple tools to construct both representable and non-representable extensions. We establish several interesting properties of the operators constructed by using Łukasiewicz t-norm and t-conorm in the operations for AIFS. Our approach will eliminate the need for complex and explicit constructions in [34–36,43].

The structure of this paper is as follows. We review operations on AIFS and IVFS in Section 2. In Section 3 we present several types of averaging aggregation functions on AIFS based on arithmetical operations on AIFS. In Section 4 we relate the mentioned aggregation functions to those defined for IVFS, and present several more general approaches to aggregation of AIFS. We will show that the use of Łukasiewicz t-norm and t-conorm in the definition of addition of AIFS guarantees consistency of aggregation of AIFS with aggregation of ordinary fuzzy sets. This section is then followed by conclusions.

2. Operations on AIFS and IVFS

We review several relevant concepts and highlight the correspondence between the notions in AIFS and IVFS [2].

Definition 1. An AIFS $\mathcal{A}$ on X is defined as $\mathcal{A} = \{\langle x, \mu_{\mathcal{A}}(x), \nu_{\mathcal{A}}(x)\rangle | x \in X\}$, where $\mu_{\mathcal{A}}(x)$ and $\nu_{\mathcal{A}}(x)$ are the degrees of membership and nonmembership of x in $\mathcal{A}$, which satisfy $\mu_{\mathcal{A}}(x),\ \nu_{\mathcal{A}}(x) \in [0,1]$ and $0 \leqslant \mu_{\mathcal{A}}(x) + \nu_{\mathcal{A}}(x) \leqslant 1$.

Definition 2. An IVFS $\mathcal{A}$ on X is defined as $\mathcal{A} = \{\langle x, [l_{\mathcal{A}}(x), r_{\mathcal{A}}(x)]\rangle | x \in X\}$, where $l_{\mathcal{A}}(x)$ and $r_{\mathcal{A}}(x)$ are the lower and upper ends of the membership interval, and satisfy $0 \leqslant l_{\mathcal{A}}(x) \leqslant r_{\mathcal{A}}(x) \leqslant 1$.

Obviously an ordinary fuzzy set can be written as $\{\langle x, \mu_{\mathcal{A}}(x), 1-\mu_{\mathcal{A}}(x)\rangle | x \in X\}$, or as $\{\langle x, [\mu_{\mathcal{A}}(x), \mu_{\mathcal{A}}(x)]\rangle | x \in X\}$. AIFS can be represented by means of IVFS and vice versa.

To ease the notation we now suppress the dependence of the membership/non-membership values on x, i.e., we will consider Atanassov's intuitionistic fuzzy values (AIFV), the pairs $A = \langle \mu_A, \nu_A\rangle$, and IV fuzzy values (IVFV) $[l_A, r_A]$.

Several indices are used to characterize AIFV.

Definition 3. The *Score* and *Accuracy* of an AIFV A are defined by

$$Score(A) = \mu_A - \nu_A,$$
$$Accuracy(A) = \mu_A + \nu_A.$$

The degree of indeterminancy of A is

$$\pi(A) = 1 - (\mu_A + \nu_A).$$

We list the indices in Table 1 together with their counterparts in the IVFS representation. We use $\frac{l+r}{2}$ to denote the center of the interval $[l, r]$.

Definition 4 (*[7,19]*). An aggregation function $f : [0,1]^n \to [0,1]$ is a function non-decreasing in each argument and satisfying $f(0,\ldots,0) = 0$ and $f(1,\ldots,1) = 1$.

An aggregation function is idempotent if $f(t,t,\ldots,t) = t$ for all $t \in [0,1]$. This is equivalent to averaging behavior of an aggregation function, i.e., $\min(x) \leqslant f(x) \leqslant \max(x)$ for all $x \in [0,1]^n$.

Let us denote by L the lattice of non-empty intervals $L = \{[a,b] | (a,b) \in [0,1]^2,\ a \leqslant b\}$ with the partial order $\leqslant_L$ defined as $[a,b] \leqslant_L [c,d] \iff a \leqslant c$ and $b \leqslant d$. The top and bottom elements are respectively $1_L = [1,1]$, $0_L = [0,0]$. The corresponding

Table 1
Various indices and operations on AIFV and IVFV.

Index	AIFV representation	IVFV representation
Membership	$\langle \mu, \nu\rangle$	$[l, r] = [\mu, 1-\nu]$
Degree of indeterminancy	$\pi = 1 - (\mu + \nu)$	$r - l = length([l,r])$
Score	$\mu - \nu$	$r + l - 1 = 2(\frac{l+r}{2}) - 1$
Accuracy	$\mu + \nu = 1 - \pi$	$l - r + 1 = 1 - length([l,r])$

partial order in the AIFV representation $\leqslant_{L_{AIFV}}$ is defined as $\langle\mu_1,\nu_1\rangle \leqslant_{L_{AIFV}} \langle\mu_2,\nu_2\rangle \iff \mu_1 \leqslant \mu_2$ and $\nu_1 \geqslant \nu_2$. The top and bottom elements are $1_{L_{AIFV}} = \langle 1,0\rangle$ and $0_{L_{AIFV}} = \langle 0,1\rangle$ respectively.

Definition 5. $f_L : L^n \to L$ is an aggregation function if it is monotone with respect to $\leqslant_L$ and satisfies $f_L(0_L,\ldots,0_L) = 0_L$ and $f_L(1_L,\ldots,1_L) = 1_L$ [12].

A function f_{AIFV} is an aggregation function on AIFVs if it is monotone with respect to $\leqslant_{L_{AIFV}}$ and satisfies $f_{AIFV}(0_{L_{AIFV}},\ldots,0_{L_{AIFV}}) = 0_{L_{AIFV}}$ and $f_{AIFV}(1_{L_{AIFV}},\ldots,1_{L_{AIFV}}) = 1_{L_{AIFV}}$.

There are many ways to construct such aggregation functions for IVFV. One way is to apply the usual aggregation functions to the ends of the intervals. Such functions are called representable in [15].

Definition 6. f_L is a representable aggregation function if there are aggregation functions f_1, f_2 such that $f_1 \leqslant f_2$, and f_L can be expressed as $f_L(A_1,\ldots,A_n) = B$ with $B = [f_1(l_1,\ldots,l_n), f_2(r_1,\ldots,r_n)]$ and $A_i = [l_i,r_i]$, $i = 1,\ldots,n$. A *natural extension* of an aggregation function f to L is f_L with $f_1 = f_2 = f$.

Various constructions of IV aggregation functions, including t-norms and t-conorms, uninorms and OWA are discussed in [15]. In the special case of OWA, Yager [38] uses $f_1 = f_2 = OWA$. We will denote this function by OWA_L.

3. Aggregation functions on the set of AIFV

The following operations are defined for AIFV [4].

$$A + B = \langle \mu_A + \mu_B - \mu_A\mu_B, \nu_A\nu_B\rangle, \tag{1}$$
$$A \cdot B = \langle \mu_A\mu_B, \nu_A + \nu_B - \nu_A\nu_B\rangle. \tag{2}$$

From these formulas one obtains the following equations [16]

$$nA = A + \cdots + A = \langle 1-(1-\mu_A)^n, \nu_A^n\rangle,$$
$$A^n = A \cdot \cdots \cdot A = \langle \mu_A^n, 1-(1-\nu_A)^n\rangle,$$

for any $n = 1,2,\ldots$, which can then be extended for positive real n.

When an aggregation function requires the sort operation (in the OWA function and the discrete Choquet integral), one needs to define a *total order* on AIFVs. The following total order on AIFVs was used in [31,35,34,36].

$$A < B \text{ if and only if} \tag{3}$$

(i) $Score(A) < Score(B)$ or
(ii) $Score(A) = Score(B)$ and $Accuracy(A) < Accuracy(B)$.

Clearly, in the IVFV representation, this corresponds to ordering according to the center of the membership interval, and then (if the centers coincide) according to the length of the interval ($length([l_A,r_A]) > length([l_B,r_B]) \Rightarrow A < B$).

Let us mention one undesirable property of such an ordering; it is not preserved under multiplication by a scalar: $A < B$ does not necessarily imply $\lambda A < \lambda B$ where λ is a scalar, as can be seen from the Example 1 below. This has a profound implication on the lack of monotonicity of aggregation functions for AIFV defined below with respect to the chosen total ordering, as shown in Proposition 1.

Example 1. Take $A = \langle 0.5, 0.4\rangle$, $B = \langle 0.3, 0.2\rangle$ and $\lambda = 0.6$. Since $Score(A) = Score(B) = 0.1$ and $Accuracy(A) = 0.9$, $Accuracy(B) = 0.5$, then $B < A$. But $\lambda A = \langle 1-(1-0.5)^{0.6}, 0.4^{0.6}\rangle \cong \langle 0.3402, 0.5771\rangle$, $\lambda B = \langle 1-(1-0.3)^{0.6}, 0.2^{0.6}\rangle \cong \langle 0.1927, 0.3807\rangle$, and $Score(\lambda A) \cong -0.2369$, $Score(\lambda B) \cong -0.188$, so $\lambda A < \lambda B$. Thus $B < A$ does not imply $\lambda B < \lambda A$.

The following aggregation functions were defined on AIFV in [35,36] using operations of addition and multiplication by a scalar. These operations were called Intuitionistic Fuzzy Weighted Averaging and Intuitionistic Fuzzy Ordered Weighted Averaging respectively in [35].

Definition 7. Intuitionistic Weighted Arithmetic Mean with respect to a weighting vector **w**, $IWAM_w$, is defined as

$$IWAM_w(A_1,\ldots,A_n) = w_1A_1 + w_2A_2 + \cdots + w_nA_n = \left\langle 1 - \prod_{i=1}^n (1-\mu_{A_i})^{w_i}, \prod_{i=1}^n \nu_{A_i}^{w_i}\right\rangle. \tag{4}$$

In Eq. (4) and at other places, $\mathbf{w} = (w_1,w_2,\ldots,w_n)$; $w_i \in [0,1]$; $\sum_{i=1}^n w_i = 1$.

Definition 8. Intuitionistic Ordered Weighted Averaging with respect to a weighting vector **w**, $IOWA_w$, is defined as

$$IOWA_w(A_1,\ldots,A_n) = w_1 A_{\sigma(1)} + w_2 A_{\sigma(2)} + \cdots + w_n A_{\sigma(n)} = \left\langle 1 - \prod_{i=1}^{n}(1-\mu_{A_{\sigma(i)}})^{w_i}, \prod_{i=1}^{n} \nu_{A_{\sigma(i)}}^{w_i} \right\rangle, \tag{5}$$

where $A_{\sigma(i)}$ is the ith largest value according to the total order (3): $A_{\sigma(1)} \geqslant \cdots \geqslant A_{\sigma(n)}$.

The explicit expressions in (4) and (5) were established in [35] from (1) and (2) by induction.

Similarly, the induced IOWA and induced Choquet integrals can be defined as in [31,43].

One problem with such definitions is that they are not consistent with aggregation operations on the ordinary fuzzy sets (when $\mu = 1 - \nu$), nor with the natural extension of aggregation to IVFS (see Definition 6). This can be easily seen from the following example.

Example 2. Take $A = \langle 0.1, 0.9\rangle$, $B = \langle 0.5, 0.5\rangle$ and WAM with the weights $\mathbf{w} = (0.8, 0.2)$. Then

$$WAM_w(0.1, 0.5) = 0.8 \cdot 0.1 + 0.2 \cdot 0.5 = 0.18,$$

and the same value is obtained by using the natural extension of WAM to L, but

$$IWAM_w(A,B) = \langle 1 - 0.9^{0.8}0.5^{0.2}, 0.9^{0.8}0.5^{0.2}\rangle = \langle 0.1998, 0.8002\rangle.$$

Another undesirable feature of (4) and (5) is that whenever one of the arguments $A_i = \langle 1, 0\rangle$ and the corresponding weight is not 0, we have $IWAM_w(A_1,\ldots,A_n) = IOWA_w(A_1,\ldots,A_n) = \langle 1, 0\rangle$, which is rather counterintuitive for an averaging operation. On the other hand, if one of the arguments is $\langle 0, 1\rangle$ and its weight is not zero, it is not accounted for at all, which is again counterintuitive. The latter feature helps to prove that IWAM and IOWA in (4) and (5) are not monotone with respect to the order based on the score and accuracy.

Proposition 1. *Aggregation operators for AIFV defined by (4) and (5) are not monotone with respect to the ordering in (3).*

Proof. Take $A_1 = \langle 0, 1\rangle$, $A_2 = \langle 0.5, 0.4\rangle$, $A_3 = \langle 0.3, 0.2\rangle$ and $w = (0.4, 0.6)$. We know from Example 1 that $A_3 < A_2$. Monotonicity requires that $IWAM(A_1,A_3) < IWAM(A_1,A_2)$, but $IWAM(A_1,A) = 0.6A$ for any intuitionistic value A. From Example 1, $0.6A_2 < 0.6A_3$, so monotonicity does not hold. For IOWA the same argument applies. □

Of course, both (4) and (5), and the operators defined in the next section are monotone with respect to the *partial order* $\leqslant_{L_{AIFV}}$. However it raises an interesting question of whether there are alternative definitions of aggregation operators which are monotone with respect to the total order (3). We will provide an affirmative answer at the end of Section 4.

4. Alternative definitions of aggregation functions on AIFV

Let us write the operation $A + B$ on AIFV as follows

$$A + B = \langle S(\mu_A, \mu_B), T(\nu_A, \nu_B)\rangle, \tag{6}$$

where $T = T_P$ is the product t-norm and $S = S_P$ is its dual t-conorm (called probabilistic sum), defined by $S_P(x,y) = 1 - T_P(1 - x, 1 - y)$. Here we can use any pair of dual t-norm and t-conorm. We first concentrate on continuous Archimedean t-norms and t-conorms, and in particular on the product t-norm, because it was used in many existing definitions. An Archimedean t-norm is strict if it is continuous and strictly increasing.

It is well known (see [21]) that a strict Archimedean t-norm is expressed via its additive generator g as follows

$$T(x,y) = g^{-1}(g(x) + g(y)),$$

and the same applies to its dual t-conorm,

$$S(x,y) = h^{-1}(h(x) + h(y)),$$

with $h(t) = g(1 - t)$. We remind that an additive generator of a continuous Archimedean t-norm is a strictly decreasing function $g : [0,1] \to [0,\infty]$ such that $g(1) = 0$. For nilpotent operations the inverse changes to the pseudoinverse. The additive generator is not unique, it is defined up to an arbitrary positive multiplier [19,21]. For T_P, an additive generator is $g(t) = -\log(t)$.

Now, let $C = A + B$. Then $g(\nu_C) = g(\nu_A) + g(\nu_B)$ and $h(\mu_C) = h(\mu_A) + h(\mu_B)$. Further, if $D = \lambda A$, then $g(\nu_D) = \lambda g(\nu_A)$, $h(\mu_D) = \lambda h(\mu_A)$.

Let us now denote by $\langle\hat{\mu}, \hat{\nu}\rangle = \langle h(\mu), g(\nu)\rangle$, the transformed membership/non-membership pair. In this notation, for AIFV $C = A + B$ and $D = \lambda A$,

$$\langle \hat{\mu}_C, \hat{\nu}_C \rangle = \langle \hat{\mu}_A + \hat{\mu}_B, \hat{\nu}_A + \hat{\nu}_B \rangle,$$
$$\langle \hat{\mu}_D, \hat{\nu}_D \rangle = \langle \lambda\hat{\mu}_A, \lambda\hat{\nu}_A \rangle.$$

We deduce that if $C = IWAM_w(A_1, \ldots, A_n)$ then

$$\langle \hat{\mu}_C, \hat{\nu}_C \rangle = \left\langle \sum_{i=1}^{n} w_i \hat{\mu}_{A_i}, \sum_{i=1}^{n} w_i \hat{\nu}_{A_i} \right\rangle,$$

and explicitly,

$$IWAM_w(A_1, \ldots, A_n) = \left\langle h^{-1}\left(\sum_{i=1}^{n} w_i h(\mu_{A_i})\right), g^{-1}\left(\sum_{i=1}^{n} w_i g(\nu_{A_i})\right) \right\rangle. \tag{7}$$

If $C = IOWA_w(A_1, \ldots, A_n)$ then

$$IOWA_w(A_1, \ldots, A_n) = \left\langle h^{-1}\left(\sum_{i=1}^{n} w_i h(\mu_{A_{\sigma(i)}})\right), g^{-1}\left(\sum_{i=1}^{n} w_i g(\nu_{A_{\sigma(i)}})\right) \right\rangle. \tag{8}$$

Similarly one obtains the expressions for other aggregation functions based on linear combination and re-ordering of the inputs, like the Choquet integral and induced OWA. By taking $g = -\log$ we recover Eqs. (4), (5) from (7) and (8) respectively.

In fact Eqs. (7), (8) show that the degrees of membership and non-membership of the combined AIFVs are simply weighted quasi-arithmetic means of the respective degrees of the components (additionally, in (8) the arguments are reordered). The properties of these functions discussed in [35,36,43,34] are now simple consequences of this fact.

Proposition 2. *The IWAM and IOWA operators in (7) and (8) are idempotent, monotone with respect to the partial order $\leqslant_{L_{AIFV}}$, and are bounded by*

$$A_- \leqslant_{L_{AIFV}} IWAM_w(A_1, \ldots, A_n) \leqslant_{L_{AIFV}} A_+,$$
$$A_- \leqslant_{L_{AIFV}} IOWA_w(A_1, \ldots, A_n) \leqslant_{L_{AIFV}} A_+,$$

with $A_- = \langle \min \mu_{A_i}, \max \nu_{A_i} \rangle$, $A_+ = \langle \max \mu_{A_i}, \min \nu_{A_i} \rangle$. The IOWA is symmetric.

Proof. The proof follows from the fact that weighted quasi-arithmetic means in both components of (7) and (8) are idempotent monotone functions. The symmetry in (8) is obvious. □

We note that direct proofs in [34,43,35] are several pages long. Next we consider the weighted quasi-arithmetic mean (WQAM)

$$M_{\mathbf{w}}(x_1, \ldots, x_n) = \phi^{-1}\left(\sum_{i=1}^{n} w_i \phi(x_i)\right)$$

with a generator ϕ. We remind that $\phi : [0,1] \to [-\infty, \infty]$ is a continuous strictly monotone function.

The intuitionistic WQAM was defined in [43] as follows (the authors considered only power functions $\phi(t) = t^p$, $p > 0$, and called it the generalized intuitionistic fuzzy weighted averaging (GIFWA))

$$IWQAM_{w,p}(A_1, \ldots, A_n) = \left\langle \left(1 - \prod^{p})^{w_i}\right)^{1/p}, 1 - \left(1 - \prod(1 - (1 - \nu_{A_i})^p)^{w_i}\right)^{1/p} \right\rangle. \tag{9}$$

It can be written in our notation in this way

$$\begin{aligned} IWQAM_w(A_1, \ldots, A_n) &= \left\langle \phi^{-1}\left(h^{-1}\left(\sum w_i h(\phi(\mu_{A_i}))\right)\right), 1 - \phi^{-1}\left(h^{-1}\left(\sum w_i h(\phi(1 - \nu_{A_i}))\right)\right) \right\rangle \\ &= \left\langle u^{-1}\left(\sum w_i u(\mu_{A_i})\right), v^{-1}\left(\sum w_i v(\nu_{A_i})\right) \right\rangle, \end{aligned} \tag{10}$$

with $u = h \circ \phi$ and $v = h \circ \phi \circ N$, where N is the standard negation. Note that this formula is similar to (7), just with a different pair of additive generators; note $u(t) = v(1 - t)$. In fact, it gives the same expression as (7), as long as we use a different pair of t-norm and t-conorm in (6) instead of the product, namely a strict t-norm with the generator v and its dual. If ϕ is the power function as in [43], it is the pair of dual t-norm and t-conorm defined by using the generator $g(t) = -\log(1 - (1 - t)^\lambda)$.

In general, the operator (9) inherits the same problems of inconsistency with the operations on ordinary fuzzy sets mentioned at the end of Section 3 (to see this, it is sufficient to take $\phi = Id$ to get IWAM as the special case we already considered). Later we will show that by using a different pair of t-norm and t-conorm in (6), we can achieve consistency.

Let us now extend these constructions to other aggregation functions. Take any aggregation function f, and following the same approach, define an aggregation function for AIFV f_{AIFV}. There are several ways of doing this. First, directly by using the following definition:

Definition 9. Let f be an aggregation function, and g is an additive generator of the t-norm in (6). f_{AIFV} is an aggregation function on AIFV corresponding to f if $f_{AIFV}(A_1,\ldots,A_n) = C$ with

$$\langle \hat{\mu}_C, \hat{v}_C \rangle = \langle f_\sigma(\hat{\mu}_{A_1},\ldots,\hat{\mu}_{A_n}), f_\sigma(\hat{v}_{A_1},\ldots,\hat{v}_{A_n}) \rangle, \quad (11)$$

where the index σ indicates that f may depend on a permutation σ of the participating $A_1,\ldots,A_n$.

The dependence on σ is needed specifically for IOWA and Choquet integrals by (5), where the ordering is done with respect to AIFV and not with respect to μ and v separately.

Proposition 3. *The expression (11) is well defined, i.e., f_{AIFV} is monotone with respect to $\leqslant_{L_{AIFV}}$ and its result is always an AIFV.*

Proof. Monotonicity is straightforward by noticing that both components of (11) are monotone with respect to all μ_{A_i} and v_{A_i}, $i = 1,\ldots,n$. To show that $\mu_C + v_C \leqslant 1$ it is convenient to write (11) in the IVFV representation

$$[\hat{l}_C, \hat{r}_C] = [f_\sigma(\hat{l}_{A_1},\ldots,\hat{l}_{A_n}), f_\sigma(\hat{r}_{A_1},\ldots,\hat{r}_{A_n})],$$

where $\hat{l} = h(l)$ and $\hat{r} = h(r)$. Since $l_{A_i} \leqslant r_{A_i}$, $h(l_{A_i}) \leqslant h(r_{A_i})$ for all i, and because f_σ is non-decreasing in all components, we get $\hat{l}_C \leqslant \hat{r}_C$ and therefore $l_C \leqslant r_C$. □

By taking the aggregation functions from [31,33–35,43] (WAM, OWA, Choquet integrals), and generator $g = -\log$ we obtain the respective formulas as special cases.

However, WQAM and generalized OWA as defined in [43] by (9) do not fit this definition. Eq. (9) is consistent with (10) whereas (11) generally leads to a different formula

$$IWQAM_w(A_1,\ldots,A_n) = \left\langle h^{-1}\left(\phi^{-1}\left(\sum_{i=1}^n w_i\phi(h(\mu_{A_i}))\right)\right), g^{-1}\left(\phi^{-1}\left(\sum_{i=1}^n w_i\phi(g(v_{A_i}))\right)\right)\right\rangle.$$

An alternative, which accommodates and extends definitions in [34,43] to a special class of generated aggregation functions, is the following:

Definition 10. Let f be a generated aggregation function with a generator ϕ, so that f is defined by $\phi(f(x_1,\ldots,x_n)) = \sum_{i=1}^n w_i\phi(x_i)$. $f_{AIFV_\mathbf{w}}$ is an aggregation function on AIFV if $f_{AIFV_\mathbf{w}}(A_1,\ldots,A_n) = C$ with

$$\langle \hat{\mu}_C, \hat{v}_C \rangle = \left\langle \sum_{i=1}^n w_{\sigma(i)}\hat{\mu}_i, \sum_{i=1}^n w_{\sigma(i)}\hat{v}_i \right\rangle, \quad (12)$$

and $\hat{\mu} = u(\mu)$, $\hat{v} = v(v)$, $u = h\circ\phi$, $v = h\circ\phi\circ N$, $h = g\circ N$ and g is an additive generator of a strict Archimedean t-norm. $w_{\sigma(i)}$; $i = 1,\ldots,n$, denote weights which depend on the ordering of pairs $\langle \mu_i, v_i\rangle$; $i = 1,\ldots,n$.

By specifying how $w_{\sigma(i)}$ depends on the ordering of the arguments, we get weighted quasi-arithmetic means, generalized OWA and generalized Choquet integrals as special cases. Note that if $\phi = Id$, Eq. (12) is equivalent to Eq. (11).

So far we have presented constructions that generalize those in [31,33–35,43], however the problem of consistency with the operations on ordinary fuzzy sets remains. To ensure consistency, let us now look at the following alternative. Take in (6) S_L, T_L, as the Łukasiewicz t-conorm and t-norm respectively, $S_L(x,y) = \min(1,x+y)$, $T_L(x,y) = \max(0,x+y-1)$ with $g(t) = 1-t$. Then we have for $\lambda \in [0,1]$

$$\lambda A = \langle \lambda\mu_A, 1-\lambda(1-v_A)\rangle.$$

We immediately obtain that the ordering of AIFV described in Section 3 is invariant under multiplication by a scalar $\lambda \in [0,1]$, because $Score(A) = Score(B) \Rightarrow Score(\lambda A) = Score(\lambda B)$, $Score(A) < Score(B) \Rightarrow Score(\lambda A) < Score(\lambda B)$, $Accuracy(A) = Accuracy(B) \Rightarrow Accuracy(\lambda A) = Accuracy(\lambda B)$, $Accuracy(A) < Accuracy(B) \Rightarrow Accuracy(\lambda A) < Accuracy(\lambda B)$. It is sufficient to observe that such an operation can be written in the IVFS representation as $\lambda[l,r] = [\lambda l, \lambda r]$.

Taking into account that $h^{-1} = h = Id$ and $g^{-1} = g = 1 - Id$ on $[0,1]$, and that $\sum_{i=1}^n w_i = 1$ ensures the argument of g^{-1} is in $[0,1]$, we consequently obtain from (7), (8)

$$IWAM_w(A_1,\ldots,A_n) = \left\langle \sum_{i=1}^n w_i\mu_{A_i}, 1-\sum_{i=1}^n w_i(1-v_{A_i}) \right\rangle = \left\langle \sum_{i=1}^n w_i\mu_{A_i}, \sum_{i=1}^n w_i v_{A_i} \right\rangle, \quad (13)$$

and

$$IOWA_w(A_1,\ldots,A_n) = \left\langle \sum_{i=1}^n w_i\mu_{A_{\sigma(i)}}, \sum_{i=1}^n w_i v_{A_{\sigma(i)}} \right\rangle. \quad (14)$$

The expression (13) is a natural extension of the WAM according to the Definition 6, and hence is a representable f_L. However (14) cannot be written as a representable f_L, because the order in which $\mu_{A_{\sigma(i)}}$ and $v_{A_{\sigma(i)}}$ are arranged depends on both μ and v according to (3). In contrast, representable f_L imply that aggregation of μ values does not depend on v values and vice versa.

Consequently $IOWA_w$ is not the same as the operator OWA_L in [38], which is a natural extension of OWA according to Definition 6 with $f_1 = f_2 = OWA$.

What is remarkable is that the use of Łukasiewicz t-norm and t-conorm in (6) is the only way consistency with the ordinary fuzzy sets can be achieved.

Proposition 4. *The operations defined by (7) and (8) are consistent with the operations on ordinary fuzzy sets if and only if the* t-*norm and* t-*conorm in (6) are Łukasiewicz ones.*

Proof. The necessity is straightforward, by letting $\nu_i = 1 - \mu_i$ in (13), (14). Sufficiency: for consistency of (7) with operations on ordinary fuzzy sets we need to ensure

$$h^{-1}\left(\sum h(\mu_{A_i})\right) = \sum w_i \mu_{A_i} \text{ for all } \mu_{A_i} \in [0,1], \tag{15}$$

(since $\nu = 1 - \mu$, the respective condition for non-membership is automatically verified). The unique solution h to the functional Eq. (15) is an affine function [1]. Because h is a generator of a t-conorm $h(0) = 0$, and hence it is a linear function. The multiplier is irrelevant since additive generators are defined up to an arbitrary positive multiplier. Hence $h = Id$ on [0,1] or its multiple.

The same result holds for (8), since the order (3) coincides with the usual order on real numbers in that case. □

Corollary 1. *The operations defined by (7) and (10) are consistent with the natural extensions of WAM and WQAM to f_L if and only if the* t-*norm* and t-*conorm* in *(6) are Łukasiewicz ones.*

Proof. Straightforward by using Proposition 4 and noticing that $u = h \circ \phi$ in (10). □

However, this result does not hold for natural extensions of OWA and other operations that require sorting based on order (3). Since the order (3) uses both μ and ν, the operations (8) are not representable.

Another useful feature of our construction that uses Łukasiewicz t-norm and t-conorm, is that the resulting aggregation functions are monotone not only with respect to the partial order $\leqslant_L$ ($\leqslant_{L_{AIFV}}$), but also with respect to the total order in (3).

Proposition 5. *Operations (10), (13) and (14) are monotone with respect to the order (3).*

Proof. Immediate for (10) and (13), since these operations are natural extensions of WAM and WQAM to f_L. For (14) note that in both sums the ordering is the same, and the operator can be written as

$$IOWA_w(A_1,\ldots,A_n) = \sum w_i A_{\sigma(i)} = \sum w_i \langle \mu_{A_{\sigma(i)}}, \nu_{A_{\sigma(i)}} \rangle,$$

which is monotone with respect to (3), because now multiplication by a scalar and addition preserve the ordering (3). □

This results does not hold for a natural extension of OWA, as can be seen from the following example.

Example 3. Take $A = [0,1]$, $B = [0.1,0.9]$ in the IVFS representation, so that $A < B$ according to (3). Let $w = (1,0)$, in which case $OWA_L(A,B) = [0.1,1]$. Now if $C = [0.1,0.95]$ then $A < B < C$ but $IOWA_L(B,C) = [0.1,0.95] < OWA_L(A,B)$.

Finally, our approach can be extended to other averaging aggregation functions, such as quasi-arithmetic means, Choquet integral and induced OWA. We define the aggregation functions by using Definition 9, which in the case of generated functions is equivalent to Definition 10.

Let us illustrate this construction on two examples.

Example 4. Let f be a weighted quasi-arithmetic mean with a generator ϕ. We have

$$IWQAM_w(A_1,\ldots,A_n) = \left\langle \phi^{-1}\left(\sum w_i \phi(\mu_{A_i})\right), 1 - \phi^{-1}\left(\sum w_i \phi(1-\nu_{A_i})\right)\right\rangle = \left\langle f(\mu_{A_1},\ldots,\mu_{A_n}), f^d(\nu_{A_1},\ldots,\nu_{A_n})\right\rangle,$$

where f^d denotes the dual of f with respect to the standard negation. When $\phi = Id$ we get (13) as a special case (we remind that a WAM is self-dual). In the IVFS representation we have

$$IWQAM_w(A_1,\ldots,A_n) = [f(l_{A_1},\ldots,_{A_n}), f(r_{A_1},\ldots,r_{A_n})].$$

This means that $IWQAM_w$ is representable and a natural extension of WQAM. The choices $\phi(t) = t^p, p \neq 0$, $\phi(t) = \log(t)$ $\phi(t) = p^t$, $p \neq 1$ and $\phi(t) = \exp(\exp(t))$ lead to natural extensions of weighted power means, geometric mean, exponential mean and double exponential means. See [7] for other types of quasi-arithmetic means that can be used.

Example 5. Consider the logarithmic mean $f(x,y) = \frac{y-x}{\log y - \log x}$, which is not a quasi-arithmetic mean. Using Definition 9, we obtain the logarithmic mean for AIFS

$$ILM(A_1, A_2) = \left\langle \frac{\mu_2 - \mu_1}{\log \mu_2 - \log \mu_1}, 1 - \frac{\nu_1 - \nu_2}{\log(1-\nu_2) - \log(1-\nu_1)} \right\rangle.$$

Formulas for generalized logarithmic, Bonferroni, Heronian and other means can be found in [7]. We note that existing constructions for "generalized" aggregation functions for AIFS in [31,33,34,36,43,35] do not work in these cases.

5. Conclusion

We have examined various definitions of aggregation operators for AIFS, which have appeared recently in the literature and are based on the operation of addition of AIFV. We have found that in all such cases, the respective expressions can be written with the help of an additive generator of the t-norm used in the operation of addition. As a consequence, most properties of aggregation operators for AIFS defined in this way follow automatically.

We looked at generalizations of averaging aggregation operators by using generating functions (quasi-arithmetic mean type construction) and obtained the respective expressions using a much more compact and general notation. We proposed two general definitions for constructing other types of aggregation operators for AIFS extending the existing methods. Both representable and non-representable aggregation functions are obtained in this way. Finally, we have shown that only by using the Łukasiewicz t-norm rather than the product t-norm in the addition of AIFV, one obtains operators (both representable and non-representable) consistent with operators on ordinary fuzzy sets. We have established the relation between corresponding expressions for AIFS and the interval valued fuzzy sets and provided several examples.

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